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CLOSED STRUCTURES ON THE CATEGORY OF TOPOLOGICAL SPACES DETERMINED BY SYSTEMS OF FILTERS

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We give a characterization of monoidal closed structures, "*determined by systems of filters*" on the category of topological spaces and continuous maps. The method we use to introduce suitable topologies on the product set $X \times Y$ of spaces X and Y, and on the set of all continuous maps from X to Y, is essentially that of Wilker.

Introduction

The following paper deals with monoidal closed structures $(\text{Top}, - \Box -, [-, -], \mathrm{I}, r, l, a)$ on Top (in the sense of [6]).

Many authors drew their attention to the study of such structures, not only in these last years, but even before category theory became relevant (see for instance [1], [4], [2], [5]).

It is known [9], that any such structures must necessarily have, as underlying set of I, $X \Box Y$ and $[Y, Z]$, the canonical one, that is, $\{*\}$, $X \times Y$ (product set) and $\hom(Y, Z)$ respectively. So, to construct monoidal closed structures on Top, it suffices to provide the product set and the set of continuous functions with suitable topologies and to study when such spaces are related by an exponential law.

In this direction, we follow the approach of Wilker [11], in

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topologizing the above-mentioned sets by families of filters (see §1). In §2 we give a characterization of those particular systems of filters, that generate monoidal closed structures on **Top** .

All the examples given by Booth and Tillotson in [2], and by Greve in [7], are of this kind.

Furthermore they seem to be all the examples known in literature on the matter.

Notation

We shall always use the following notations: X, Y, Z denote topological spaces; $X \times Y$ indicates the product set of X and Y while $\hom(X, Y)$ denotes the set of continuous functions from X to Y . $\{*\}$ stands for a singleton space and S for a Sierpinski space, that is, a space consisting of two points $0, 1$, with $\emptyset$, $\{1\}$, $\{0, 1\}$ as its open sets.

For any topological space Y , $\mathrm{Op}\,Y$ denotes the set of open sets of Y and Φ the natural identification of $\mathrm{Op}\,Y$ with $\hom(Y, S)$. Φ is defined by $\Phi(U) = \Phi_U$, $U \in \mathrm{Op}\,Y$, where $\Phi_U : Y \to S$ is the characteristic map of U $\left(\Phi_U(y) = 1 \text{ if } y \in U ,\ \Phi_U(y) = 0 \text{ if } y \notin U\right)$. If $X \otimes Y$ and (Y, Z) are arbitrary topological spaces on the product set $X \times Y$ and the set $\hom(Y, Z)$, respectively, then, by an exponential law ψ , we mean the bijection

$$\psi : \hom(X \otimes Y, Z) \to \hom\big(X, (Y, Z)\big)$$

defined by

$$\psi(f)(x)(y) = f(x, y) ,$$

$$\psi^{-1}(g)(x, y) = g(x)(y) ,$$

for any $f : X \otimes Y \to Z$ and $g : X \to (Y, Z)$ continuous maps.

The symbol $\langle K \rangle$ with $K \subseteq Y$ and $Y \in \mathbf{Top}$, denotes the filter of open sets generated by K , that is,

$$\langle K \rangle = \{U,\ U \supseteq K \mid U \text{ open set of } Y\} \subseteq \mathrm{Op}\,Y .$$

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1.

Let Y be a topological space, and F a family of filters F on $\mathrm{Op}\ Y$ (that is, each $F \in \mathsf{F}$ is a subset of the set $\mathrm{Op}\ Y$). It is possible to define a topology on $\mathrm{Op}\ Y$, by requiring these sets F to form an open subbase. $\mathrm{Op}\ Y$ with this topology will be denoted by TY . (If F is the empty family, then TY is an indiscrete topological space.)

