3

CHANGE OF RINGS

In this chapter, we introduce a family of constructions that show off the functorial approach to great advantage. The common theme is that they allow us to pass from a category of modules over one ring to a category of modules over another ring. These constructions are based on the tensor product, which we introduce in the first section. Given a ring R, a right R-module M and a left R-module N, their tensor product $M \otimes_R N$ is an abelian group that has a certain universal property. If also M or N is a bimodule, then $M \otimes_R N$ inherits a module structure. Thus, by holding one of M and N fixed and varying the other, we obtain functors between module categories.

An important matter is whether or not these functors are exact; this leads to the concept of a flat module, which always gives an exact functor. Some fundamental examples and properties of flat modules are given in section 2.

One way of changing rings is by means of a ring homomorphism from one ring to another. Classically, such homomorphisms are inclusions and the functors are extension and restriction of scalars. In the third section, we look at these functors for a ring homomorphism in general.

3.1 THE TENSOR PRODUCT

The tensor product is a fundamental tool for constructing functors between various categories of modules. In this section, we give the definition and basic properties of the tensor product $M \otimes_R N$ of a right R-module M and a left R-module N, where R is an arbitrary ring. We then establish the functorial properties of the tensor product, which can be viewed variously as a functor of the second term N, with M being kept fixed, or of the first term M, with N kept fixed, or as a bifunctor. We discuss how the tensor product behaves on various module categories, and we investigate the adjointness between the tensor product and the Hom functor.

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Finally, we show that the category of modules over a generalized triangular matrix ring can be realized as a morphism category.

3.1.1 The definition

Let R be an arbitrary ring, let M_R be a right R-module and let ${}_RN$ be a left R-module.

A *biadditive R-balanced map* on the pair M, N is a function β from the cartesian product $M \times N$ to some abelian group A, with the following properties:

$$\beta(m + m', n) = \beta(m, n) + \beta(m', n),$$

$$\beta(m, n + n') = \beta(m, n) + \beta(m, n')$$

and

$$\beta(mr, n) = \beta(m, rn)$$

for all m, m' in M, n, n' in N and r in R.

The *tensor product* $M \otimes_R N$ is defined to be the universal object with respect to such maps (see (1.4.2)). This means that $M \otimes_R N$ is an abelian group and that there is a biadditive R-balanced map

$$\tau : M \times N \longrightarrow M \otimes_R N$$

with the property that, given any β as above, there is a unique homomorphism

$$\alpha : M \otimes_R N \longrightarrow A$$

with $\alpha\tau = \beta$.

For a category-theoretic description, consider the category whose objects are the biadditive R-balanced maps $\beta : M \times N \to A$ on the pair M, N, a morphism from $\beta : M \times N \to A$ to $\beta' : M \times N \to A'$ being a homomorphism of abelian groups $\alpha : A \to A'$ with $\alpha\beta = \beta'$. Then $\tau : M \times N \to M \otimes_R N$ is the initial object in this category. The associated diagram is

$$\begin{tikzcd} M \times N \arrow[rr, "\tau"] \arrow[dr] & & M \otimes_R N \arrow[dl] \\ & A & \end{tikzcd}$$

Such a definition does not guarantee the existence of the required universal object, which we now establish.

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3.1.2 The construction

To see that the tensor product exists, start with the free abelian group $M\#N$ generated by the cartesian product $M \times N$; elements of $M\#N$ can be uniquely expressed as sums of the form

$$z_1(m_1, n_1) + \cdots + z_k(m_k, n_k),$$

where $z_i \in \mathbb{Z}$, $m_i \in M$, $n_i \in N$ for $i = 1, \ldots, k$, and $k \geq 0$ (the vacuous sum with $k = 0$ is to be read as 0).

Next, let $B(M, N)$ be the subgroup of $M\#N$ which is generated by all expressions of the form

$$(m + m', n) - (m, n) - (m', n),$$

$$(m, n + n') - (m, n) - (m, n'),$$

and

$$(mr, n) - (m, rn),$$

with m, m' in M, n, n' in N and r in R.

We write

$$M \otimes_R N = (M\#N)/B(M, N),$$

and put $\tau(m, n) = m \otimes n$, where $\tau : M \times N \to M \otimes_R N$ is the canonical homomorphism. (Where it is felt safe to do so, we occasionally write $M \otimes N$, omitting the ring R.)

A typical element of the tensor product is then a sum of the form

$$z_1(m_1 \otimes n_1) + \cdots + z_k(m_k \otimes n_k),$$

and the identities

$$(m + m') \otimes n = m \otimes n + m' \otimes n,$$

$$m \otimes (n + n') = m \otimes n + m \otimes n',$$

and

$$mr \otimes n = m \otimes rn$$

hold among the generators of $M \otimes_R N$.

Thus τ is biadditive R-balanced as desired. It is straightforward to check that τ is universal among such maps in the sense described previously and thus that $M \otimes_R N$ is the tensor product.

Although applied mathematicians and physicists have long computed with tensors, the formal definition of the tensor product is comparatively recent.

It was given for abelian groups in [Whitney 1938], and subsequently for arbitrary rings and modules in [Bourbaki 1948]. It should be remarked that the tensor product is not in fact a product or coproduct as defined in (1.4.11) and (1.4.12), but a 'coend'. For a definition of this term and a category-theoretic analysis of the tensor product, see [Mac Lane 1971] p. 222.

3.1.3 Bimodule structures

Commonly, $M \otimes_R N$ is rather more than just an abelian group. For example, when N is an R-S-bimodule, then $M \otimes_R N$ becomes an S-module via

$$(m \otimes n)s = m \otimes ns \text{ for } m \in M,\ n \in N \text{ and } s \in S.$$

To verify this assertion, note that $M\#N$ can be made into a right S-module by setting $(m,n)s = (m,ns)$. Then $B(M,N)$ is an S-submodule of $M\#N$ and so the quotient abelian group $M \otimes_R N$ inherits a right S-module structure as indicated.

This method of constructing actions or mappings on $M \otimes_R N$ can be described informally as specifying what happens to elements of the form $m \otimes n$ and then extending the effect to linear combinations of such elements 'by linearity'. Properly speaking, the action or mapping must first be defined on the generators of $M\#N$, and then it must be verified that $B(M,N)$ is invariant, so that there is a corresponding induced action or mapping on $M \otimes_R N$. However, such verifications are usually trivial.

If M is a T-R-bimodule for some ring T, then $M \otimes_R N$ is a left T-module, and if M is a T-R-bimodule and N an R-S-bimodule, then $M \otimes_R N$ is a T-S-bimodule.

Our first computation is the following result, wherein R is viewed as an R-R-bimodule.

3.1.4 Proposition

Let M_R be a right R-module and let ${}_RN$ be a left R-module. Then

(i) $M \otimes_R R \cong M$ *as right R-modules*

and

(ii) $R \otimes_R N \cong N$ *as left R-modules.*

Proof

(i) Define homomorphisms $\eta : M \otimes_R R \to M$ and $\lambda : M \to M \otimes_R R$, by $m \otimes r \mapsto mr$ and $m \mapsto m \otimes 1$ respectively. Since $m \otimes r = mr \otimes 1$ (and $m1 = m$), η and λ are mutually inverse R-module homomorphisms.

(ii) Similar. □

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The next fact can be interpreted as saying that the tensor product is associative.

3.1.5 Proposition

Let R and S be rings and let L_R, ${}_RM_S$ and ${}_SN$ be modules as indicated. Then there is an isomorphism

$$(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N).$$

Proof

The expected isomorphism is that under which $(\ell \otimes m) \otimes n$ corresponds to $\ell \otimes (m \otimes n)$. We must check that this correspondence does in fact give a well-defined homomorphism.

Take a fixed element n of N. The map

$$\beta(n) : L \times M \to L \otimes_R (M \otimes_S N),$$

$$(\ell, m) \longmapsto \ell \otimes (m \otimes n),$$

is biadditive and R-balanced (since $r(m \otimes n) = rm \otimes n$), and so it induces a homomorphism

$$\beta'(n) : L \otimes_R M \longrightarrow L \otimes_R (M \otimes_S N).$$

Allowing n to vary, we see that the maps $\beta'(n)$ define a biadditive S-balanced map $\beta' : (L \otimes_R M) \times N \to L \otimes_R (M \otimes_S N)$, which induces a homomorphism $\theta : (L \otimes_R M) \otimes_S N \to L \otimes_R (M \otimes_S N)$ as desired.

Reversing this construction, and appealing to the universal property of the tensor product, we confirm that θ has the expected inverse. □

3.1.6 Functorial properties of tensor products

We wish to see how the tensor product behaves as a functor when one of the modules M_R or ${}_RN$ is kept fixed and the other varies through ${}_R\mathcal{M}\textsc{od}$ or $\mathcal{M}\textsc{od}_R$. As we are mainly concerned with functors on categories of right modules, we give the arguments in the case where the second module in the tensor product is regarded as being constant. The parallel results where the first module is viewed as constant are noted from time to time, the details usually being left to the reader, who will easily get them right.

Let S be an arbitrary ring and suppose that N is an R-S-bimodule. As noted above, for any right R-module M the tensor product $M \otimes_R N$ is a right S-module under the rule

$$(m \otimes n)s = m \otimes ns, \ s \in S.$$

Thus, for any homomorphism $\alpha : M' \to M$ of right R-modules, the induced homomorphism

$$\alpha \otimes id_N : M' \otimes_R N \longrightarrow M \otimes_R N,$$

defined by

$$(\alpha \otimes id_N)(m' \otimes n) = \alpha m' \otimes n,$$

is a homomorphism of right S-modules.

Write $\alpha \otimes_R N = \alpha \otimes id_N$, so that the symbol $- \otimes_R N$ represents both a mapping from the objects of $\mathcal{M}\textsc{od}_R$ to those of $\mathcal{M}\textsc{od}_S$ and a collection of mappings

$$- \otimes_R N : \operatorname{Hom}_R(M', M) \longrightarrow \operatorname{Hom}_S(M' \otimes_R N, M \otimes_R N),$$

one for each pair of R-modules M', M. This information, together with the obvious properties

$$id_M \otimes id_N = id_{M \otimes N}$$

and

$$(\beta\alpha) \otimes id_N = (\beta \otimes id_N)(\alpha \otimes id_N),$$

defines a functor from $\mathcal{M}\textsc{od}_R$ to $\mathcal{M}\textsc{od}_S$, which we again denote $- \otimes_R N$. Moreover, $- \otimes_R N$ is additive because if $\alpha : M' \to M$ and $\alpha' : M' \to M$ are right R-module homomorphisms, then

$$(\alpha + \alpha') \otimes id_N = \alpha \otimes id_N + \alpha' \otimes id_N.$$

In summary, we have the following result.

3.1.7 Lemma

$- \otimes_R N$ is an additive covariant functor from $\mathcal{M}\textsc{od}_R$ to $\mathcal{M}\textsc{od}_S$. □

3.1.8 Proposition

Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ be a direct sum of right R-modules, where Λ is any ordered set. Then

$$M \otimes_R N \cong \bigoplus_{\lambda \in \Lambda} (M_\lambda \otimes_R N).$$

Proof

When Λ is finite, the assertion follows immediately from the preceding result, since additive functors preserve finite direct sums (2.2.20).

Now suppose that Λ is infinite. An element of either $M \otimes N$ or of the direct sum $\bigoplus_{\lambda\in\Lambda}(M_\lambda \otimes_R N)$ involves nonzero terms $m_\lambda \in M_\lambda$ for only a finite subset Σ of Λ, which makes it clear that there is a homomorphism

$$\phi : M \otimes_R N \longrightarrow \bigoplus\nolimits_{\lambda\in\Lambda}(M_\lambda \otimes_R N)$$

and that ϕ is surjective.

Suppose that $\phi(x) = 0$. Then the preimage of $\phi(x)$ in $\bigoplus_\Sigma(M_\sigma \# N)$ belongs to the direct sum of relation subgroups $\bigoplus_\Sigma B(M_\sigma, N)$. Thus the preimage of x is already 0 in $(\bigoplus_\Sigma M_\sigma) \otimes N$, and hence $x = 0$. □

Any left R-module N is automatically an R-$\mathbb{Z}$-bimodule, so our discussion includes the case when N has not been attributed any particular structure as a right module. The functor $- \otimes_R N$ is then a functor from $\mathcal{M}\textsc{od}_R$ to $\mathcal{A}\textsc{b}$, the category of abelian groups.

3.1.9 Fixing the first argument

We now review the results and notation that arise when the first module of a tensor product is considered to be an operator on the second.

Let M be a T-R-bimodule for some ring T. Then there is an additive functor

$$M \otimes_R - : {}_R\mathcal{M}\textsc{od} \longrightarrow {}_T\mathcal{M}\textsc{od}.$$

On modules,

$$(M \otimes_R -)(N) = M \otimes_R N,$$

with the T-module structure given by

$$t(m \otimes n) = tm \otimes n,$$

and for a homomorphism $\zeta : N' \to N$ of left R-modules,

$$M \otimes \zeta = (id_M \otimes \zeta) : m \otimes n' \longmapsto m \otimes n'\zeta.$$

When $id_M \otimes \zeta$ is to be considered as a left T-module homomorphism, as happens in (5.1.19) for example, we write $id_M \otimes \zeta$ as a right operator, in accord with our convention (1.1.4).

However, sometimes we are given only that M is a right R-module, and we wish to consider M to be a $\mathbb{Z}$-R-bimodule in the natural way. In this case, the functor $M \otimes_R -$ takes values in ${}_\mathbb{Z}\mathcal{M}\textsc{od}$, but, for applications, it is often more convenient to compose it with the mirror functor Mir : ${}_{\mathbb{Z}}\mathcal{M}\textsc{od} \to \mathcal{M}\textsc{od}_{\mathbb{Z}}$ and view $M \otimes_R -$ as having values in $\mathcal{A}\textsc{b} = \mathcal{M}\textsc{od}_{\mathbb{Z}}$.

Thus, for a homomorphism $\zeta : N' \to N$ of left R-modules, $id_M \otimes \zeta$ is to be viewed as a left operator on $M \otimes_R N'$, which is the convention most commonly met in the literature. With this notation, $M \otimes_R -$ is a covariant functor from ${}_R\mathcal{M}\textsc{od}$ to $\mathcal{M}\textsc{od}_{\mathbb{Z}}$ that appears to 'reverse products'; in the terminology of (1.2.6), it is a contrachiral functor.

