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STRONG CONVERGENCE OF APPROXIMATING FIXED POINT SEQUENCES FOR NONEXPANSIVE MAPPINGS

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Consider a nonexpansive self-mapping T of a bounded closed convex subset of a Banach space. Banach's contraction principle guarantees the existence of approximating fixed point sequences for T. However such sequences may not be strongly convergent, in general, even in a Hilbert space. It is shown in this paper that in a real smooth and uniformly convex Banach space, appropriately constructed approximating fixed point sequences can be strongly convergent.

1. INTRODUCTION

Let X be a real Banach space and C be a closed convex subset of X. Let $T : C \to C$ be a self-mapping of C. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for all $x, y \in C$. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T (that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$). Throughout this article, we assume that $\mathrm{Fix}(T)$ is nonempty.

Recall also that a sequence $\{x_n\}$ in C is said to be an approximating fixed point sequence for T if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

There are several ways to construct an approximating fixed point sequence for a nonexpansive mapping T. We mention two below.

Firstly we can use Banach's contraction principle to obtain a sequence $\{x_n\}$ in C such that

$$x_n = t_n x_0 + (1 - t_n)Tx_n, \quad n \geqslant 1$$

where the initial guess x_0 is taken arbitrarily in C and $\{t_n\}$ is a sequence in the interval $(0, 1)$ such that $t_n \to 0$ as $n \to \infty$. Due to the assumption that $\mathrm{Fix}(T) \neq \emptyset$, this sequence $\{x_n\}$ is bounded (indeed $\|x_n - p\| \leqslant \|x_0 - p\|$ for all $p \in \mathrm{Fix}(T)$). Hence

$$\|x_n - Tx_n\| = t_n\|x_0 - Tx_n\| \to 0$$

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and $\{x_n\}$ is an aproximating fixed point sequence for T.

Secondly, we use Mann's iteration process [8] to generate a sequence $\{x_n\}$ in C by the recursive formula

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geqslant 0 \tag{1.1}$$

where the initial guess $x_0 \in C$ is arbitrary, and the sequence $\{\alpha_n\}$ lies in the interval $(0,1)$. This sequence $\{x_n\}$ is bounded since, for any $p \in \mathrm{Fix}(T)$, we have

$$\|x_{n+1} - p\| \leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\| \leqslant \|x_n - p\|.$$

That is, $\{\|x_n - p\|\}$ is a nonincreasing sequence. Moreover, it is not hard to find that the sequence $\{\|x_n - Tx_n\|\}$ is also nonincreasing; hence $\lim_n \|x_n - Tx_n\|$ exists.

However, it is not known whether this sequence $\{x_n\}$ is always an approximating fixed point sequence of T. Only partial answers have been obtained. Indeed, if the space X is uniformly convex and if the control sequence $\{\alpha_n\}$ satisfies the condition $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then Reich [12] showed that the sequence $\{x_n\}$ generated by Mann's iteration process (1.1) is an approximating fixed point sequence of T. For the sake of completeness, we include a brief proof to this fact. Let δ_X be the modulus of convexity of X. Pick a $p \in \mathrm{Fix}(T)$. Assuming $\|x_n - p\| > 0$ and noticing $\|Tx_n - p\| \leqslant \|x_n - p\|$, we deduce that

$$\|x_{n+1} - p\| \leqslant \|x_n - p\|\Big[1 - 2\alpha_n(1-\alpha_n)\delta_X\Big(\frac{\|x_n - Tx_n\|}{\|x_n - p\|}\Big)\Big].$$

Hence

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)\|x_n - p\|\delta_X\Big(\frac{\|x_n - Tx_n\|}{\|x_n - p\|}\Big) \leqslant \|x_0 - p\| < \infty. \tag{1.2}$$

Put $r = \lim_n \|x_n - p\|$. If $r = 0$, we are done. So assume $r > 0$. If $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, we obtain from (1.2) that $\lim_n \delta_X(\|x_n - Tx_n\|/r) = 0$. This implies that $\lim_n \|x_n - Tx_n\| = 0$ and $\{x_n\}$ is an approximating fixed point sequence of T.

An approximating fixed point sequence is not necessarily always weakly convergent though it is true that in a Hilbert space every weak limit point of an approximating fixed point sequence is always a fixed point of T. This fact is called the demiclosedness principle for nonexpansive mappings which indeed holds in uniformly convex Banach spaces as stated in the next lemma.

Lemma 1.1. (See [4].) *Let X be a uniformly convex Banach space, C a closed convex subset of C, and $T : C \to C$ a nonexpansive mapping with a fixed point. Then $I - T$ is demiclosed in the sense that if $\{x_n\}$ is a sequence in C and if $x_n \to x$ weakly and $(I-T)x_n \to y$ strongly for some x and y, then $(I-T)x = y$.*

In a summary, in the setting of real uniformly convex Banach spaces X, what is clear is that every weak limit point of an approximating fixed point sequence for T is a fixed point of T. However it remains unclear if the entire approximating fixed point sequence is weakly convergent. Reich [**12**] proves that if, in addition, X also has a Frechet differentiable norm and if $\{x_n\}$ is an approximating fixed point sequence generated by Mann's iteration process (1.1), then $\{x_n\}$ is weakly convergent.