DEFINITION 1.1. For any $Z \in \mathbf{Top}$, we denote by $[Y, Z]$ the set $\hom(Y, Z)$ with the initial topology with respect to all the functions of the form

$$\lambda_U : \hom(Y, Z) \to TY$$

with $U \in \mathrm{Op}\ Z$ and $\lambda_U(f) = f^{-1}(U)$, for any $f \in \hom(Y, Z)$.

The family of all the sets $W(F, U) = \{f,\ f \in \hom(Y, Z) \mid f^{-1}(U) \in F\}$, $U \in \mathrm{Op}\ Z$, $F \in \mathsf{F}$, is an open subbase of $[Y, Z]$. It is easy to see that, defining $[Y, h] = \hom(1_Y, h) : [Y, Z] \to [Y, Z']$, for any $h : Z \to Z'$, $[Y, -]$ becomes a functor $\mathbf{Top} \to \mathbf{Top}$. Furthermore $TY \cong [Y, S]$ by Φ and $\lambda_U = [Y, \Phi_U]$.

Consider now the product set $X \times Y$. For any $A \subseteq X \times Y$, and $x \in X$, we use the symbol A_x to denote the set $\{y,\ y \in Y \mid (x, y) \in A\}$.

DEFINITION 1.2. A subset A of $X \times Y$ is an F-*open set* if and only if

(i) $A_x \in \mathrm{Op}\ Y$, for any $x \in X$,

(ii) $u : X \to TY$, $u(x) = A_x$, is a continuous map.

A family F of filters on $\mathrm{Op}\ Y$ is called an *adjoining system*, in the sense of Wilker [11], if and only if the two following conditions are satisfied:

(i) each $F \in \mathsf{F}$ is a compact filter (that is, if $\cup U_i \in F$, $i \in I$, $U_i \in \mathrm{Op}\ Y$, then there is a finite subset $J \subseteq I$ such that $\cup U_i \in F$, $i \in J$);

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(ii) for any pair $U_1, U_2 \in \mathrm{Op}\, Y$ and for any $F \in \mathsf{F}$, if $U_1 \cup U_2 \in F$, then there are filters F_1, F_2, finite intersections of filters of F, such that $U_1 \in F_1$, $U_2 \in F_2$ and $F_1 \cap F_2 \subseteq F$.

Wilker proved in [*11*, p. 272, Theorem 2] that the F-*open sets* are the open sets for a topology on $X \times Y$, if and only if F is an *adjoining system*.

We denote this topological space by $X \square Y$, and by $- \square Y : \mathbf{Top} \to \mathbf{Top}$ the associated functor (where, for any $h : X \to X'$, $h \square Y = h \times 1_Y : X \square Y \to X' \square Y$).

LEMMA 1.3. *If* $X \otimes Y$ *denotes an arbitrary topology on the product set* $X \times Y$, *then the exponential law*

$$\psi_{X,Y,Z} : \hom(X \otimes Y,\ Z) \to \hom(X,\ [Y,\ Z])$$

is a bijection for any X *and* Z *in* $\mathbf{Top}$, *if and only if* ψ *is a bijection for any* X *in* $\mathbf{Top}$ *and* $Z = S$.

Proof. If $Z = S$ and $\psi_{X,Y,S} : \hom(X \otimes Y,\ S) \to \hom(X,\ TY)$ is a bijection (where $[Y,\ S]$ is identified with TY), then the open sets of $X \otimes Y$ are exactly the F-*open sets* and $X \otimes Y = X \square Y$. It follows that the inclusion map $i : \{x\} \times Y \to X \square Y$ is continuous, for any $x \in X$.

To prove that $\psi_{X,Y,Z}$ is a bijection, for any X and Z, it suffices to verify that, given $f : X \square Y \to Z$, $g : X \to [Y,\ Z]$, $x \in X$, the maps $\psi(f) : X \to [Y,\ Z]$, $\psi(f)(x) : Y \to Z$ and $\psi^{-1}(g) : X \square Y \to Z$ are continuous. $\psi(f)(x) = f(x,\ -)$, by definition of ψ, and

$$f(x,\ -) = f \cdot i_x : Y \cong \{x\} \times Y \to X \square Y \to Z$$

for any $x \in X$, so it is continuous.