This approach is especially handy when we have both a right R-module homomorphism $\alpha : M' \to M$ and a left R-module homomorphism $\zeta : N' \to N$, in which case the composite

$$\alpha \otimes \zeta = (\alpha \otimes id_N)(id_{M'} \otimes \zeta) : M' \otimes_R N' \longrightarrow M \otimes_R N$$

cannot in general be regarded as a module homomorphism.

We use the convention that $id_M \otimes \zeta$ is to be considered as a left operator for the remainder of the present chapter.

Next, we note some naturality properties of the functors $M \otimes_R -$ and $- \otimes_R N$.

3.1.10 Proposition

Let T, R and S be arbitrary rings.

(i) *Let $\zeta : N' \to N$ be a homomorphism of left R-modules or of R-S-bimodules. Then ζ induces a natural transformation*

$$- \otimes_R \zeta : - \otimes_R N' \longrightarrow - \otimes_R N.$$

(ii) *Let $\alpha : M' \longrightarrow M$ be a homomorphism of right R-modules or of T-R-bimodules. Then α induces a natural transformation*

$$\alpha \otimes_R - : M' \otimes_R - \longrightarrow M \otimes_R -.$$

Proof

The argument is a matter of interpreting the requirements for a natural transformation (1.3.1) in terms of the notation of the tensor product. Fix ζ, and define η_M for each M in $\mathcal{M}\textsc{od}_R$ by

$$\eta_M = id_M \otimes \zeta : M \otimes_R N' \longrightarrow M \otimes_R N.$$

Given a homomorphism $\alpha : M' \to M$ of right R-modules, we need to check that $(\alpha \otimes N)\eta_{M'} = \eta_M(\alpha \otimes N')$. But both sides are simply $\alpha \otimes \zeta$. □

If we allow both M and N to vary simultaneously, the preceding result can be interpreted as saying that $- \otimes_R -$ is a bifunctor from $\mathcal{M}\textsc{od}_R \times {}_R\mathcal{M}\textsc{od}$ to the category $\mathcal{A}\textsc{b}$ of abelian groups.

We now state a refined version of (3.1.4).

3.1.11 Proposition

(i) *The functor* $- \otimes_R R$ *is naturally isomorphic to the identity functor on* $\mathcal{M}\text{OD}_R$.

(ii) *The functor* $R \otimes_R -$ *is naturally isomorphic to the identity functor on* ${}_R\mathcal{M}\text{OD}$.

Proof

For each right R-module M, let

$$\eta_M : M \otimes_R R \xrightarrow{\cong} M, \qquad \eta_M(m \otimes r) = mr,$$

be the isomorphism of right R-modules defined in (3.1.4). If $\alpha : M' \to M$ is a homomorphism of right R-modules, then the formula $\alpha(m'r) = \alpha(m')r$ gives

$$\alpha\eta_{M'} = \eta_M(\alpha \otimes id_R);$$

thus η is a natural transformation of functors. □

As $M \otimes_R -$ and $- \otimes_R N$ are additive functors, the next result is immediate from (2.2.20).

3.1.12 Proposition

(a) *Suppose that* $M \cong M' \oplus M''$ *as a right* R*-module and that* N *is an* R*-*S*-bimodule. Then*

$$M \otimes_R N \cong (M' \otimes_R N) \oplus (M'' \otimes_R N)$$

as a right S*-module.*

(b) *Suppose that* $N \cong N' \oplus N''$ *as a left* R*-module and that* M *is a* T*-*R*-bimodule. Then*

$$M \otimes_R N \cong (M \otimes_R N') \oplus (M \otimes_R N'')$$

as a left T*-module.* □

Since the isomorphisms in parts (a) and (b) of the above proposition are evidently natural in N and M respectively, the definitions of a direct sum of functors (2.2.21) and of a natural isomorphism (1.3.3) lead to the following functorial interpretation of the proposition.

3.1.13 Proposition

(a) *Suppose that $M \cong M' \oplus M''$ as a right R-module. Then there is a natural isomorphism*

$$M \otimes_R - \simeq M' \otimes_R - \oplus M'' \otimes_R -$$

of functors from ${}_R\mathcal{M}\text{OD}$ to $\mathcal{A}\text{B}$.

(b) *Suppose that $N \cong N' \oplus N''$ as a left R-module. Then there is a natural isomorphism*

$$- \otimes_R N \simeq - \otimes_R N' \oplus - \otimes_R N''$$

of functors from $\mathcal{M}\text{OD}_R$ to $\mathcal{A}\text{B}$. □

The above proposition easily leads to the next, in the case of finite direct sums. However, we also need to consider direct sums based on an infinite ordered set Λ, and to introduce some notation. Recall that we write the standard free right R-module on Λ as R^Λ. The *standard free left* R-module is written as ${}^\Lambda R$, and we view ${}^\Lambda R$ as the 'space of row vectors' over ${}^\Lambda R$ indexed by Λ. When the index set is the finite set $\{1, \ldots, n\}$, we use the more familiar notations R^n and ${}^n R$ for the free right and left modules respectively. The direct sum of copies of a general R-module M is written M^Λ or M^n, regardless of chirality.

3.1.14 Proposition

(a) *The two functors*

$$\mathcal{M}\text{OD}_R \times \mathcal{O}\text{RD} \longrightarrow \mathcal{M}\text{OD}_R$$

given respectively by

$$(M, \Lambda) \longmapsto M \otimes_R {}^\Lambda R$$

and

$$(M, \Lambda) \longmapsto M^\Lambda$$

are naturally isomorphic.

(b) *The two functors*

$${}_R\mathcal{M}\text{OD} \times \mathcal{O}\text{RD} \longrightarrow {}_R\mathcal{M}\text{OD}$$

given respectively by

$$(N, \Lambda) \longmapsto R^\Lambda \otimes_R N$$

and

$$(N, \Lambda) \longmapsto N^\Lambda$$

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are naturally isomorphic.

Proof

It suffices to discuss (a). The key point is that the sequences (m_λ), with each $m_\lambda \in M$, $\lambda \in \Lambda$, that make up M^Λ, have only finitely many terms m_λ nonzero. Writing the standard generators of ${}^\Lambda R$ as e_λ, we map $M \otimes_R {}^\Lambda R$ to M^Λ by sending $\sum_i(\sum_\lambda m_i \otimes r_{i\lambda}e_\lambda)$ to $(\sum_i m_i r_{i\lambda})_\lambda$. Clearly this defines a surjective natural transformation, whose injectivity follows from the interchangeability of the finite summations:

$$\sum_i(\sum_\lambda m_i \otimes r_{i\lambda}e_\lambda) = \sum_\lambda(\sum_i m_i r_{i\lambda}) \otimes e_\lambda. \qquad \square$$

We also note an important special case of the above proposition. Let $M_{m,n}(R)$ denote the additive group of $m \times n$ matrices with entries in the ring R. Clearly, $M_{m,n}(R)$ is an $M_m(R)$-$M_n(R)$-bimodule, and we know that R^m is an $M_m(R)$-R-bimodule and that the standard free left R-module ${}^nR$ is an R-$M_n(R)$-bimodule. A direct verification gives the following result.

3.1.15 Corollary

$R^m \otimes_R {}^nR \cong M_{m,n}(R)$ as an $M_m(R)$-$M_n(R)$-bimodule. $\square$

3.1.16 Return of the dyads

In classical tensor analysis, an $m \times n$ matrix that can be expressed as the matrix product of a column vector by a row vector (that is, an $m \times 1$ matrix by a $1 \times n$ matrix) is called a *dyad*, especially when $m = n$. Thus the above result tells us that any matrix is a sum of dyads.

We next investigate the effect of a tensor product when it acts on subcategories of $\mathcal{M}\text{od}_R$. Recall that $\mathcal{M}_R$ is the category of finitely generated right R-modules and that $\mathcal{P}_R$ is the category of finitely generated projective right R-modules.

3.1.17 Lemma

Let N be an R-S-bimodule.

(i) *The functor $- \otimes_R N$ from $\mathcal{M}\text{od}_R$ to $\mathcal{M}\text{od}_S$ induces a functor*

$$- \otimes_R N : \mathcal{M}_R \longrightarrow \mathcal{M}_S$$

if and only if N is finitely generated as a right S-module.

(ii) *The functor $- \otimes_R N$ from $\mathcal{M}\text{od}_R$ to $\mathcal{M}\text{od}_S$ induces a functor*

$$- \otimes_R N : \mathcal{P}_R \longrightarrow \mathcal{P}_S$$

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if and only if N is finitely generated and projective as a right S-module.

Proof

Necessity is clear in both cases, since $N \cong R \otimes_R N$ must be in the target category. On the other hand, if N has a finite set $\{n_1, \ldots, n_k\}$ of generators as an S-module, and M has generators $\{m_1, \ldots, m_\ell\}$ as an R-module, then $\{m_1 \otimes n_1, \ldots, m_\ell \otimes n_k\}$ generates $M \otimes_R N$, which gives (i).

For (ii), note that any module P in $\mathcal{P}_R$ is a direct summand of R^n for some integer n. But then $P \otimes_R N$ is a direct summand of $R^n \otimes_R N \cong N^n$, which is projective precisely when N is, by a standard result ([BK: IRM] Theorem 2.5.5). □

3.1.18 The adjointness of the functors Hom *and* $\otimes$

The homomorphism functors and the tensor product are connected by an adjointness relation, the existence of which serves to explain and simplify some calculations, especially in the Morita theory.

It will be convenient to depart from our customary labels for rings in order to avoid some confusing substitutions in applications. Let A, B and C be arbitrary rings, and take bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ as indicated.

Then $L \otimes_B M$ is an A-C-bimodule, and we can form the abelian group $\mathrm{Hom}_{A-C}(L \otimes_B M, N)$ of A-C-bimodule homomorphisms from $L \otimes_B M$ to N. It is easy to verify that this group is natural with respect to bimodule homomorphisms of each variable, being contravariant in the first two variables and covariant in the third. Thus we have a trifunctor

$$\mathrm{Hom}_{A-C}(- \otimes_B -, -) : {}_A\mathcal{B}\mathcal{IMOD}_B \times {}_B\mathcal{B}\mathcal{IMOD}_C \times {}_A\mathcal{B}\mathcal{IMOD}_C \longrightarrow \mathcal{A}\mathcal{B}.$$

We can also give $\mathrm{Hom}(M_C, N_C)$ the structure of an A-B-bimodule by the usual rule:

$$(a\mu b)(m) = a(\mu(bm)) \ \text{ for } m \in M,\ \mu \in \mathrm{Hom}(M_C, N_C).$$

We can then form the group $\mathrm{Hom}_{A-B}(L, \mathrm{Hom}(M_C, N_C))$, so obtaining a trifunctor

$$\mathrm{Hom}_{A-B}(-, \mathrm{Hom}(-_C, -_C)) : {}_A\mathcal{B}\mathcal{IMOD}_B \times {}_B\mathcal{B}\mathcal{IMOD}_C \times {}_A\mathcal{B}\mathcal{IMOD}_C \longrightarrow \mathcal{A}\mathcal{B}.$$

A third variation on this theme is to view $\mathrm{Hom}({}_AL, {}_AN)$ as a B-C-bimodule and form the group $\mathrm{Hom}_{B-C}(M, \mathrm{Hom}({}_AL, {}_AN)$. Notice that the order of the bimodules L, M and N is changed. Despite this alteration, we again have a trifunctor

$$\mathrm{Hom}_{B-C}(-, \mathrm{Hom}({}_A-, {}_A-)) : {}_A\mathcal{B}\mathcal{IMOD}_B \times {}_B\mathcal{B}\mathcal{IMOD}_C \times {}_A\mathcal{B}\mathcal{IMOD}_C \longrightarrow \mathcal{A}\mathcal{B}.$$

The Adjointness Theorem tells us that all three general constructions yield essentially the same trifunctor. More precisely, the following result holds.

3.1.19 The Adjointness Theorem for Hom **and** $\otimes$

Let A, B and C be arbitrary rings. Then there are natural isomorphisms

$$\eta : \mathrm{Hom}_{A-C}(L \otimes_B M, N) \longrightarrow \mathrm{Hom}_{A-B}(L, \mathrm{Hom}(M_C, N_C)),$$

and

$$\psi : \mathrm{Hom}_{A-C}(L \otimes_B M, N) \longrightarrow \mathrm{Hom}_{B-C}(M, \mathrm{Hom}({}_AL, {}_AN))$$

of trifunctors from ${}_A\mathcal{B}\mathrm{IMOD}_B \times {}_B\mathcal{B}\mathrm{IMOD}_C \times {}_A\mathcal{B}\mathrm{IMOD}_C$ *to* $\mathcal{A}\mathrm{B}$.

In particular, for M in ${}_B\mathcal{B}\mathrm{IMOD}_C$, *the functor*

$$- \otimes_B M : {}_A\mathcal{B}\mathrm{IMOD}_B \longrightarrow {}_A\mathcal{B}\mathrm{IMOD}_C$$

is left adjoint to

$$\mathrm{Hom}(M_C, -) : {}_A\mathcal{B}\mathrm{IMOD}_C \longrightarrow {}_A\mathcal{B}\mathrm{IMOD}_B,$$

and for L in ${}_A\mathcal{B}\mathrm{IMOD}_B$, *the functor*

$$L \otimes_B - : {}_B\mathcal{B}\mathrm{IMOD}_C \longrightarrow {}_A\mathcal{B}\mathrm{IMOD}_C$$

is left adjoint to

$$\mathrm{Hom}({}_AL, -) : {}_A\mathcal{B}\mathrm{IMOD}_C \longrightarrow {}_B\mathcal{B}\mathrm{IMOD}_C.$$

Proof

We establish the first assertion only, the argument for the second being similar.

Take bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ and let α be in $\mathrm{Hom}_{A-C}(L \otimes_B M, N)$. Then a map $\eta\alpha : L \to \mathrm{Hom}(M_C, N_C)$ is defined by setting

$$((\eta\alpha)(\ell))(m) = \alpha(\ell \otimes m) \text{ for each } \ell \in L \text{ and } m \in M.$$

Since C acts on $L \otimes M$ by $(\ell \otimes m)c = \ell \otimes mc$ and α is a right C-module homomorphism (and the tensor product is linear in each variable), every $(\eta\alpha)(\ell)$ is a right C-module homomorphism. It is also clear that $\eta\alpha$ is an additive map on L.

The verification that $\eta\alpha$ is an A-B-bimodule homomorphism is a matter of careful book-keeping. We need to check that for all $\ell \in L$, $a \in A$ and $b \in B$ we have

$$(\eta\alpha)(a\ell b) = a((\eta\alpha)(\ell))b.$$

For any $m \in M$,

$$\begin{aligned}((\eta\alpha)(a\ell b))(m) &= \alpha(a\ell b \otimes m) &&= \alpha(a(\ell \otimes bm)) \\ &= a(\alpha(\ell \otimes bm)) &&= a(\eta\alpha(\ell))(bm) = (a((\eta\alpha)(\ell))b)(m),\end{aligned}$$

which is the desired result.

We define the inverse τ of η as follows. Let $\gamma \in \operatorname{Hom}_{A-B}(L, \operatorname{Hom}(M_C, N_C))$, and put $(\tau\gamma)(\ell \otimes m) = \gamma(\ell)(m)$. Routine checking confirms that $\tau\gamma$ arises from a B-balanced biadditive map on $L \times M$ in the expected way and that $\tau\gamma$ is an A-C-bimodule homomorphism.

It is clear that η and τ are mutually inverse isomorphisms for each triple L, M and N, and further routine verification shows them to be natural in each variable. □

3.1.20 An equivalence of categories

Finally, we use the tensor product to generalize the characterization of triangular matrix rings that was given in (1.3.18).

Let R and S be arbitrary rings and let W be an R-S-bimodule. We define the *morphism category relative to* W (a variant of the fibre category of (1.2.9)) to be the category $\mathcal{M}\text{OR}(W; R, S)$ whose objects are triples (α, M, N) in which M is a right R-module, N is a right S-module and $\alpha : M \otimes_R W \to N$ is a homomorphism of right S-modules.

A morphism from (α', M', N') to (α, M, N) is a pair (μ, ν) of homomorphisms, $\mu : M' \to M$ and $\nu : N' \to N$, such that $\nu\alpha' = \alpha(\mu \otimes id)$, which means that the following diagram must commute:

$$\begin{array}{ccc} M' \otimes_R W & \xrightarrow{\alpha'} & N' \\ {\scriptstyle \mu \otimes id}\downarrow & & \downarrow{\scriptstyle \nu \otimes id} \\ M \otimes_R W & \xrightarrow{\alpha} & N \end{array}$$

Thus with $W = R = S$, we regain the morphism category $\mathcal{M}\text{OR}(\mathcal{M}\text{OD}_R)$.

The rings R and S and the bimodule W also define the generalized triangular matrix ring $T = \begin{pmatrix} R & W \\ 0 & S \end{pmatrix}$.

3.1.21 Theorem

The categories $\mathcal{M}\text{OD}_T$ and $\mathcal{M}\text{OR}(W; R, S)$ are equivalent.

Proof

As the details of the argument are very similar to those of the proof of (1.3.18), we merely indicate the correspondence between right T-modules L and homomorphisms $\alpha : M \otimes_R W \to N$.

Suppose that L is a right T-module. Since the direct product $R \times S$ is a subring of T, L can be decomposed into a direct sum $M \oplus N$ with M a right R-module and N a right S-module. For m in M and w in W, we can write

$$(m, 0)\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} = (0, \alpha(m, w))$$

with $\alpha(m, w) \in N$. It is easy to check that α is biadditive and R-balanced, and so gives rise to a homomorphism $\beta : M \otimes_R W \to N$.

Conversely, given such a homomorphism β, we can define a T-module structure on $M \oplus N$. The remaining details are for the reader. $\square$

Exercises

3.1.1 Let $\beta : M \to M''$ and $\xi : N \to N''$ be surjective homomorphisms of right and left R-modules respectively. Show that $\operatorname{Ker} \beta \otimes \xi$ is the subgroup of $M \otimes N$ generated by the set

$$\{k \otimes n, m \otimes \ell \mid k \in \operatorname{Ker} \beta,\ \ell \in \operatorname{Ker} \xi,\ m \in M,\ n \in N\}.$$

3.1.2 Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of the ring R. Show that $R/\mathfrak{a} \otimes_R R/\mathfrak{b} \cong R/(\mathfrak{a}+\mathfrak{b})$. (The preceding exercise is relevant.)

Hence compute $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z}$ for various integers a and b.

3.1.3 **Dedekind domains**

Let $\mathcal{O}$ be a Dedekind domain. This exercise requires some facts about $\mathcal{O}$-modules which are summarized in (2.3.20).

(a) Let P be a finitely generated projective $\mathcal{O}$-module. Then, by Steinitz' Theorem, P is isomorphic to a projective module in the standard form $P \cong \mathcal{O}^{r-1} \oplus \mathfrak{a}$ where the rank r and the ideal class $\{\mathfrak{a}\}$ are uniquely determined by P. Compute $P \otimes_{\mathcal{O}} Q$ for any finitely generated projective modules P, Q.

(b) Using the Primary Decomposition Theorem and Exercise 3.1.2 above, describe $M \otimes_{\mathcal{O}} N$ for any pair M and N of finitely generated torsion $\mathcal{O}$-modules.

(c) Using the fact that any finitely generated $\mathcal{O}$-module has the form $T \oplus P$ with T torsion and P projective, describe $M \otimes_{\mathcal{O}} N$ for any pair M and N of finitely generated $\mathcal{O}$-modules.

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3.1.4 **Associativity of the tensor product**

Let R and S be rings and let L_R, ${}_RM_S$ and ${}_SN$ be modules as indicated.

Show that the isomorphism

$$(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N)$$

of (3.1.5) is natural in each variable.

Deduce that the functors $(- \otimes_R -) \otimes_S -$ and $- \otimes_R (- \otimes_S -)$ from $\mathcal{M}od_R \times {}_R\mathcal{B}imod_S \times {}_S\mathcal{M}od$ to $\mathcal{A}b$ are naturally isomorphic.

3.1.5 Let A, B, and C be rings with bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ as indicated.

Using the Adjointness Theorem (3.1.19) together with the characterizations of projective and injective modules in terms of the Hom functor given in (2.1.8), show that

(i) if L_B and M_C are both projective, then $L \otimes_B M$ is projective as a right C-module (here, take $A = \mathbb{Z}$);

(ii) if M_C is projective and ${}_AN$ is injective, then $\operatorname{Hom}(M_C, N_C)$ is left A-injective (here, take $B = \mathbb{Z}$.)

3.1.6 (Tricky!)

Let T be the ring of upper triangular 2×2-matrices over R. By (3.1.21), there is an equivalence between the categories $\mathcal{M}od_T$ and $\mathcal{M}or(\mathcal{M}od_R)$. Show that the projective T-modules correspond to split surjections $\alpha : P \to Q$ with both P and Q R-projective.

Dualize this statement to one about injectives.

3.2 EXACTNESS OF THE TENSOR PRODUCT

We next investigate whether or not a tensor product functor $- \otimes_R N$ is an exact functor. The answer depends both on the left R-module N that is kept fixed and on the category of right R-modules on which it operates. The modules that always give exact functors form an important class of modules, namely, the flat modules. We give some criteria for a module to be flat, and we prove Villamayor's Lemma, a fundamental result which shows that a flat module enjoys a weak form of the splitting property that defines projective modules. We also give a summary of the properties of the functors Tor_n^R that are used to repair the non-exactness of the tensor product, and we show that the tensor product induces a pairing on $\mathcal{M}od_R$ for a commutative ring R.

The novelty of our proofs is that they are 'elementary' in that they avoid the use of the machinery of homological algebra.

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We also show that, for R commutative, the tensor product defines a binary operation on the category of R-modules.

Unless otherwise stated, the ring R is arbitrary. Our main object of attention is a left module, since we wish to investigate operators on categories of right modules.

3.2.1 Flat modules

Let $\mathcal{C}$ be a G-exact category of right R-modules, that is, $\mathcal{C}$ is a subcategory of $\mathcal{M}od_R$ together with a specified set $\mathrm{Ex}(\mathcal{C})$ of short exact sequences as in (2.4.1), and let N be an R-S-bimodule. The additive functor $-\otimes_R N$ may or may not be an exact functor on $\mathcal{C}$ as defined in (2.4.6). It will be so if $\mathcal{C}$ is split exact, but it is not in general, as the following example for $\mathcal{M}od_{\mathbb{Z}}$ shows.

For any nonzero integer n, there is an exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ n\ } \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

in which the first map is multiplication by n. Tensoring with $\mathbb{Z}/n\mathbb{Z}$ (as a $\mathbb{Z}$-module) gives the sequence

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{\ 0\ } \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

which evidently fails to be exact at the first term $\mathbb{Z}/n\mathbb{Z}$.

A left R-module N for which $-\otimes_R N : \mathcal{M}od_R \to \mathcal{M}od_{\mathbb{Z}}$ is an exact functor is, by definition, a *flat* module. Thus, the left R-module N is flat if for any short exact sequence

$$0 \longrightarrow M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M'' \longrightarrow 0$$

of right R-modules the sequence

$$0 \longrightarrow M' \otimes_R N \xrightarrow{\ \alpha\otimes id\ } M \otimes_R N \xrightarrow{\ \beta\otimes id\ } M'' \otimes_R N \longrightarrow 0$$

is also a short exact sequence. The definition of a flat right R-module is analogous.

We have an almost tautologous lemma.

3.2.2 Lemma

Suppose that the R-S-bimodule N is flat as a left R-module and finitely generated as a right S-module (where R and S are arbitrary rings). Then the functor

$$-\otimes_R N : \mathcal{M}_R \longrightarrow \mathcal{M}_S$$

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is exact. □

Before we make a detailed analysis of the properties of flat modules, we discuss the extent to which exactness is preserved by modules in general.

3.2.3 Proposition

(a) *Let N be a left R-module and let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence of right R-modules. Then the sequence

$$M' \otimes_R N \xrightarrow{\alpha \otimes id} M \otimes_R N \xrightarrow{\beta \otimes id} M'' \otimes_R N \longrightarrow 0$$

is exact.

(b) *Let M be a right R-module and let*

$$0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$$

be a short exact sequence of left R-modules. Then the sequence

$$M \otimes_R N' \xrightarrow{id \otimes \alpha} M \otimes_R N \xrightarrow{id \otimes \beta} M \otimes_R N'' \longrightarrow 0$$

is exact.

Proof

(a) The fact that $M'' \otimes N$ is generated by symbols $m'' \otimes n$ shows immediately that $\beta \otimes id$ is surjective. Since

$$0 = \beta\alpha \otimes id = (\beta \otimes id)(\alpha \otimes id),$$

we have

$$\mathrm{Im}(\alpha \otimes id) \subseteq \mathrm{Ker}(\beta \otimes id).$$

We must verify that this inclusion is equality. Write $I = \mathrm{Im}(\alpha \otimes id)$ and let

$$\lambda : (M \otimes_R N)/I \longrightarrow M'' \otimes_R N$$

be the homomorphism induced by β. It suffices to show that λ is an isomorphism, which we do by constructing an inverse.

Recall that $M'' \otimes_R N = M''\#N/B(M'', N)$ (3.1.2). Define

$$\mu : M''\#N \longrightarrow (M \otimes_R N)/I$$

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by

$$\mu(m'', n) = m \otimes n + I,$$

where m is any element of M such that $\beta m = m''$.

If also $\beta m_1 = m''$, then exactness at M implies that $m - m_1$ is in $\operatorname{Im} \alpha$ and hence

$$m \otimes n + I = m_1 \otimes n + I.$$

Thus μ is well-defined. Since μ is trivial on $B(M'', N)$, it induces a homomorphism $M'' \otimes_R N \longrightarrow (M \otimes_R N)/I$ which is evidently inverse to λ.

The proof of (b) is analogous. □

3.2.4 Corollary

A left R-module N is flat if and only if for each injective right R-module homomorphism $\alpha : M' \to M$, the group homomorphism

$$\alpha \otimes id_N : M' \otimes_R N \longrightarrow M \otimes_R N,$$

is also an injection. □

The existence of flat modules is guaranteed by the next result.

3.2.5 Corollary

Suppose that N is a projective left R-module. Then N is flat.

Proof

That any free module R^Λ is flat follows from (3.1.14), since any injective right R-homomorphism $M' \to M$ induces an injection $M'^\Lambda \to M^\Lambda$. Now, for the module N, we have $N \oplus Q \cong {}^\Lambda R$ for some left R-module Q and free module ${}^\Lambda R$. Therefore, by (3.1.13),

$$- \otimes_R N \oplus - \otimes_R Q \simeq - \otimes_R {}^\Lambda R.$$

So $- \otimes_R N$ is exact by Exercise 2.4.6. □

3.2.6 The functors Tor_n^R

The failure of the tensor product to be an exact functor can be measured by a sequence of derived functors $\operatorname{Tor}_n^R(-,-)$ for $n \geq 1$. These may be constructed as follows.

Given a right R-module M and a left R-module N, where R is an arbitrary ring, we take a short exact sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow N \longrightarrow 0$$

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of left R-modules with F flat (for example, projective), and define

$$\operatorname{Tor}_1^R(M,N) = \operatorname{Ker}(M \otimes_R L \to M \otimes_R F)$$

and

$$\operatorname{Tor}_n^R(M,N) = \operatorname{Tor}_{n-1}^R(M,L) \text{ for } n > 1.$$

It can be shown that the abelian groups $\operatorname{Tor}_n^R(M,N)$ are independent of the choice of the short exact sequence for N, and that they could equally be defined by taking a similar short exact sequence for M and considering the kernel after tensoring with N. Further, $\operatorname{Tor}_n^R(M,N)$ is a covariant additive functor in each of its arguments, and, given a short exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

of left R-modules, there is a long exact sequence

$$\begin{array}{cccccccc}
 & & & \cdots & \longrightarrow & \operatorname{Tor}_{n+1}^R(M,N'') & \longrightarrow \\
\longrightarrow & \operatorname{Tor}_n^R(M,N') & \longrightarrow & \operatorname{Tor}_n^R(M,N) & \longrightarrow & \operatorname{Tor}_n^R(M,N'') & \longrightarrow \\
\longrightarrow & \cdots & & & & & \\
 & & & \cdots & \longrightarrow & \operatorname{Tor}_1^R(M,N'') & \longrightarrow \\
\longrightarrow & M \otimes_R N' & \longrightarrow & M \otimes_R N & \longrightarrow & M \otimes_R N'' & \longrightarrow & 0,
\end{array}$$

and likewise for the other variable.