For the universal property of initial topology, $\psi(f) : X \to [Y,\ Z]$ is continuous if and only if the composite

$$X \xrightarrow[\psi(f)]{} [Y,\ Z] \xrightarrow[\lambda_U]{} TY$$

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is continuous, for any $U \in \mathsf{Op}\ Z$; being $\psi_{X,Y,S}$ a bijection, such composite is continuous if and only if

$$\psi^{-1}_{X,Y,S}\big(\lambda_U \cdot \psi(f)\big) : X \,\square\, Y \to S$$

is continuous.

Since this map is the characteristic map Φ_A , associated to the open set $A = f^{-1}(U) \in \mathsf{Op}\ X \,\square\, Y$, the result follows.

If $g : X \to [Y, Z]$, let $\tilde{g}$ denote the function $\psi^{-1}(g) : X \,\square\, Y \to Z$. Since g is continuous, the composite

$$X \xrightarrow[g]{} [Y, Z] \xrightarrow[\lambda_U]{} TY$$

is continuous, for any $U \in \mathsf{Op}\ Z$; applying $\psi^{-1}_{X,Y,S}$ we obtain a continuous map of $X \,\square\, Y$ in S , which is exactly the characteristic map associated to the set $\tilde{g}^{-1}(U)$. Since $\tilde{g}^{-1}(U)$ is an open set for any $U \in \mathsf{Op}\ Z$, the continuity of $\tilde{g}$ follows. $\square$

PROPOSITION 1.4. *There is an adjunction* $(- \,\square\, Y, [Y, -], \psi)$ *if and only if* F *is an adjoining system for* Y .

Proof. This is a consequence of [*11*, p. 272, Theorem 2] and of Lemma 1.3, where ψ is - of course - natural in X and Z . $\square$

Suppose now we have an adjoining system of filters, $\mathsf{F}(Y)$, for any $Y \in \mathsf{Top}$. $- \,\square\, Y$ and $[Y, -]$ are the partial functors of the bifunctors $- \,\square\, - : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Top}$ and $[-, -] : \mathsf{Top}^\circ \times \mathsf{Top} \to \mathsf{Top}$, if and only if $[-, S] : \mathsf{Top}^\circ \to \mathsf{Top}$ defined by

$$[-, S](Y) = [Y, S] \ , \quad Y \in \mathsf{Top} \ ,$$

$$[-, S](h) = [h, S] = \hom\big(h, 1_S\big) : [Y, S] \to [Y', S] \ , \quad h : Y' \to Y \ ,$$

is a contravariant functor.

In fact, by the universal property of initial topology and by the following commutative diagram

$$\begin{array}{ccccc} [Y,\ Z] & \xrightarrow{\hom(h,\, 1_Z)} & [Y',\ Z] & \xrightarrow{\lambda'_U} & TY' \cong [Y',\ S] \\ & \searrow^{\lambda_U} & & & \big\uparrow [h,\ S] \\ & & & & TY \cong [Y,\ S] \end{array}$$

(where $h : Y' \to Y$, $U \in \mathrm{Op}\ Z$), each $[-,\ Z]$, $Z \in \mathrm{Top}$ is functorial. Since $[h,\ Z] = \hom(h,\ 1_Z)$ is continuous, and the diagram

$$\begin{array}{ccc} \hom(X \,\Box\, Y,\ Z) & \cong & \hom(X,\ [Y,\ Z]) \\ {\scriptstyle \hom(1_X \times h,\ 1_Z)} \big\downarrow & & \big\uparrow {\scriptstyle \hom(1_X,\ [h,\ Z])} \\ \hom(X \,\Box\, Y',\ Z) & \cong & \hom(X,\ [Y',\ Z]) \end{array}$$

is commutative, then the function $1_X \times h : X \,\Box\, Y' \to X \,\Box\, Y$ is continuous, and each $X \,\Box\, -$, $X \in \mathrm{Top}$ is a functor.