Much analysis is required to verify these claims; a good account is given in [Rotman 1979].

Note that the flat left R-modules can be characterized as those modules F for which the functors $\operatorname{Tor}_n^R(-,F)$ $(n \geq 1)$ are identically zero (and correspondingly for right flat modules). Some very elementary calculations with the Tor functors are indicated in Exercise 3.2.7 below, and in Exercise 6.2.1, which justifies the notation 'Tor'. Flat modules made a reticent entrance into the mathematical landscape in Exercise 3 of Ch. VI of [Cartan & Eilenberg 1956], where they are defined in terms of the vanishing of the functors Tor_n. The pivotal role that flat modules now play will become clear in subsequent chapters of this text.

3.2.7 Criteria for flatness and Villamayor's Lemma

We now obtain some useful criteria for flatness and thence Villamayor's Lemma. For the next few results and proofs, it is convenient to say that a functor F *respects the injectivity* of a module monomorphism μ if $F\mu$ is also a monomorphism.

3.2.8 Proposition

(a) *For a left R-module N, the following assertions are equivalent.*

(i) ${}_RN$ *is flat.*

(ii) *The functor* $-\otimes_R N$ *respects the injectivity of any R-module monomorphism* $\alpha : M'_R \to M_R$ *with* M'_R *finitely generated. That is,*

$$\alpha \otimes id : M' \otimes_R N \longrightarrow M \otimes_R N$$

is also injective.

(iii) *The functor* $-\otimes_R N$ *respects the injectivity of any R-module monomorphism* $M'_R \to M_R$.

(iv) *The functor* $- \otimes_R N : \mathcal{M}\text{od}_R \to \mathcal{A}\text{b}$ *is exact.*

(b) *For a right R-module M, the following assertions are equivalent.*

(i) M_R *is flat.*

(ii) *The functor* $M\otimes_R -$ *respects the injectivity of any R-module monomorphism* ${}_RN' \to {}_RN$ *with* ${}_RN'$ *finitely generated.*

(iii) *The functor* $M\otimes_R -$ *respects the injectivity of any R-module monomorphism* ${}_RN' \to {}_RN$.

(iv) *The functor* $M \otimes_R - : {}_R\mathcal{M}\text{od} \to \mathcal{A}\text{b}$ *is exact.*

Proof

The only outstanding points to check are the implications (ii) $\Rightarrow$ (iii); we deal with (a) only. Consider an injective homomorphism $\alpha : M'_R \to M_R$ with M'_R not necessarily finitely generated, and let $x = \sum_{i=1}^{k}(m'_i \otimes n_i)$ be in $\mathrm{Ker}(\alpha \otimes id)$. Then x also lies in $\mathrm{Ker}(\alpha' \otimes id)$, where α' is the restriction of α to the submodule of M generated by $m'_1, \dots, m'_k$. This forces $x = 0$, so that $\mathrm{Ker}(\alpha \otimes id) = 0$ as required. □

The following criterion is designed to simplify one's workload in establishing that a given module is flat.

3.2.9 The Flat Test

A left R-module N is flat if and only if $-\otimes_R N$ *respects the injectivity of all inclusions* $\mu : \mathfrak{a} \to R$ *of finitely generated right ideals in R. The corresponding statement holds with left and right interchanged.*

Proof

Necessity of the condition is immediate from the preceding proposition. Sufficiency is proved in four stages. First, we note that the argument just given to prove (ii) $\Rightarrow$ (iii) above also shows that $-\otimes_R N$ respects the injectivity of inclusions $\mu : \mathfrak{a} \to R$ of arbitrary right ideals in R. Second, we argue by induction on n to show that $-\otimes_R N$ respects injectivity for $\mu : M \to R^n$, the case $n = 1$ having just been settled.

For the inductive step, we use the standard embedding of R^{n-1} in R^n. Put $M'' = M/(M \cap R^{n-1})$. Since $M'' \cong (M + R^{n-1})/R^{n-1}$ and $R^n/R^{n-1} \cong R$, there is an injective homomorphism from M'' to R. So there is a commuting diagram with exact rows, the lower of which is split:

$$\begin{tikzcd}
 & (M \cap R^{n-1}) \otimes_R N \arrow[r] \arrow[d] & M \otimes_R N \arrow[r] \arrow[d] & M'' \otimes_R N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^{n-1} \otimes_R N \arrow[r] & R^n \otimes_R N \arrow[r] & R \otimes_R N \arrow[r] & 0
\end{tikzcd}$$

Then the injectivity of the two outside vertical arrows implies that of the middle one.

The next step is to show that $-\otimes_R N$ respects the injectivity of a monomorphism $\mu : M \to F$ where F is an arbitrary free module. Again, it is enough to check the case where the domain M is finitely generated. But then, if $(\mu \otimes id)x = 0$ in $F \otimes_R M$, there is a finitely generated free submodule G of the codomain F such that $(\mu \otimes id)x = 0$ in $G \otimes_R M$ already, since any member of the relation group $B(F, M)$ (3.1.2) can involve only finitely many generators of F. By the previous step, $x = 0$.

For the final step, let $\mu : M' \to M$ be an arbitrary injective homomorphism of right R-modules, and let F be a free right R-module with $\beta : F \to M$ an R-homomorphism onto M, having kernel K, say. Then, in the pull-back

$$\begin{tikzcd}
\overline{F} \arrow[r, "\overline{\beta}"] \arrow[d, "\overline{\mu}"'] & M' \arrow[d, "\mu"] \\
F \arrow[r, "\beta"] & M
\end{tikzcd}$$

$\overline{\beta}$ also has kernel K, and so there is a commuting diagram with exact rows

$$\begin{tikzcd}
K \otimes_R N \arrow[r] \arrow[d, "id"'] & \overline{F} \otimes_R N \arrow[r] \arrow[d, "\overline{\mu} \otimes id"'] & M' \otimes_R N \arrow[r] \arrow[d, "\mu \otimes id"'] & 0 \\
K \otimes_R N \arrow[r] & F \otimes_R N \arrow[r] & M \otimes_R N \arrow[r] & 0
\end{tikzcd}$$

Since $\overline{\mu} \otimes id$ is now known to be injective, so too is $\mu \otimes id$, and the proof is complete. □

3.2.10 Corollary

Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of left R-modules. If N' and N'' are flat, then so too is N.

Proof

Considering any inclusion $\mathfrak{a} \to R$ of a right ideal in R, simply chase the following commutative diagram of exact rows and columns to show that the middle row commences with an injection.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathfrak{a} \otimes_R N' \arrow[r] \arrow[d] & R \otimes_R N' \arrow[r] \arrow[d] & R/\mathfrak{a} \otimes_R N' \arrow[r] \arrow[d] & 0 \\
 & \mathfrak{a} \otimes_R N \arrow[r] \arrow[d] & R \otimes_R N \arrow[r] \arrow[d] & R/\mathfrak{a} \otimes_R N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{a} \otimes_R N'' \arrow[r] \arrow[d] & R \otimes_R N'' \arrow[r] \arrow[d] & R/\mathfrak{a} \otimes_R N'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

3.2.11 Villamayor's Lemma

Let M be a right R-module. Then the following assertions are equivalent.

(i) *M is flat.*

(ii) *Given any pair (ϵ, L), where $\epsilon : F \to M$ is a surjective homomorphism of right R-modules with F free, and L is a finitely generated submodule of $\operatorname{Ker} \epsilon$, then there is an R-homomorphism $\rho : F \to \operatorname{Ker} \epsilon$ which restricts to the identity map on L and which has finitely generated image ρF.*

Proof

First assume that M is flat. We argue by induction on the number of generators $u_1, \ldots, u_n$ of the submodule L.

$n = 1$. Write $u_1 = \sum_{i=1}^{k} x_i r_i$ with $x_1, \ldots, x_n$ elements of a free generating set for F. Let $\mathfrak{a}$ be the left ideal of R generated by $r_1, \ldots, r_k$. Since M is flat, the obvious map $M \otimes_R \mathfrak{a} \to M \otimes_R R \cong M$ is an injection. So in the following commutative diagram with exact rows, the right vertical arrow is injective.

$$\begin{array}{ccccccccc} & & (\operatorname{Ker}\epsilon) \otimes_R \mathfrak{a} & \longrightarrow & F \otimes_R \mathfrak{a} & \longrightarrow & M \otimes_R \mathfrak{a} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}\epsilon & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since u_1 is the image of the element $\sum_{i=1}^k x_i \otimes r_i$ of $F \otimes_R \mathfrak{a}$, a diagram chase shows that

$$\sum_{i=1}^k x_i \otimes r_i = \operatorname{Im}(\sum_{j=1}^h y_j \otimes s_j)$$

for elements $y_j \in \operatorname{Ker}\epsilon$ and $s_j \in \mathfrak{a}$. Write $s_j = \sum_{i=1}^k a_{ij} r_i$ for $j = 1, \dots, h$, with each a_{ij} in R, and put $v_i = \sum_{j=1}^h y_j a_{ij}$ for each i. Then

$$u_1 = \operatorname{Im}(\sum_{i=1}^k v_i \otimes r_i) = \sum_{i=1}^k v_i r_i \text{ with each } v_i \in \operatorname{Ker}\epsilon.$$

The R-homomorphism $\rho_1 : F \to \operatorname{Ker}\epsilon$, sending x_i to v_i for $i = 1, \dots, k$ and each other generator of F to zero, has finitely generated image and $\rho_1 u_1 = u_1$.

$n > 1$. Let $\rho_1 : F \to \operatorname{Ker}\epsilon$ be as above, and let $\rho' : F \to \operatorname{Ker}\epsilon$ be the homomorphism obtained by applying the induction hypothesis to the submodule of $\operatorname{Ker}\epsilon$ generated by $u_2 - \rho_1 u_2, \dots, u_n - \rho_1 u_n$. Then the R-homomorphism

$$\rho = \rho_1 + \rho'(id - \rho_1) : F \longrightarrow \operatorname{Ker}\epsilon$$

has the desired property.

For the converse, we wish to apply the Flat Test. Therefore let $\mathfrak{a}$ be a left ideal of R (with inclusion $\iota : \mathfrak{a} \to R$) and let $\epsilon : F \to M$ be some surjection from a free module F. We obtain the same diagram as above, but this time seek to establish the injectivity of the right-hand vertical arrow.

Since the free module F is flat by the left-handed version of (3.2.5), $id \otimes \iota : F \otimes_R \mathfrak{a} \to F \otimes_R R \cong F$ is injective. Thus, by an easy diagram chase, the injectivity of the right-hand arrow will follow once we can show that if $\sum_{i=1}^k x_i \otimes a_i$ is in $F \otimes_R \mathfrak{a}$ and $\epsilon(\sum_{i=1}^k x_i a_i) = 0$, then $\sum_{i=1}^k x_i a_i$ lies in $(id \otimes \iota)((\operatorname{Ker}\epsilon) \otimes_R \mathfrak{a})$.

Now by hypothesis there exists an R-homomorphism $\rho : F \to \operatorname{Ker}\epsilon$ which restricts to the identity map on $\sum_{i=1}^k x_i a_i$. Then $\sum_{i=1}^k \rho x_i \otimes a_i \in (\operatorname{Ker}\epsilon) \otimes_R \mathfrak{a}$

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and

$$\sum_{i=1}^{k} \rho(x_i)a_i = \rho(\sum_{i=1}^{k} x_i a_i) = \sum_{i=1}^{k} x_i a_i,$$

as required. $\square$

Note that here L is not quite a direct summand of F. We now show that in favourable circumstances we do obtain a direct summand. The first result is obvious, the next rather deeper.

3.2.12 Corollary

Suppose that $\epsilon : F \to M$ is surjective, with F free and M flat. If $\operatorname{Ker}\epsilon$ is finitely generated, then $F \cong \operatorname{Ker}\epsilon \oplus M$. $\square$

Although the finitely generated submodule L in Villamayor's Lemma need not be a summand of F, the next result shows that it is contained in a finitely generated summand provided that F is 'big enough'. This result is used in (5.2.7) to help in the derivation of another important characterization of flatness.

3.2.13 Lemma

Suppose that (ϵ, L) is a pair as in Villamayor's Lemma, and also that M is flat and that $F = \operatorname{Fr}_R(X)$, where for each $x \in X$ there exist infinitely many $y \in X$ such that $\epsilon y = \epsilon x$. Then there is a finitely generated direct summand L' of F with

$$L \subseteq L' \subseteq \operatorname{Ker}\epsilon.$$

Proof

With $\rho : F \to \operatorname{Ker}\epsilon$ as above, let X_1 be the finite set of elements of X occurring in the expressions of the generators of ρF as linear combinations of elements of X. By hypothesis, we can partition X as $X = X_1 \sqcup X_2 \sqcup X_3$ with X_1 and X_2 in bijective correspondence such that corresponding elements have the same images under ϵ. Write $F_i = \operatorname{Fr}_R(X_i)$, so that $L \subseteq F_1$ and $F = F_1 \oplus F_2 \oplus F_3$, and let $\pi_1 : F \to F_1$ denote projection to the first summand. The bijection $X_1 \to X_2$ defines an injective homomorphism $\sigma : F_1 \to F_2 \hookrightarrow F$ with $\pi_1\sigma = 0$ and $\epsilon\sigma = \epsilon$. Define

$$\tau = id + \sigma(\rho - id) : F_1 \to F,$$

with image $L' = \tau F_1$. (Roughly speaking, $\tau - id$ is just $\rho - id$ shifted away from F_1.) Evidently, $\pi_1\tau = id$, so that the inclusion of L' in F is split by

$\tau\pi_1 : F \to \tau F_1$. Because the restriction of τ to L is the identity map, $L \subseteq L'$. Finally, since $\epsilon\sigma = \epsilon$ and $\epsilon\rho = 0$, we have $L' \subseteq \operatorname{Ker} \epsilon$. □

3.2.14 A pairing on $\mathcal{M}\mathit{od}_R$

As we remarked following (3.1.10), the tensor product $-\otimes_R-$ can be regarded as a bifunctor from $\mathcal{M}\mathit{od}_R \times {}_R\mathcal{M}\mathit{od}$ to $\mathcal{A}\mathit{b}$. For a general ring R and arbitrary modules M_R and ${}_RN$, there is no natural method by which $M \otimes_R N$ can be endowed with an R-module structure. However, if both M and N belong to the category ${}_R\mathcal{B}\mathit{imod}_R$ of R-R-bimodules (and R-R-bimodule homomorphisms), then $M \otimes_R N$ is also in ${}_R\mathcal{B}\mathit{imod}_R$ (see (3.1.3)).

This observation permits us to use the tensor product to define a binary operation in $\mathcal{M}\mathit{od}_R$ when the ring R is commutative. An R-R-bimodule is said to be *balanced* if $rm = mr$ for all m in M and r in R. Clearly, any right (or left) R-module M can be viewed as a balanced R-R-bimodule by setting $rm = mr$ for all r in R and m in M. Thus we can choose to identify both $\mathcal{M}\mathit{od}_R$ and ${}_R\mathcal{M}\mathit{od}$ with the category $\mathcal{B}\mathit{al}_R$ of balanced R-bimodules.

Using this identification, we can regard $-\otimes_R-$ as a bifunctor from $\mathcal{M}\mathit{od}_R \times \mathcal{M}\mathit{od}_R$ to $\mathcal{M}\mathit{od}_R$. Such a bifunctor which is additive in both variables is sometimes called a *pairing* on $\mathcal{M}\mathit{od}_R$. It is clear that this pairing induces a pairing on $\mathcal{M}_R$, the category of finitely generated R-modules.

The next result shows that this pairing is, in essence, commutative. The proof is an easy exercise.

3.2.15 Proposition

Let M and N be balanced bimodules over the commutative ring R. Then there is a natural isomorphism $M \otimes_R N \cong N \otimes_R M$ of balanced R-bimodules. □

The above pairing can be viewed as a generalization of the product of fractional ideals of a commutative domain $\mathcal{O}$. Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}$, and recall that a fractional ideal of $\mathcal{O}$ is a finitely generated (balanced) $\mathcal{O}$-submodule of $\mathcal{K}$, and that the product $\mathfrak{a}\mathfrak{b}$ of fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ is the submodule of $\mathcal{K}$ generated by all products ab with $a \in \mathfrak{a}, b \in \mathfrak{b}$ (2.3.20). Then $\mathfrak{a}\mathfrak{b}$ is also a fractional ideal. The connection between the pairing and the product is given by the following useful computation.

3.2.16 Proposition

Let $\mathcal{O}$ be a commutative domain with field of fractions $\mathcal{K}$, and let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of $\mathcal{O}$.

Then there is an isomorphism $\mathfrak{a} \otimes_{\mathcal{O}} \mathfrak{b} \to \mathfrak{ab} \subseteq \mathcal{K}$, *given by* $a \otimes b \mapsto ab$.

Proof

Clearly, the map $a \otimes b \mapsto ab$ is a well-defined surjection, so we need only check injectivity. Suppose first that $\mathfrak{b} = \mathcal{O}x$ is principal. Then $\mathfrak{b}$ is projective and so flat as an $\mathcal{O}$-module, and the natural inclusion $\mathfrak{a} \to \mathcal{K}$ gives an injective homomorphism $\mathfrak{a} \otimes_{\mathcal{O}} \mathcal{O}x \to \mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}x$. Arguing as in (3.1.4), we have $\mathfrak{a} \otimes_{\mathcal{O}} \mathcal{O}x \cong \mathfrak{a}x$ and $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}x \cong \mathcal{K}x = \mathcal{K}$ via $a \otimes x \mapsto ax$.

In general, suppose that $a_1, \ldots, a_k$ are in $\mathfrak{a}$ and $b_1, \ldots, b_k$ are in $\mathfrak{b}$, and that $\sum_i a_i \otimes b_i \mapsto 0$ in $\mathcal{K}$. Finding a common denominator $x \in \mathcal{K}$, we can write $b_i = c_i x$ with $c_i \in \mathcal{O}$ for all i. Then

$$\sum_i a_i \otimes b_i = \sum_i a_i c_i \otimes x \longmapsto 0,$$

so $\sum_i a_i c_i = 0$ in $\mathfrak{a}$. Thus $\sum_i a_i \otimes c_i = 0$ in $\mathfrak{a} \otimes_{\mathcal{O}} \mathcal{O}$ and, after multiplication by x, $\sum_i a_i \otimes b_i = 0$ in $\mathfrak{a} \otimes \mathfrak{b}$. $\square$

Exercises

3.2.1 Let R be the ring of dual numbers $A[\epsilon]$ over a ring A, and recall that A is both a left and right R-module, with ϵ acting as 0 on A. By considering the short exact sequence of right R-modules

$$0 \longrightarrow \epsilon A[\epsilon] \xrightarrow{\ \alpha\ } A[\epsilon] \xrightarrow{\ \beta\ } A \longrightarrow 0,$$

where α is the inclusion and β the obvious surjection, show that A is not R-flat.

3.2.2 Let $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$ be a direct sum of left R-modules, where Λ is any ordered set. Show that N is flat if and only if each N_λ is flat.

3.2.3 An R-module M is said to be *finitely related* if there is a short exact sequence of the form

$$0 \longrightarrow R^m \longrightarrow R^\Lambda \longrightarrow M \longrightarrow 0.$$

Show that if also M is flat, then M is projective.

3.2.4 Given a subring $\mathcal{O}$ of a ring R, we introduce two $\mathcal{O}$-*relative* G-*exact categories* $\mathcal{M}_{R,\mathcal{O}}$ and $\mathcal{P}_{R,\mathcal{O}}$ as follows.

The category $\mathcal{M}_{R,\mathcal{O}}$ has as underlying category $\mathcal{M}_R$ and we take the set $\mathrm{Ex}(\mathcal{M}_{R,\mathcal{O}})$ of admissible short exact sequences to be all the short exact sequences in $\mathcal{M}_R$ which are split as exact sequences of $\mathcal{O}$-modules. The underlying category of $\mathcal{P}_{R,\mathcal{O}}$ is the full subcategory of $\mathcal{M}_R$ whose objects are in $\mathcal{P}_\mathcal{O}$, the class $\mathrm{Ex}(\mathcal{P}_{R,\mathcal{O}})$ consisting of all short exact sequences with terms in $\mathcal{P}_\mathcal{O}$ (so $\mathcal{P}_{R,\mathcal{O}}$ is repletely exact).

Show that $-\otimes_R N$ is exact on $\mathcal{M}_{R,\mathcal{O}}$ and $\mathcal{P}_{R,\mathcal{O}}$ for any left R-module N.

(In applications, $\mathcal{M}_{R,\mathcal{O}}$ is useful when $\mathcal{O}$ is a commutative domain and R is an $\mathcal{O}$-order, while $\mathcal{P}_{R,\mathcal{O}}$ is useful when $\mathcal{O}$ is arbitrary and R is the group ring $\mathcal{O}G$ for a suitable group G ([Berrick 1982] p. 94).)

3.2.5 **Firm modules over nonunital rings**

For a nonunital ring R with unitalization $\overline{R}$, recall from (1.3.2)(iv) that an R-module is just the same thing as an $\overline{R}$-module. In particular, R itself is an R-bimodule. We define a right R-module V to be *firm* if multiplication induces a right R-module isomorphism $V \otimes_{\overline{R}} R \to V$; similarly for firm left R-modules. The ring R itself is called a *firm nonunital ring* if it is firm as a right R-module. Show that this is equivalent to R being firm as a left R-module, and to there being a canonical (nonunital) ring isomorphism $R \otimes_{\overline{R}} R \to R$, where $R \otimes_{\overline{R}} R$ has ring structure defined by

$$(r_1 \otimes r_2)(r_1' \otimes r_2') = r_1(r_2 r_1') \otimes r_2' = r_1 \otimes (r_2 r_1') r_2'.$$

Now suppose that R is a firm nonunital ring, and let V be a firm right R-module and W a firm left R-module equipped with a surjective R-bimodule homomorphism $\sigma : W \otimes_\mathbb{Z} V \to R$. Define a multiplication on the abelian group $S = V \otimes_R W$ by

$$(v \otimes w)(v' \otimes w') = v \cdot \sigma(w, v') \otimes w' = v \otimes \sigma(w, v') \cdot w'.$$

Show that S is thereby also a firm nonunital ring.

3.2.6 Let $\mathcal{O}$ be any commutative ring. Using (3.2.14), show that the tensor product induces biadditive functors from

(i) $\mathcal{M}_\mathcal{O} \times \mathcal{M}_\mathcal{O}$ to $\mathcal{M}_\mathcal{O}$,
(ii) $\mathcal{F}_\mathcal{O} \times \mathcal{F}_\mathcal{O}$ to $\mathcal{F}_\mathcal{O}$,
(iii) $\mathcal{P}_\mathcal{O} \times \mathcal{P}_\mathcal{O}$ to $\mathcal{P}_\mathcal{O}$,
and
(iv) $\mathcal{M}_\mathcal{O} \times \mathcal{T}_{\mathcal{ORO}}$ to $\mathcal{T}_{\mathcal{ORO}}$.

3.2.7 **Calculations with** Tor

For this exercise, we assume the properties of the functor Tor as outlined in (3.2.6).

(i) Let M be an abelian group (that is, a $\mathbb{Z}$-module.) Show that, for any positive integer a,

$$\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/\mathbb{Z}a) = \{m \in M \mid ma = 0\}$$

and that

$$\mathrm{Tor}_n^{\mathbb{Z}}(M, -) \equiv 0 \text{ for } n > 1.$$

(ii) Let $R = A[\epsilon]$ be the ring of dual numbers over a coefficient ring A and let M be a right R-module. After Exercise 3.2.1, there is a short exact sequence (of left or right R-modules)

$$0 \longrightarrow A \longrightarrow R \longrightarrow A \longrightarrow 0$$

in which the inclusion is multiplication by ϵ and the surjection is the natural one.

Show that $M \otimes_R A \cong M/M\epsilon$ and that $\mathrm{Tor}_1^R(M, A) \cong M_\epsilon/M\epsilon$ where $M_\epsilon = \{m \in M \mid m\epsilon = 0\}$. Deduce that $\mathrm{Tor}_1^R(A, A) \cong A$, as a group.

Show also that $\mathrm{Tor}_{n-1}^R(M, A) \cong \mathrm{Tor}_n^R(M, A)$ for $n > 1$.

(iii) Prove (3.2.10) using the long exact sequence.

3.2.8 Let

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

be any short exact sequence of left R-modules, with N'' flat, and let M be any right R-module.

(a) Choose a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F flat, and construct a 3×3 commutative diagram as in (3.2.10). By chasing the diagram, show that $M \otimes_R N'$ maps injectively into $M \otimes_R N$.

(b) From the case where N is free, observe that $\mathrm{Tor}_1^R(M, N'') = 0$.

(c) Also deduce from (a) that if N is flat, then so is N', a converse to (3.2.10).

3.3 CHANGE OF SCALARS

Now that the tensor product and the language of category theory are available to us, we can analyse in detail the relationships between various categories of modules that arise from homomorphisms between rings.

Given a ring homomorphism

$$f : R \longrightarrow S,$$

where R and S are arbitrary rings, we define two functors

$$f^\# : \mathcal{M}\mathrm{OD}_S \longrightarrow \mathcal{M}\mathrm{OD}_R$$

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and

$$f_{\#} : \mathcal{M}\mathit{od}_R \longrightarrow \mathcal{M}\mathit{od}_S,$$

which are called respectively restriction and extension of scalars. These functors generalize the notions of restriction and extension of scalars between, for example, real and complex vector spaces (where f is the inclusion map of $\mathbb{R}$ in $\mathbb{C}$), to modules over arbitrary rings.

An important special case occurs when the ring homomorphism is an automorphism of R, which we prefer to write as α rather than f. In this situation, an R-module M can be 'twisted' to obtain a new R-module M^{α} (3.3.22), which turns out to be the restriction $\alpha^{\#}M$ (3.3.25). Twisted modules are encountered frequently in the theory of skew polynomial rings.

The functorial analysis of change of scalars through the tensor product was initiated in [Cartan & Eilenberg 1956], Ch. II §6. Our emphasis on the preservation of exact sequences and the twisting of module structures anticipates the requirements of K-theory.

As usual, we concentrate on right modules. It will be obvious that our definitions and results have analogues for left modules (see (3.3.21)). We do not always state these analogues separately. Note that we always write ring homomorphisms on the left of their arguments.

The reader is warned that there is considerable variation in the notation and terminology for restriction and extension. For example, [Quillen 1973] §4 (5), uses f_* where we use $f^{\#}$, and f^* for our $f_{\#}$. The variation stems from the convention that covariant behaviour is usually indicated by a subscript and contravariant by a superscript. In applications to geometry and topology, the interest is in the action of restriction and extension on categories of varieties, schemes or topological spaces rather than modules [Fulton & Lang 1985], II §1. However, the functors that link rings to these objects are themselves contravariant [Hartshorne 1997], I (3.8), II (2.3), with a resulting switch in the variance of restriction and extension. We mention the alternative terminologies from time to time.

3.3.1 Restriction

Let M be a right S-module and suppose that there is a ring homomorphism $f : R \to S$.