DEFINITION 1.5. A family of adjoining systems $(F(Y))_{Y \in \mathrm{Top}}$ is a *functorial adjoining system* if and only if $[-,\ S] : \mathrm{Top}^{\circ} \to \mathrm{Top}$ is a functor.

For characterization of filters of a functorial adjoining system see Wilker [*11*, p. 275-276].

2.

In this section we give a characterization of those functorial adjoining systems that determine a monoidal closed structure $(\mathrm{Top},\ - \Box -,\ [-,\ -],\ \{*\})$ on Top .

A topology on the set $\mathrm{Op}\ Y$, $Y \in \mathrm{Top}$, is called a *topological topology* [*8*] if and only if it makes finite intersection and arbitrary union continuous operations.

LEMMA 2.1. *If* $F(Y)$ *is an adjoining system for* Y , *then* TY *is a topological topology on* $\mathrm{Op}\ Y$. *Conversely, if* TY *is a topological topology on* $\mathrm{Op}\ Y$, *and* TY *has a subbase* F , *whose elements are filters, then* F *is an adjoining system for* Y .

Proof. If $F(Y)$ is an adjoining system, then $- \Box Y \dashv [Y,\ -]$.

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Since $\{*\} \square Y \cong Y$, it follows by [10, p. 34, Theorem 4.1] or by [8], that $[Y, S] \cong TY$ is a topological topology.

Conversely, the pair (Y, TY) with TY topological topology on $\mathsf{Op}\ Y$, determines, again by [10, p. 34, Theorem 4.1], a pair of adjoint functors $H \dashv G$, where $H \cong - \square Y$ and $G \cong [Y, -]$. From Proposition 1.4, F must be an adjoining system on $\mathsf{Op}\ Y$. □

Let $(\mathsf{F}(Y))_{Y \in \mathsf{Top}}$ be a family of adjoining systems.

LEMMA 2.2. *The exponential law*

$$\psi_{X,Y,Z} : [X \square Y, Z] \to [X, [Y, Z]]$$

is a homeomorphism, for any X, Y, Z *in* Top, *if and only if* ψ *is a homeomorphism for any* X, Y *in* Top *and* $Z = S$.

Proof. Let $\psi_{X,Y,S}$ be a continuous map for any $X, Y \in \mathsf{Top}$. For the universal property of initial topology, to prove that $\psi_{X,Y,Z}$ is continuous, we have just to show that the composite

$$[X \square Y, Z] \xrightarrow[\psi_{X,Y,Z}]{} [X, [Y, Z]] \xrightarrow[\lambda_V]{} TX$$

is continuous for any λ_V, V open set of $[Y, Z]$, $Z \in \mathsf{Top}$. Let $W = W(F, U)$, $F \in \mathsf{F}(Y)$, $U \in \mathsf{Op}\ Z$ be a subbasic open set of $[Y, Z]$. It is easy to see that the characteristic map Φ_W coincides with the following composite map

(2.2.1)
$$\begin{array}{ccc} [Y, Z] & \xrightarrow{\ \Phi_W\ } & S \\ {\scriptstyle [Y, \Phi_U]} \searrow & & \nearrow {\scriptstyle \Phi_F} \\ & [Y, S] \cong TY & \end{array} .$$

Let us consider now the diagram

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$$\begin{array}{ccccc}
[X \,\square\, Y,\ Z] & \xrightarrow{\psi_{X,Y,Z}} & [X,\ [Y,\ Z]] & \xrightarrow{\lambda_W} & TX \cong [X,\ S] \\
\Big\downarrow \lambda_U & & \Big\downarrow [X,\ \lambda_U] & \nearrow \lambda_F & \\
T(X \,\square\, Y) & & [X,\ TY] & & \\
\wr\Vert & & \wr\Vert & & \\
[X \,\square\, Y,\ S] & \xrightarrow[\psi_{X,Y,S}]{} & [X,\ [Y,\ S]] & &
\end{array} \quad .$$