Given elements r of R and m of M, define

$$m \cdot r = m \cdot (fr).$$

It is clear that this rule gives M the structure of a right R-module, which we denote by $f^{\#}M$. It is evident that a homomorphism

$$\alpha : M' \longrightarrow M$$

of right S-modules is also a homomorphism

$$f^{\#}\alpha : f^{\#}M' \longrightarrow f^{\#}M$$

of right R-modules and that

$$f^{\#} : \mathcal{M}\text{ODS} \longrightarrow \mathcal{M}\text{ODR}$$

is an additive covariant functor, which we call *restriction* or *restriction of scalars*

The terminology is most appropriate when f is an injection, because we are then 'restricting the scalars' in the literal sense. When f is surjective, the reader may meet the terms *pull-back* to describe $f^{\#}$ and *coinduced module* to describe $f^{\#}M$. Finally, in K-theory $f^{\#}$ gives rise to various homomorphisms of K-groups; many of these homomorphisms are called *transfer* maps.

Here are some elementary properties of the restriction functor.

3.3.2 Lemma

(i) *$(id_R)^{\#}$ is the identity functor* $\mathrm{Id}_R : \mathcal{M}\text{OD}_R \to \mathcal{M}\text{OD}_R$.

(ii) *Given ring homomorphisms $f : R \to S$ and $g : S \to T$, there is a natural isomorphism between the functors $(gf)^{\#}$ and $f^{\#}g^{\#}$.* □

If we put $(R)^{\#} = \mathcal{M}\text{OD}_R$, then $(-)^{\#}$ is a contravariant functor from the category $\mathcal{R}\text{ING}$ of all rings to the category $\mathcal{C}\text{AT}$ of all categories.

Next, we consider the behaviour of the restriction functor on various subcategories of $\mathcal{M}\text{OD}_S$. To save notation, we use the same symbol $f^{\#}$ to denote a number of functors arising from $f^{\#}$.

3.3.3 Proposition

(i) *The functor $f^{\#}$ induces a functor*

$$f^{\#} : \mathcal{P}\text{ROJ}_S \longrightarrow \mathcal{P}\text{ROJ}_R$$

if and only if $f^{\#}S$ is a projective right R-module.

(ii) *The functor $f^{\#}$ induces a functor*

$$f^{\#} : \mathcal{M}_S \longrightarrow \mathcal{M}_R$$

if and only if $f^{\#}S$ is a finitely generated right R-module.

(iii) *The functor $f^\#$ induces a functor*

$$f^\# : \mathcal{P}_S \longrightarrow \mathcal{P}_R$$

if and only if $f^\# S$ is a finitely generated projective right R-module.
(iv) *The functor $f^\#$ induces a functor*

$$f^\# : \mathcal{F}_S \longrightarrow \mathcal{F}_R$$

if and only if $f^\# S$ is a free right R-module of finite rank.

In each case, if the functor $f^\#$ exists, it is exact.

Proof

Since all the subcategories that occur are full subcategories of $\mathcal{M}od_R$ or $\mathcal{M}od_S$, we need only verify that $f^\#$ has the desired action on modules.
(i). Suppose first that $f^\# S$ is R-projective. If M is in $\mathcal{P}roj_S$, then $M \oplus N \cong S^\Lambda$ for some module N and index set Λ ([BK: IRM] Theorem 2.5.8), and clearly $f^\# M \oplus f^\# N \cong (f^\# S)^\Lambda$ also, which shows that $f^\# M$ is projective ([BK: IRM] Theorem 2.5.5).

The converse is obvious since S is in $\mathcal{P}roj_S$.
(ii) – (iv). Similar to (i); note that if an S-module M has a finite set of generators $\{m_1, \ldots, m_\ell\}$ and S has a finite set of generators $\{s_1, \ldots, s_k\}$ as a right R-module, then the set of products $\{m_i s_j\}$ is a finite set of generators of the R-module $f^\# M$.

The final assertion follows from the observation that restriction must preserve exact sequences of modules since it has no effect on the underlying abelian groups. Note also that all the categories mentioned are repletely G-exact subcategories of $\mathcal{M}od_R$ or $\mathcal{M}od_S$. □

3.3.4 Extension

Again, let $f : R \to S$ be a ring homomorphism. To define the extension functor $f_\# : \mathcal{M}od_R \to \mathcal{M}od_S$, we first consider S as a *left* R-module by the rule $r \cdot s = (fr)s$, so that S becomes an R-S-bimodule. (Strictly speaking, this amounts to replacing S by its image $f^\#(S)$ under the restriction functor for left modules – see (3.3.21).)

Then we put

$$f_\# = - \otimes_R S : \mathcal{M}od_R \longrightarrow \mathcal{M}od_S;$$

explicitly, for a right R-module M,

$$f_\#(M) = M \otimes_R S$$

and for a homomorphism

$$\alpha : M' \longrightarrow M$$

of right R-modules,

$$f_{\#}(\alpha) = \alpha \otimes id_S : f_{\#}(M') \longrightarrow f_{\#}(M).$$

By (3.1.7), $f_{\#}$ is an additive covariant functor from $\mathcal{M}od_R$ to $\mathcal{M}od_S$.

The functor $f_{\#}$ is called *extension of scalars* and the module $f_{\#}(M)$ the *extended module*. In particular, for a right ideal $\mathfrak{a}$ of R, $f_{\#}\mathfrak{a}$ is easily seen to be naturally isomorphic to the right ideal of S generated by the image $f(\mathfrak{a})$ of $\mathfrak{a}$; $f_{\#}\mathfrak{a}$ is commonly known as the *extension* or *extended ideal* of $\mathfrak{a}$.

As with restriction, the terminology is perhaps most appropriate when f is an injection. In special cases, alternative terms may be used. For example, when $S = R_\Sigma$ is a localization of R, $f_{\#}$ is also called *localization*, which we discuss in detail in Chapter 6. When H is a subgroup of a group G and $f : AH \to AG$ is the obvious inclusion of group rings (see Exercise 3.3.8), $f_{\#}$ is called *induction*. Here A can be any coefficient ring.

Here are some of the basic properties of the extension functor.

3.3.5 Lemma

(i) $(id_R)_{\#}$ *is the identity functor* $\mathrm{Id}_R : \mathcal{M}od_R \to \mathcal{M}od_R$.

(ii) *Given ring homomorphisms* $f : R \to S$ *and* $g : S \to T$, *then there is a natural isomorphism*

$$g_{\#}f_{\#} \simeq (gf)_{\#}.$$

Proof

The first assertion is obvious. The second follows from the fact that, for any R-module M, there is an associativity isomorphism from $M \otimes_R (S \otimes_S T)$ to $(M \otimes_R S) \otimes_S T$ (3.1.5), which is easily seen to be natural in M. □

3.3.6 Corollary

In (ii) *above, suppose that* S *is flat when viewed as a left* R*-module by restriction and that* T *is flat as a left* S*-module. Then* T *is flat as a left* R*-module.*

Proof

We appeal to (3.2.8). Since ${}_RS$ and ${}_ST$ are flat, $f_{\#}$ and $g_{\#}$ are exact functors. Thus $(gf)_{\#}$ is an exact functor and so ${}_RT$ is flat. □

If we write $f_{\#}(R) = \mathcal{M}\textsc{od}_R$ and put $g_{\#}f_{\#} = (gf)_{\#}$, then $(-)_{\#}$ is a covariant functor from $\mathcal{R}\textsc{ing}$ to $\mathcal{C}\textsc{at}$.

As with restriction, we use the same symbol $f_{\#}$ for several functors arising from $f_{\#}$.

3.3.7 Proposition

(i) *For any index set* Λ, $f_{\#}(R^{\Lambda}) \cong S^{\Lambda}$.

(ii) $f_{\#}$ *induces additive functors*

$$f_{\#} : \mathcal{P}\textsc{roj}_R \longrightarrow \mathcal{P}\textsc{roj}_S,$$

$$f_{\#} : \mathcal{M}_R \longrightarrow \mathcal{M}_S,$$

$$f_{\#} : \mathcal{P}_R \longrightarrow \mathcal{P}_S$$

and

$$f_{\#} : \mathcal{F}_R \longrightarrow \mathcal{F}_S.$$

Proof

The functor $f_{\#}(-) = - \otimes_R S$ is additive (3.1.7) and has $f_{\#}(R) \cong S$ by (ii) of (3.1.4), so (i) follows from the left-handed version of (3.1.8). Explicitly, the isomorphism is defined by the map

$$(r_\lambda) \otimes s \longmapsto (f(r_\lambda)s),$$

which has inverse

$$(s_\lambda) \longmapsto \sum (e_\lambda \otimes s_\lambda),$$

where e_λ has entry 1 in the λ th place and zero elsewhere (recall that only a finite number of entries of either R^{Λ} or S^{Λ} can be nonzero).

The remaining assertions in (ii) follow from (3.1.17). □

3.3.8 Exactness

The functors

$$f_{\#} : \mathcal{P}\textsc{roj}_R \longrightarrow \mathcal{P}\textsc{roj}_S,$$

$$f_{\#} : \mathcal{P}_R \longrightarrow \mathcal{P}_S$$

and

$$f_{\#} : \mathcal{F}_R \longrightarrow \mathcal{F}_S$$

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are all exact, since in each case the domain is a split G-exact category. In general,

$$f_{\#} : \mathcal{M}_R \longrightarrow \mathcal{M}_S$$

will not be exact, which can be seen by taking $f : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ to be the standard surjection and considering the effect of $f_{\#}$ on the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ n\ } \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

as in (3.2.1).

3.3.9 An identification

Suppose that R is a subring of S and that $f : R \to S$ is the inclusion. Given a right R-module M, it is tempting to view the extended module $f_{\#}M$ simply as the right S-module MS generated by the members of M. This temptation must usually be resisted, since there may be no S-module that contains the elements of M. For example, take R to be $\mathbb{Z}$, S to be $\mathbb{Q}$ and M to be $\mathbb{Z}/2\mathbb{Z}$; then $f_{\#}(M) = 0$.

However, it is legitimate to identify $f_{\#}M$ with MS when M is a flat R-module, since f then induces an injection

$$M \xrightarrow{\ \cong\ } M \otimes_R R \xrightarrow{\ id \otimes f\ } M \otimes_R S;$$

more properly, we identify each $m \in M$ with its image $m \otimes 1 \in M \otimes_R S$, so that

$$M \otimes_R S = \{\sum_{i=1}^{k} m_i s_i \mid m_i \in M, s_i \in S, k \geq 1\} = MS.$$

Likewise, if N is a flat left R-module, we can write $f_{\#}N = S \otimes_R N$ as SN.

3.3.10 The quotient functor

When the ring homomorphism $f : R \to S$ is a surjection, $f_{\#}$ is called the *quotient functor*, which we now describe in more detail.

Let $\mathfrak{a}$ be the kernel of f. Given an R-module M, write

$$f_Q(M) = M/M\mathfrak{a},$$

where $M\mathfrak{a}$ is the submodule of M generated by all products mx with $x \in \mathfrak{a}$ and $m \in M$.

It is clear that $f_Q(M)$ is an S-module under the rule

$$\overline{m} \cdot fr = \overline{mr},$$

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and that an R-module homomorphism $\alpha : M \to M''$ defines an S-module homomorphism $f_Q(\alpha) : f_Q(M) \to f_Q(M'')$ by the rule

$$f_Q(\alpha)(\overline{m}) = (\overline{\alpha m}).$$

Thus we have an additive covariant functor

$$f_Q : \mathcal{M}\textsc{od}_R \longrightarrow \mathcal{M}\textsc{od}_S.$$

3.3.11 Proposition

The functors $f_\#$ and f_Q are naturally isomorphic.

Proof

Let M be a right R-module. By (3.2.3), there is an exact sequence

$$M \otimes_R \mathfrak{a} \longrightarrow M \otimes_R R \longrightarrow M \otimes_R S \longrightarrow 0.$$

By (3.1.4), the obvious map $\eta_M : m \otimes r \mapsto mr$ from $M \otimes_R R$ to M is an isomorphism of right R-modules, and this isomorphism clearly maps the image of $M \otimes_R \mathfrak{a}$ in $M \otimes_R R$ onto $M\mathfrak{a}$ in M.

Thus η_M induces an isomorphism between $f_\# M$ and $f_Q M$, which can be verified directly or, more eruditely, by appealing to the Five Lemma (2.3.23). Since the construction is natural in M, we have a natural isomorphism between $f_\#$ and f_Q. □

The relationship between the extension functor and the restriction functor in general is given by the following result.

3.3.12 Proposition

There are natural transformations

(i)

$$\eta : f_\# f^\# \longrightarrow \mathrm{Id}_S$$

and

(ii)

$$\nu : \mathrm{Id}_R \longrightarrow f^\# f_\#,$$

where Id_S and Id_R are the identity functors on $\mathcal{M}\textsc{od}_S$ and $\mathcal{M}\textsc{od}_R$ respectively.

(iii) *For any right S-module N, we have*

$$\eta_N \nu_{f^\# N} = id_N.$$

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Proof

(i) Let N be a right S-module. Then $f_{\#}f^{\#}N = N \otimes_R S$ and we define

$$\eta_N : N \otimes_R S \longrightarrow N$$

by

$$\eta_N(n \otimes s) = ns;$$

as in (3.1.4), this is a well-defined homomorphism which is clearly natural in N.

(ii) On the other hand, given the right R-module M, we define the homomorphism

$$\nu_M : M \longrightarrow M \otimes_R S$$

by

$$\nu_M m = m \otimes 1,$$

which is again natural.

(iii) This is immediate from the above formulas. □

There are also relations between these functors and tensor products, when the rings act on the appropriate sides of the modules. The first is called a *projection* (or *reciprocity*) *formula.* (See Exercise 3.3.12 for the usage of 'reciprocity' in the representation theory of groups.) Its proof follows easily from the naturality of the various constructions involved, using (3.1.11) and Exercise 3.1.4.

3.3.13 Proposition

For a ring homomorphism $f : R \to S$ and modules M in $\mathcal{M}\text{OD}_R$ and N in ${}_S\mathcal{B}\text{IMOD}_S$ there is a natural isomorphism

$$f^{\#}(f_{\#}M \otimes_S N) \cong M \otimes_R f^{\#}N$$

of right R-modules.

In particular, when $N = S$,

$$f^{\#}f_{\#}M \cong M \otimes_R f^{\#}S. \qquad \square$$

The next result follows from (3.2.15), again using the naturality results in (3.1.11) and Exercise 3.1.4.

3.3.14 Proposition

For a homomorphism $f : R \to S$ of commutative rings and modules M in $\mathcal{M}od_R$ and N in $\mathcal{B}al_R$, there is a natural isomorphism

$$(f_\# M) \otimes_S (f_\# N) \cong f_\#(M \otimes_R N)$$

of right S-modules. □

A further relationship between restriction and extension can be most neatly expressed in the language of category theory.

3.3.15 Proposition

The extension functor is left adjoint to the restriction functor. In other words, given a ring homomorphism $f : R \to S$ and modules M in $\mathcal{M}od_R$ and N in $\mathcal{M}od_S$, then there is a natural isomorphism

$$\mathrm{Hom}_S(f_\# M, N) \cong \mathrm{Hom}_R(M, f^\# N).$$

Proof

Recall from the Adjointness Theorem (3.1.19) that for arbitrary rings A, B and C and bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ as indicated, there is a natural isomorphism

$$\eta : \mathrm{Hom}_{A-C}(L \otimes_B M, N) \longrightarrow \mathrm{Hom}_{A-B}(L, \mathrm{Hom}(M_C, N_C))$$

of trifunctors from ${}_A\mathcal{B}imod_B \times {}_B\mathcal{B}imod_C \times {}_A\mathcal{B}imod_C$ to $\mathcal{A}b$.

Take $A = \mathbb{Z}$, $B = R$ and $C = S$, and let M be S viewed as a (fixed) R-S-bimodule in the usual way. We then obtain a natural isomorphism

$$\eta : \mathrm{Hom}_S(L \otimes_R S, N) \longrightarrow \mathrm{Hom}_R(L, \mathrm{Hom}(S_S, N_S))$$

of bifunctors from $\mathcal{M}od_R \times \mathcal{M}od_S$ to $\mathcal{A}b$.

The functor $N_S \mapsto \mathrm{Hom}(S_S, N_S)$ is naturally isomorphic to the identity functor on $\mathcal{M}od_S$ (Exercise 2.1.3), so the result follows on noting that $f_\# L = L \otimes_R S$ and changing notation. □

3.3.16 $\mathbb{R}$ ***and*** $\mathbb{C}$

To illustrate what has gone before, we consider the special case in which $f : \mathbb{R} \to \mathbb{C}$ is the usual inclusion of the field of real numbers in the complex numbers.

Let V be a real vector space, say of finite dimension n, with basis

$$\{e_1, \ldots, e_n\}.$$

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Then $f_{\#}V$ is a complex vector space of the same dimension, with basis

$$\{e_1 \otimes 1, \ldots, e_n \otimes 1\},$$

and $f^{\#}f_{\#}V$ is the real space of dimension $2n$ with basis

$$\{e_1 \otimes 1, e_1 \otimes i, \ldots, e_n \otimes 1, e_n \otimes i\},$$

where $i = \sqrt{-1}$.

In the other direction, a complex vector space W of dimension n becomes a real space $f^{\#}W$ of dimension $2n$ and then a complex space $f_{\#}f^{\#}W$ of dimension $2n$. These matters are developed further in [Adams 1969], Chapter 3.

The relationship generalizes in the following way.

3.3.17 Proposition

Suppose that a ring homomorphism $f : R \to S$ is an injection which gives S the structure of a free right R-module R^k of finite rank, with basis $\{s_1, \ldots, s_k\}$.

(i) *If $M \cong R^h$ is a free right R-module, with basis $\{m_1, \ldots, m_h\}$, then $f^{\#}f_{\#}M$ is a free right R-module of rank hk, with basis $\{m_i \otimes s_j\}$.*

(ii) *If $N \cong S^n$, then $f^{\#}N \cong R^{nk}$ and $f_{\#}f^{\#}N \cong S^{nk}$.* □

3.3.18 Skew fields unbalanced

The structure of S as a left R-module is irrelevant in the above result. It is worth remarking at this point that the left R-module structure of S may well differ from the right module structure. For example, [Schofield 1985] gives examples of inclusions of skew fields $\mathcal{D} \to \mathcal{D}'$ such that the left and right dimensions of $\mathcal{D}'$ over $\mathcal{D}$ take any preassigned pair of values, provided neither is 1.

Next, we consider the relationship between the functors which arise when we have two ring homomorphisms f and g from R to S.

3.3.19 Theorem

Let f and $g : R \to S$ be ring homomorphisms. Then the following statements are equivalent.

(i) *There is an element λ of S such that*

$$f(r)\lambda = \lambda g(r)$$

for all elements r of R.

(ii) *There is a natural transformation of restriction functors*

$$\eta : g^{\#} \longrightarrow f^{\#}.$$

(iii) *There is a natural transformation of extension functors*

$$\nu : g_{\#} \longrightarrow f_{\#}.$$

Furthermore, λ *is a unit of* R *if and only if either one (and hence both) of* η *and* ν *is an isomorphism.*

Proof

(ii) $\Rightarrow$ (i). We know that there is an R-module homomorphism

$$\eta_S : g^{\#}S \longrightarrow f^{\#}S.$$

Write $\lambda = \eta_S(1_S)$.

Now, for any element n of an S-module N, there is a homomorphism

$$\rho(n) : S \longrightarrow N$$

given by

$$\rho(n)s = ns,$$

and because η is a natural transformation, there is a commutative diagram

$$\begin{array}{ccc} g^{\#}S & \xrightarrow{\eta_S} & f^{\#}S \\ {\scriptstyle g^{\#}\rho(n)}\downarrow & & \downarrow{\scriptstyle f^{\#}\rho(n)} \\ g^{\#}N & \xrightarrow{\eta_S} & f^{\#}N \end{array}$$

which means that $\eta_N(\rho(n)s) = \rho(n)(\eta_S(s))$.

With $s = 1_S$, we see that

$$\eta_N n = n\lambda, \tag{3.1}$$

so that the natural transformation η is completely determined by the element λ of S.

Now, the fact that η_S respects the R-module structures on S means that $\eta_S(1_S \cdot r) = \eta_S(1_S) \cdot r$, that is,

$$g(r)\lambda = \lambda f(r).$$

To establish the converse implication (i) $\Rightarrow$ (ii), it is enough to verify that, for every right R-module N, the formula

$$\eta_N^\lambda n = n\lambda \text{ for } n \text{ in } N,$$

defines a natural transformation $\eta^\lambda : g^\# \to f^\#$ when λ satisfies the given condition (i). This is left to the reader, as is the equivalence of (i) and (iii).

For the final assertion, recall from Equation 3.1 above that any η as in (ii) is $\eta = \eta^\lambda$ for some $\lambda \in S$. The claim now follows from the observations that $\eta^\mu \eta^\lambda = \eta^{\lambda\mu}$ and $\eta^{(1_S)} = \mathrm{Id}_S$. □

We note an important consequence of the theorem. Recall that a ring endomorphism α of R is an inner automorphism if $\alpha r = \lambda r \lambda^{-1}$ for some unit λ of R.

3.3.20 Corollary

For an endomorphism α of a ring R, the following statements are equivalent.

(i) *α is an inner automorphism of R.*
(ii) *$\alpha^\#$ is naturally isomorphic to* Id_R.
(iii) *$\alpha_\#$ is naturally isomorphic to* Id_R. □

3.3.21 The definitions for left modules

For the convenience of the reader, we quickly review the definitions of restriction and extension for left modules.

Given a ring homomorphism $f : R \to S$ and a left S-module N, the restriction of N is the left R-module $f^\# N$, with action given by the rule

$$r \cdot n = (fr) \cdot n.$$

With right-handed restriction, the ring S is itself an S-R-bimodule, and the extended module $f_\# N$ is defined to be the left S-module $S \otimes_R N$.

It is clear that all our results for the right-handed restriction and extension functors have left-handed versions.

3.3.22 The twisting of modules

When α is an automorphism of a ring R, we may view the extension and restriction functors in a different light. Given a right R-module M, we define the *twisted module* M^α in $\mathcal{M}\text{od}_R$ as follows. Elements of M^α are symbols m^α, addition is given by

$$(m+n)^\alpha = m^\alpha + n^\alpha,$$

and the action of R is given by

$$m^\alpha \cdot r = (m \cdot \alpha r)^\alpha.$$

Similarly, a left R-module N can be twisted to obtain the left R-module ${}^{\alpha}N$, in which $r \cdot {}^{\alpha}n = {}^{\alpha}(\alpha r \cdot n)$.

Applying these operations to R itself, we obtain new R-R-bimodules R^{α} and ${}^{\alpha}R$.

3.3.23 Lemma

(i) *The map $\zeta : x \mapsto (\alpha x)^{\alpha}$ from R to R^{α} is an isomorphism of right R-modules.*

(ii) *The map $\xi : x \mapsto {}^{\alpha}(\alpha x)$ from R to ${}^{\alpha}R$ is an isomorphism of left R-modules.*

Proof

For r in R, $\zeta(xr) = (\alpha x \cdot \alpha r)^{\alpha}$, which is $(\alpha x)^{\alpha} \cdot r$ by the definition. Thus $\zeta(xr) = \zeta(x) \cdot r$. Clearly, ζ is an additive bijection, so (i) is proven. The argument for (ii) is much the same. □

The above result generalizes to a description of the effect of twisting on projective modules. We use some results on the relationship between projective modules and idempotent matrices that are considered in detail in [BK: IRM] (2.5.9)ff.

3.3.24 Lemma

Let $\alpha : R \to R$ be a ring automorphism, and let α_k denote the induced automorphism of the matrix ring $M_k(R)$ for each $k \geq 1$ (thus $\alpha_1 = \alpha$). Suppose that P is a finitely generated projective right R-module. Then the following hold.

(i) *$P \cong \eta R^k$ for some $k \times k$ idempotent matrix η over R.*

(ii) *There is a right R-module isomorphism $P^{\alpha} \cong \alpha_k^{-1}(\eta) R^k$.*

(iii) *P^{α} is also a finitely generated projective right R-module.*

Proof

(i) There is a surjective R-module homomorphism $\pi : R^k \to P$ for some k which is split by an R-module homomorphism $\sigma : P \to R^k$. Then $\sigma\pi$ is an idempotent endomorphism of R^k which can be represented by a $k \times k$ idempotent matrix η (1.3.4), and clearly $P \cong \eta R^k$.

(ii) Let $\alpha_{\oplus k} = \alpha \oplus \cdots \oplus \alpha : R^k \to R^k$ be the R-module homomorphism induced by α, and define

$$\theta : P^{\alpha} \longrightarrow \alpha_k^{-1}(\eta) R^k$$

by

$$\theta((\eta x)^\alpha) = \alpha_{\oplus k}^{-1}(\eta x) = \alpha_k^{-1}(\eta)\alpha_{\oplus k}^{-1}(x).$$

It is clear that θ is a well-defined additive bijection. Moreover, for any r in R, we have

$$\begin{aligned} \theta((\eta x)^\alpha \cdot r) &= \theta((\eta x \cdot \alpha(r))^\alpha) \\ &= \alpha_{\oplus k}^{-1}(\eta x \cdot \alpha(r)) \\ &= \alpha_{\oplus k}^{-1}(\eta x) \cdot r \\ &= \theta((\eta x)^\alpha) \cdot r, \end{aligned}$$

showing θ to be an R-module homomorphism as desired.

(iii) Since $\alpha_k^{-1}(\eta)$ is again idempotent, $\alpha_k^{-1}(\eta)R^k$ is a direct summand of R^k and hence projective. □

The next result shows that a twisted module can be viewed equally as being a restricted or an extended module.

3.3.25 Proposition

(a) *Let M be a right R-module. Then there are isomorphisms, natural in M, between the following right R-modules:*

(i) *the twisted module M^α,*

(ii) *the restricted module $\alpha^\# M$,*
and

(iii) *the extended module $(\alpha^{-1})_\# M$.*

(b) *Let N be a left R-module. Then there are isomorphisms, natural in N, between the following left R-modules:*

(i) *the twisted module ${}^\alpha N$,*

(ii) *the restricted module $\alpha^\# N$,*
and

(iii) *the extended module $(\alpha^{-1})_\# N$.*

Proof

We give the argument for (a) only. We first consider the relation between the twisted module and the restricted module.

For clarity, we write a typical element of $\alpha^\# M$ as ${}^\# m$, so that the R-action on $\alpha^\# M$ is given by ${}^\# m \cdot r = {}^\#(m \cdot \alpha r)$. Then the function $\alpha^\# M \to M^\alpha$ sending ${}^\# m$ to m^α is clearly an isomorphism.

Next, to compare M^α with the extended module $(\alpha^{-1})_\# M$, recall that the extended module is generated by elements of the form $m \otimes t$, where $ms \otimes t = m \otimes (\alpha^{-1}s)t$. Because α is an automorphism, it follows that each element

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of $(\alpha^{-1})_{\#}M$ may in fact be written in the form $m \otimes 1$. Then the function $M^{\alpha} \to (\alpha^{-1})_{\#}M$ given by $m^{\alpha} \mapsto m\otimes 1$ is evidently a bijection. The R-actions are respectively

$$m^{\alpha} \cdot r = (m \cdot \alpha r)^{\alpha} \text{ on } M^{\alpha},$$

and

$$(m \otimes 1) \cdot r = m \otimes r = m \otimes \alpha^{-1}(\alpha r) = m \cdot \alpha r \otimes 1 \text{ on } (\alpha^{-1})_{\#}M,$$

so that the map is also a homomorphism. □

3.3.26 Group rings

The representation theory of groups plays an important role in K-theory, both as a tool for use in constructing the theory, and as a source of problems within the theory. We therefore give a brief summary of the terminology and some special constructions which arise when we apply 'change of rings' to modules over group rings.

Let A be a commutative coefficient ring and let G be a group, usually but not necessarily finite. The group ring AG is the free (left) A-module generated by the set G, with multiplication inherited from G. Thus an element $x \in AG$ looks like $x = \sum_{g\in G} x_g g$ with only a finite set of nonzero coefficients x_g, and multiplication is given by

$$\left(\sum_{g\in G} x_g g\right)\left(\sum_{h\in G} y_h h\right) = \sum_{gh\in G} x_g y_h gh.$$

In particular, AG is a balanced A-module, and the identity for the multiplication is $1_A 1_G$.