It is commutative for the naturality of ψ and for (2.2.1), recalling that $\lambda_W \cong [X,\ \Phi_W]$, $\lambda_F \cong [X,\ \Phi_F]$ and $[X,\ \lambda_U] \cong [X,\ [Y,\ \Phi_U]]$. It follows that $\lambda_W \cdot \psi_{X,Y,Z}$ is a continuous map, for any $X,\ Y,\ Z \in$ **Top** . If $\tilde{V}$ is a finite intersection of W_j , that is, $\tilde{V} = \cap W_j = \cap W(F_j,\ U_j)$ $(j = 1,\ \ldots,\ n)$, $F_j \in F(Y)$, $U_j \in$ **Op Z** , we can consider the diagram

$$(2.2.2) \qquad \begin{array}{ccccc}
[X \,\square\, Y,\ Z] & \xrightarrow{\psi_{X,Y,Z}} & [X,\ [Y,\ Z]] & \xrightarrow{\lambda_{\tilde{V}}} & TX \cong [X,\ S] \\
\Delta \Big\downarrow & & \Big\downarrow \Delta & & \Big\uparrow \cap \\
\Pi_j[X \,\square\, Y,\ Z] & \xrightarrow[\Pi_j \psi_{X,Y,Z}]{} & \Pi_j[X,\ [Y,\ Z]] & \xrightarrow[\Pi_j \lambda_{W_j}]{} & \Pi_j TX \cong \Pi_j[X,\ S]
\end{array}$$

where Δ is the canonical diagonal map, and Π denotes the topological product.

The commutativity of such a diagram follows from the naturality of Δ and the definition of $\tilde{V}$.

Since the diagonal map and the intersection map are continuous (Lemma 2.1), it follows that $\lambda_{\tilde{V}} \cdot \psi_{X,Y,Z}$ is continuous too, for any $X,\ Y,\ Z$ in **Top** .

Now, if V is an arbitrary union of $\tilde{V}_\alpha$, $\tilde{V}_\alpha = \cap W_{j^\alpha}$, $(j^\alpha : 1,\ \ldots,\ n^\alpha)$, we can prove the continuity of $\lambda_V \cdot \psi_{X,Y,Z}$ with a diagram similar to (2.2.2), where the intersection map is replaced with the union map. A similar proof applies to the inverse map $\psi^{-1}_{X,Y,Z}$, through the following commutative diagram:

$$\begin{array}{ccccc}
[X,\ [Y,\ Z]] & \xrightarrow{\psi^{-1}_{X,Y,Z}} & [X \Box Y,\ Z] & \xrightarrow{\lambda_U} & T(X \Box Y) \\
{\scriptstyle [X,\ [Y,\ \Phi_U]]}\downarrow & & \downarrow{\scriptstyle [X \Box Y,\ \Phi_U]} & \nearrow{\scriptstyle \wr} & \\
[X,\ [Y,\ S]] & \xrightarrow[\psi^{-1}_{X,Y,S}]{} & [X \Box Y,\ S] & &
\end{array}$$

$U \in \mathrm{Op}\ Z$. □

PROPOSITION 2.3. *A functorial adjoining system* $\left(F(Y)\right)_{Y \in \mathrm{Top}}$ *determines a monoidal closed structure* $(\mathrm{Top},\ - \Box -,\ [-,\ -],\ \{*\},\ r,\ 1,\ a)$ *on* Top *if and only if*

(i) $F(\{*\})$ *induces on* $\mathrm{Op}\{*\}$ *a topological topology* $T(\{*\}) \cong S$ *;*

(ii) $\psi_{X,Y,S} : [X \Box Y,\ S] \cong [X,\ [Y,\ S]]$ *is a homeomorphism for any* $X,\ Y \in \mathrm{Top}$.