Any right AG-module will be a right A-module, which we always take to be a balanced A-module.

A group homomorphism $f : H \to G$ gives an evident ring homomorphism $f : AH \to AG$ (of the same name). In this context, the extension functor $f_{\#} : M_{AH} \to (M \otimes_{AH} AG)_{AG}$ is called the *induction functor*, $M \otimes_{AH} AG$ being the *induced module*. The terminology for restriction is unchanged. Both the terms induction and restriction are most appropriate when the group homomorphism f is an inclusion, but their usage is extended to general homomorphisms.

Given a right AH-module M, the induced AG-module is more often written M^G, the ring A being understood. Similarly, the restriction of a right AG-module N is written N_H.

The special nature of a group ring allows further operations on AG-modules.

The map $\iota : G \to G$ given by inversion, $\iota(g) = g^{-1}$, is an anti-automorphism and so gives an anti-automorphism of AG. Thus the group ring AG is isomorphic to its opposite ring AG°, and so a right AG-module N can be regarded as a left AG-module by the action $gn = ng^{-1}$ for g in G and n in N. However, N will not be an AG-AG-bimodule unless G is abelian. (This is a special case of the method for switching from right to left modules given in [BK: IRM] (1.2.6); the connection between opposites and anti-automorphisms for rings is explored in [BK: IRM] Exercise 1.2.13, and, for groups, in Exercise 1.1.5 of the present text.)

This manoeuvre permits the definition of a pairing on $\mathcal{M}\textsc{od}_{AG}$ with values in $\mathcal{M}\textsc{od}_A$, namely $(N, P) \mapsto N \otimes_{AG} P$, where P becomes a left AG-module as above.

It is also possible to define a pairing on $\mathcal{M}\textsc{od}_{AG}$ with values in $\mathcal{M}\textsc{od}_{AG}$, by tensoring over A rather than AG. Given right AG-modules N and P, $N \otimes_A P$ becomes a right AG-module by the *diagonal action* of G. This is given on the generators of the tensor product by

$$(n \otimes p)g = ng \otimes pg \ \text{ for } n \text{ in } N,\ p \text{ in } P \text{ and } g \text{ in } G,$$

and extends by linearity to both $N \otimes_A P$ and AG.

A full exploration of the above constructions belongs to works on representation theory, for example, [Curtis & Reiner 1966], [Curtis & Reiner 1981], [Curtis & Reiner 1987] or [Serre 1977]. We give some formal results as exercises below.

Exercises

3.3.1 Let $f : R \to S$ be a surjective ring homomorphism, with kernel $\mathfrak{a}$. Verify that

(i) $f^\# f_\# M \cong M/M\mathfrak{a}$ for an R-module M,
(ii) $f_\# f^\# N \cong N$ for an S-module N,
(iii) $f_\# f^\# \simeq \mathrm{Id}_S : \mathcal{M}\textsc{od}_S \to \mathcal{M}\textsc{od}_S$.

3.3.2 Let $f : R \to S$ be a ring homomorphism. Given a right R-module N, show that the R-S-bimodule structure of S gives a right S-module structure on $\mathrm{Hom}_R(S, N)$.

Prove that $\mathrm{Hom}_R(S, -) : \mathcal{M}\textsc{od}_R \to \mathcal{M}\textsc{od}_S$ is a covariant functor which is right adjoint to the restriction functor $f^\#$, in that there is a natural isomorphism

$$\mathrm{Hom}_R(f^\# M, N) \cong \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N)).$$

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([Cartan & Eilenberg 1956] call the module $\mathrm{Hom}_R(S, N)$ the *contravariant f-extension* of N, in distinction to the *covariant f-extension* $N \otimes_R S$.)

3.3.3 Let A be a ring and let α be a (ring) endomorphism of A. The *skew polynomial ring* $A[T, \alpha]$ consists of all right polynomials

$$a_0 + Ta_1 + T^2 a_2 + \cdots + T^k a_k, \quad k \geq 0,$$

with coefficients $a_0, a_1, \ldots, a_k$ in A, and multiplication given by

$$a \cdot T = T \cdot \alpha(a) \ \text{ for } a \in A.$$

A detailed discussion of skew polynomial rings can be found in [BK: IRM] §3.2.

Show that α can be extended to an endomorphism $\widehat{\alpha}$ of the skew polynomial ring $A[T, \alpha]$.

Verify that there are natural transformations

$$\widehat{\alpha}_{\#} \longrightarrow \mathrm{Id}_{A[T,\alpha]} \quad \text{and} \quad \widehat{\alpha}^{\#} \longrightarrow \mathrm{Id}_{A[T,\alpha]}.$$

Show that these transformations are natural isomorphisms if and only if the corresponding statement is already true for α and Id_A.

3.3.4 Let $R = R_1 \times \cdots \times R_n$ be a direct product of rings, viewed as a direct sum of nonunital rings, with $\pi_i : R \to R_i$ and $\sigma_i : R_i \to R$, $i = 1, \ldots, n$, the corresponding surjective and injective nonunital ring homomorphisms.

Show that $(\pi_i)^{\#} = (\sigma_i)_{\#}$ and $(\pi_i)_{\#} = (\sigma_i)^{\#}$ for all i.

Prove that there is a natural isomorphism

$$\mathrm{Id}_R \simeq (\sigma_1)_{\#}(\pi_1)_{\#} \oplus \cdots \oplus (\sigma_n)_{\#}(\pi_n)_{\#},$$

where the direct sum of functors is defined as in (2.2.21).

Using (2.3.17) (or directly), give an alternative proof of (a generalization of) (1.3.16):

$$\mathcal{M}\textsc{od}_R \ \textit{is equivalent to} \ \mathcal{M}\textsc{od}_{R_1} \times \cdots \times \mathcal{M}\textsc{od}_{R_n}.$$

Deduce the corresponding decompositions for $\mathcal{M}_R$ and $\mathcal{P}_R$.

These decompositions of categories give an alternative view of some elementary results on the structure of modules over a direct product of rings – see [BK: IRM] §2.6, particularly (2.6.7), (2.6.8).

3.3.5 Let $\mathfrak{a}$ be a right ideal of the ring R, and α an automorphism of R. Further, let $\zeta : x \mapsto (\alpha x)^{\alpha}$ from R to R^{α} be the isomorphism of right R-modules given in (3.3.23).

Show that $\mathfrak{a}^{\alpha} = \zeta(\alpha^{-1}\mathfrak{a})$ and hence that $\mathfrak{a}^{\alpha} \cong \alpha^{-1}\mathfrak{a}$ as a right

R-module. Deduce that $(R/\mathfrak{a})^{\alpha} \cong R^{\alpha}/\mathfrak{a}^{\alpha} \cong R/\alpha^{-1}\mathfrak{a}$ as a right R-module.

3.3.6 Let γ be an automorphism of the field $\mathcal{K}$, and let $f \in \mathcal{K}[T]$ be a polynomial.

Show that $(f\mathcal{K}[T])^{\gamma} = f\mathcal{K}[T]$ if and only if $f \in \mathcal{F}[T]$, where

$$\mathcal{F} = \{x \in \mathcal{K} \mid \gamma x = x\}$$

is the fixed field of γ.

Let $\mathcal{K} = \mathbb{Q}(X)$, the field of rational functions over $\mathbb{Q}$, and define the automorphism α of $\mathcal{K}$ by $\alpha X = X + 1$. Find an ideal $\mathfrak{a}$ of $\mathcal{K}[T]$ such that $(\mathfrak{a})^{\alpha^i} \neq \mathfrak{a}$ for any $i \geq 1$.

3.3.7 Let $\mathcal{O}$ be the ring of integers of a quadratic number field $\mathbb{Q}(\sqrt{d})$, and let $\gamma : a + b\sqrt{d} \mapsto a - b\sqrt{d}$ be the conjugation automorphism.

(a) Let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}$, so that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ for some integer prime p. The relationship between p and $\mathfrak{p}$ is found by considering the prime factorization (2.3.20 - A) of $p\mathcal{O}$ as an ideal over the Dedekind domain $\mathcal{O}$, which must fall into one of the following three cases ([BK: IRM] (5.3.2)). In each case, verify that the effect of γ on $\mathfrak{p}$ is as claimed.

  (i) $\mathfrak{p} = p\mathcal{O}$ (p is inert). Then $\mathfrak{p}^{\gamma} = \mathfrak{p}$.
  (ii) $\mathfrak{p}^2 = p\mathcal{O}$ (p is ramified). Then $\mathfrak{p}^{\gamma} = \mathfrak{p}$.
  (iii) $\mathfrak{p} = \mathfrak{p}_i$, $i = 1, 2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct prime ideals of $\mathcal{O}$ with $\mathfrak{p}_1\mathfrak{p}_2 = p\mathcal{O}$ (p is split). Then $\mathfrak{p}_i^{\gamma} = \mathfrak{p}_{3-i}$.

(b) Let P be a projective $\mathcal{O}$-module with ideal class $\{\mathfrak{a}\} \in \mathrm{Cl}(\mathcal{O})$. Show that P^{γ} has ideal class $\{\gamma^{-1}\mathfrak{a}\}$.
Deduce that $P \cong P^{\gamma}$ if and only if $\{\mathfrak{a}\} = \{\gamma^{-1}\mathfrak{a}\}$ (see Steinitz' Theorem (2.3.20 - D)).
Using the calculation of the ideal class group of the ring of integers of $\mathbb{Q}(\sqrt{-71})$ given in [BK: IRM] Theorem 5.3.20, or otherwise, find an example with $P \ncong P^{\gamma}$.

(c) Show that $(\mathcal{O}/\mathfrak{p}^n)^{\gamma} = \mathcal{O}/(\mathfrak{p}^{\gamma})^n$ for any prime ideal $\mathfrak{p}$ and integer n. Using the structure theory for $\mathcal{O}$-modules given in (2.3.20), find M^{γ} for an arbitrary finitely generated $\mathcal{O}$-module M.

3.3.8 **Group rings**

The remaining exercises in this section explore the behaviour of the extension and restriction functors on group rings.

Let H be a subgroup of a group G and suppose that the index of H in G is finite, say $[G : H] = k$, so that G can be written as a disjoint union $G = Hg_1 \sqcup \cdots \sqcup Hg_k$ of cosets of H. Following standard usage, we take $g_1 = 1$.

Let A a commutative ring and let M be a right AH-module. Show that, as an abelian group, $M^G = (M \otimes_{AH} 1) \oplus \cdots \oplus (M \otimes_{AH} g_k)$.

Suppose further that H is normal in G. For $i = 1, \ldots, k$, let $\gamma(i)$ be the ring automorphism of AH induced by the conjugation automorphism $\gamma(i) : h \mapsto g_i h g_i^{-1}$ on H, so that $\gamma(1) = id$.

Verify that $(M^G)_H \cong M \oplus M^{\gamma(2)} \oplus \cdots \oplus M^{\gamma(k)}$ as an AH-module.

3.3.9 Let G be a group and let G act on the direct product $G \times G$ by the diagonal action: $(x, y)g = (xg, yg)$ for $x, y, g \in G$. Show that for a given pair (x, y) there are unique elements $x', g' \in G$ with $(x, y)g' = (x', 1)$.

Deduce that, with the diagonal action on the tensor product,

$$AG \otimes_A AG = \sum_{x \in G} (x \otimes 1) AG.$$

Contrast this with the equation $AG \otimes_{AG} AG = AG$.

3.3.10 Let H be a subgroup of a group G and let A be an arbitrary commutative coefficient ring. Let M be a right AH-module and let N be a right AG-module.

Show that the map

$$\alpha : (N_H \otimes_A M) \otimes_{AH} AG \longrightarrow N \otimes_A (M \otimes_{AH} AG),$$

$$\alpha : (n \otimes m) \otimes g \longmapsto ng \otimes_A (m \otimes g),$$

induces an isomorphism of right AG-modules

$$(N_H \otimes_A M)^G \cong N \otimes_A M^G$$

which is natural in both M and N.

Show also that restriction respects products: if N and P are AG-modules, then $N_H \otimes_A P_H \cong (N \otimes_A P)_H$.

Note. The tensor products over A must be regarded as modules over AH or AG by the diagonal action.

Contrast these formulas with those obtained in (3.3.13) and (3.3.14) for arbitrary rings R and S; in those results, tensor products were taken over R or S.

3.3.11 Let H be a subgoup of G, let A be a commutative ring and let N be an AH-module. Verify that $\operatorname{Hom}_{AH}(AG, N)$ is a right AG-module, the *coinduced module*.

Suppose that $[G : H] = k$ is finite. Show that $\operatorname{Hom}_{AH}(AG, N) \cong N^k$ as a right AH-module.

3.3.12 Let H be a subgroup of a group G, and let A be a commutative ring. Use the Adjointness Theorem (3.1.19) to obtain the isomorphism of A-modules

$$\operatorname{Hom}_{AH}(M, N_H) \cong \operatorname{Hom}_{AG}(M^G, N).$$

Hint. Note that $N \cong \operatorname{Hom}_{AG}(AG, N)$ as a right AG-module, and that the isomorphism gives an isomorphism of the restrictions of both sides.

Remark. The above formula is called the *Frobenius Reciprocity Law*, although the original statement was in the context of complex character theory. A number of similar formulas, such as the isomorphism $(N_H \otimes_A M)^G \cong N \otimes_A M^G$ of Exercise 3.3.10, are also called *reciprocity laws*. A discussion can be found in [Curtis & Reiner 1981], §10.

3.3.13 Let G be a group and let $\epsilon : G \to 1$ be the unique homomorphism from G to the trivial group 1. For any commutative coefficient ring A, the induced homomorphism $\epsilon : AG \to A$ is called the *augmentation homomorphism* and its kernel $\mathfrak{A}$ is the *augmentation ideal* of AG.

Show that $\mathfrak{A}$ is generated by the set $\{g - 1 \mid g \in G\}$.

Let M be any right AG-module. Prove the universal properties which show that

(i) $\epsilon_{\#}M$ is the 'largest quotient' of M on which AG acts trivially;
(ii) $\operatorname{Hom}_{AG}(A, M)$ is the 'largest submodule' of M on which AG acts trivially, the module of G-*invariants* of M.

Hint. Exercise 3.3.1 helps.