Proof. Any adjoining functorial system (see Definition 1.5) determines two bifunctors $- \Box - : \mathrm{Top} \times \mathrm{Top} \to \mathrm{Top}$ and $[-,\ -] : \mathrm{Top}^{\circ} \times \mathrm{Top} \to \mathrm{Top}$ such that, for any $Y \in \mathrm{Top}$, $- \Box Y \dashv [Y,\ -]$.

Let $T(\{*\})$ be homeomorphic to a Sierpinski space, and let μ denote such homeomorphisms; we have necessarily $\mu(\{*\}) = 1$. Since $[\{*\},\ X]$ is, up to bijection, the set X with the initial topology with respect to the maps

$$\lambda_U : [\{*\},\ X] \to T(\{*\}) \cong S$$

with $U \in \mathrm{Op}\ X$, and since $\lambda_U^{-1}(\{*\})$, $\{*\} \in T(\{*\})$ is exactly the open set U , it follows that $[\{*\},\ X] \cong X$, for any $X \in \mathrm{Top}$. We denote by υ_X such a natural homeomorphism, defined by $\upsilon_X(x_0)(*) = x_0$, $x_0 \in X$. Then, from Lemma 2.2 and from [6, p. 495, Theorem 5.10], it follows that $(\mathrm{Top},\ - \Box -,\ [-,\ -],\ \{*\},\ r,\ 1,\ a)$ is a monoidal closed structure on Top , where $r,\ 1,\ a$ are enrichments in Top of the canonical natural isomorphisms of Set . □

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Suppose now we use the same symbol both for an open set of Y and for its characteristic associated map.

The space $[X, [Y, S]]$ has the initial topology with respect to the maps

$$\lambda_U : \hom(X, [Y, S]) \to TX$$

with $U \in TY$.

If $\psi : [X \square Y, S] \cong [X, [Y, S]]$, it follows that the family $\psi^{-1}_{X,Y,S}\left(\lambda_U^{-1}(V)\right)$, $U \in TY$ and $V \in TX$, is an open subbase of $[X \square Y, S]$; furthermore for properties of initial topology (see also diagram (2.2.2)), it suffices to consider $U \in F(Y)$, $V \in F(X)$.

Let us denote by $H(F, G)$, $F \in F(Y)$, $G \in F(X)$, the following subset of $\mathrm{Op}(X \square Y)$ (that is, up to bijection, just the set $\psi^{-1}_{X,Y,S}\left(\lambda_F^{-1}(G)\right)$):

$$H(F, G) = \{A, A \in \mathrm{Op}(X \square Y) \mid Q_A = \{x, x \in X \mid A_x \in F\} \in G\} .$$

PROPOSITION 2.4. *If* $F(Y)$ *and* $F(X)$ *are adjoining systems on* $\mathrm{Op}\, Y$ *and* $\mathrm{Op}\, X$ *respectively, then the family* $\{H(F, G) \mid F \in F(Y), G \in F(X)\}$ *is an adjoining system on* $\mathrm{Op}\, X \square Y$.

Proof. It is easily seen that $H(F, G)$ is a filter on $\mathrm{Op}\, X \square Y$, for any $F \in F(Y)$ and $G \in F(X)$. To prove that $H(F, G)$ is a compact filter, suppose that $\cup A_i \in H(F, G)$, $i \in I$ and $A_i \in \mathrm{Op}\, X \square Y$.

If $Q = \{x, x \in X \mid \cup(A_i)_x \in F, i \in I\}$, then, by definition of $H(F, G)$, it follows that $Q \in G$.

Since F is a compact filter, for any $\bar{x} \in Q$ there is a finite subset $J_{\bar{x}}$ of I such that $\cup(A_i)_{\bar{x}} \in F$, $i \in J_{\bar{x}}$.

If $Q_{\bar{x}} = \{x, x \in X \mid \cup(A_i)_x \in F, i \in J_{\bar{x}}\}$, then $Q = \cup Q_{\bar{x}}$, $\bar{x} \in Q$. Since G is a compact filter, it follows that there is a finite subset L of Q such that $\cup Q_{\bar{x}} \in G$, $\bar{x} \in L$. If we define $K = \cup J_{\bar{x}}$, $\bar{x} \in L$, we obtain a finite subset of I and $\cup A_i \in H(F, G)$ with $i \in K$.

Now we are going to prove that the family

$$\{H(F, G) \mid F \in \mathsf{F}(Y), G \in \mathsf{F}(X)\}$$

verifies condition (ii) of the definition of adjoining systems. If $A_1 \cup A_2 \in H(F, G)$, with $A_1, A_2 \in \mathrm{Op}\, X \square Y$, let us denote by Q the set $\{x, x \in X \mid (A_1)_x \cup (A_2)_x \in F\}$; then $Q \in G$.

Since F is an adjoining system it follows that, for any $\bar{x} \in Q$, $(A_1)_{\bar{x}} \in (F_1)_{\bar{x}}$ and $(A_2)_{\bar{x}} \in (F_2)_{\bar{x}}$ where $(F_1)_{\bar{x}}$ and $(F_2)_{\bar{x}}$ are finite intersections of members of $\mathsf{F}(Y)$ and $(F_1)_{\bar{x}} \cap (F_2)_{\bar{x}} \subseteq F$.

If we define

$$P_{\bar{x}} = \{x, x \in X \mid (A_1)_x \in (F_1)_{\bar{x}}\}$$

and

$$R_{\bar{x}} = \{x, x \in X \mid (A_2)_x \in (F_2)_{\bar{x}}\} ,$$

then it is easy to see that $Q = \bigcup (P_{\bar{x}} \cap R_{\bar{x}})$, $\bar{x} \in Q$.

Since G is a compact filter, it follows that there is a finite subset K of Q such that $\bigcup (P_{\bar{x}} \cap R_{\bar{x}}) \in G$, $\bar{x} \in K$. As the general finite case follows by induction, only two elements x_1 and x_2 will suffice. From $(P_{x_1} \cap R_{x_1}) \cup (P_{x_2} \cap R_{x_2}) \in G$, and property (ii) of the adjoining system G, it follows that $P_{x_1} \cap R_{x_1} \in G_1$ and $P_{x_2} \cap R_{x_2} \in G_2$, where G_1 and G_2 are finite intersections of filters belonging to $\mathsf{F}(X)$ and $G_1 \cap G_2 \subseteq G$.

Recall that, if F is a finite intersection of F_α, with $F_\alpha \in \mathsf{F}(Y)$ and G is a finite intersection of G_β with $G_\beta \in \mathsf{F}(X)$, then $H(F, G) = \bigcap H(F_\alpha, G_\beta)$.

Consider now the filters

$$H\left((F_j)_{x_i}, G_i\right) , \quad i = 1, 2 , \quad j = 1, 2 ;$$

it follows that $A_1 \in H\left((F_1)_{x_1}, G_1\right) \cap H\left((F_1)_{x_2}, G_2\right)$ and

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$A_2 \in H\big((F_2)_{x_1}, G_1\big) \cap H\big((F_2)_{x_2}, G_2\big)$. To verify that

$$\cap H\big((F_j)_{x_i}, G_i\big) \subseteq H(F, G) \ , \quad i = 1, 2 \ , \quad j = 1, 2 \ ,$$

let A be an open set of $X \,\Box\, Y$ and $A \in \cap H\big((F_j)_{x_i}, G_i\big)$. If L denotes the set $\{x, x \in X \mid A_x \in F\}$, we must prove that $L \in G$.

Let us define the sets

$$L_{j,i} = \{x, x \in X \mid A_x \in (F_j)_{x_i}\} \ , \quad i = 1, 2 \ , \quad j = 1, 2 \ ;$$

it follows that $L_{j,i} \in G_i$ and $L_{1,i} \cap L_{2,i} \subseteq G_i$, $i = 1, 2$. Furthermore, if we define $L^* = \cup\big(L_{1,i} \cap L_{2,i}\big)$, $i = 1, 2$, then $L^* \in G_1 \cap G_2$, hence $L^* \in G$.

It suffices now to prove that $L^* \subseteq L$. If $x \in L^*$, then $x \in L_{1,1} \cap L_{2,1}$ or $x \in L_{1,2} \cap L_{2,2}$. Suppose $x \in L_{1,1} \cap L_{2,1}$; then, since $A_x \in (F_1)_{x_1} \cap (F_2)_{x_1}$, it follows $A_x \in F$ and $x \in L$. □

The topological space on $\mathrm{Op}\, X \,\Box\, Y$, having the family $\{H(F, G), F \in \mathsf{F}(Y), G \in \mathsf{F}(X)\}$, as an open subbase, will be denoted by $\langle X, Y \rangle$. Then condition *(ii)* of Proposition 2.3 can be replaced by

(ii)' $T(X \,\Box\, Y) \cong \langle X, Y \rangle$

for any $X, Y \in \mathbf{Top}$.

It is interesting to observe that it is possible to give a characterization of *all monoidal closed structures* on **Top** , generalizing Proposition 2.3 to the case of arbitrary topological topology TY (not given by filters), and recalling the theorem of Isbell on the characterization of adjoint functors in **Top** [8].

3.

For any family $\mathsf{F}(Y)$ of filters on $\mathrm{Op}\, Y$, let us denote by $\mathsf{F}'(Y)$ the system of all filters which can be written as finite intersection of filters belonging to $\mathsf{F}(Y)$. Of course $\mathsf{F}(Y)$ is an adjoining system if and only if $\mathsf{F}'(Y)$ is an adjoining system.

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All the examples of monoidal closed structures on Top, given by Booth and Tillotson in [2], and by Greve in [7] are *determined by filters*. In such examples, if A is the opportune class of spaces used to define the structure, then the associated functorial adjoining system is given by families of generated filters of the form

$$\{\langle f(A)\rangle \mid A \in \mathsf{A},\ f : A \to Y\}_{Y \in \mathbf{Top}} .$$

In particular, the canonical symmetric monoidal closed structure (topology of separate continuity on $X \times Y$, and pointwise convergence on $\hom(Y, Z)$) is determined by the functorial adjoining system $\{\langle y\rangle \mid y \in Y\}_{Y \in \mathbf{Top}}$ of all principal ultrafilters of open sets. Moreover $\{\langle y\rangle \mid y \in Y\}' = E(Y)$, where $E(Y)$ is the minimal adjoining system of [11].

Consider now the trivial functorial adjoining system $J\big(J(Y) = \emptyset$, for any $Y \in \mathbf{Top}\big)$ and the maximal functorial adjoining system M of [11]. Condition *(i)* of Proposition 2.3 is satisfied by M, but not by J. Condition *(ii)* is satisfied by J; as for M, the only result we can get is, by Proposition 2.4, $T(X \square Y) \geq \langle X, Y\rangle$, for any $X, Y \in \mathbf{Top}$.

If we assume all spaces to be Hausdorff, then M determines a monoidal structure $\big(\mathbf{Haus}, - \times_S -, [-, -]_{co}, \{*\}\big)$, where co denotes the compact-open topology ([4]). To prove this result it suffices to recall that each compact filter F on Op Y, $Y \in \mathbf{Haus}$, is compactly generated [8, p. 331]; then, by [11, p. 278, Theorem 7], it follows that the topological space on $\hom(Y, Z)$ determined by M coincides with $[Y, Z]_{co}$, for any $Y, Z \in \mathbf{Top}$.

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