In general, an approximating fixed point sequence may fail to be strongly convergent even in the Hilbert space setting [**3**].

It is the purpose of this note to prove that an appropriately constructed approximating fixed point sequence can be strongly convergent in a smooth and uniformly convex Banach space. For more recent investigations on strong convergence for nonexpansive and maximal monotone mappings, see [**5, 6, 7, 9, 10, 11, 13, 14, 15, 17**] and the references therein.

2. Projections in uniformly convex Banach spaces

Let X be a real uniformly convex Banach space X. Thus, for every $\varepsilon > 0$, $\delta_X(\varepsilon) > 0$, where δ_X is the modulus of convexity of X defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \|x-y\| \geqslant \varepsilon\right\}.$$

Let C be a nonempty closed convex subset of X. Like the Hilbert space case, we can define the nearest point projection P_C from X onto C by assigning to each $x \in X$ the only point $P_C x$ in C with the property

$$\|x - P_C x\| = \inf\{\|x-y\| : y \in C\}.$$

This projection P_C, though continuous (indeed uniformly continuous on bounded sets), is however inconvenient to use because it is not nonexpansive anymore (hence $I - P_C$ lacks monotonicity), as contrast to the nonexpansivity of nearest point projections in a Hilbert space. Instead, another kind of projections has been introduced to replace the nearest point projections, which is however still denoted by the same notation P_C. That is, in the rest of the paper, by P_C we mean the projection from X onto C introduced as follows.

Let $J : X \to X^*$ be the duality map of X defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

Assume X is smooth so that J is single-valued on X and hence we can define a function φ on $X \times X$ by (see [**1, 5**])

$$\varphi(x,y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2, \quad x, y \in X. \tag{2.1}$$

It is easily seen that

$$\left(\|x\|-\|y\|\right)^2 \leqslant \varphi(x,y) \leqslant \left(\|x\|+\|y\|\right)^2, \quad x,y \in X.$$

Since for each fixed y, $\varphi(\cdot,y)$ is a continuous strictly convex function on X, there is a unique point $z \in C$ which solves the minimisation

$$\varphi(z,y) = \min\{\varphi(x,y) : x \in C\}. \tag{2.2}$$

This unique point z in C is called the (generalised) projection of y onto C. That is, we define the projection operator $P_C : X \to C$ by setting

$$P_C y = z, \tag{2.3}$$

where z is the only point in C satisfying (2.2). (Note that if X is a Hilbert space, $\varphi(x,y) = \|x-y\|^2$. Hence the projection P_C defined in (2.3) coincides with the nearest point projection onto C in the Hilbert space setting.)

The next proposition gathers some basic properties of P_C which will be used in the proof of the main result in the next section.

PROPOSITION 2.1. *Assume that X is a smooth and uniformly convex Banach space and C is a nonempty closed convex subset of X.*

(i) *Given sequences $\{x_n\}$ and $\{y_n\}$ in X. If one of them is bounded, then $\varphi(x_n,y_n) \to 0$ if and only if $\|x_n - y_n\| \to 0$.*

(ii) *Given $y \in X$ and $z \in C$. Then $z = P_C y$ if and only if there holds the inequality:*

$$\langle v - z, J(z) - J(y)\rangle \geqslant 0 \quad \forall v \in C. \tag{2.4}$$

(iii) *The following inequality holds:*

$$\varphi(x, P_C y) + \varphi(P_C y, y) \leqslant \varphi(x,y) \quad \forall x \in C,\ y \in X. \tag{2.5}$$

PROOF: (i) The necessity part is proved in **[5]** under the stronger condition that the space X be uniformly smooth. The uniform smoothness can be indeed weakened to smoothness. To see this, we notice that if $\varphi(x_n,y_n) \to 0$ and if one of the sequences $\{x_n\}$ and $\{y_n\}$ is bounded, then both $\{x_n\}$ and $\{y_n\}$ are bounded. Let $r > 0$ be such that the closed ball $B_r = \{u \in X : \|u\| \leqslant r\}$ contains all the points of $\{x_n\}$, $\{y_n\}$ and $\{x_n - y_n\}$. By Xu **[16]**, we have a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ and satisfying the property:

$$\|u+v\|^2 \geqslant \|u\|^2 + 2\langle v, J(u)\rangle + g(\|v\|), \quad \forall u,v \in B_r.$$

In particular,

$$\begin{aligned}\|x_n\|^2 &= \|y_n + (x_n - y_n)\|^2 \\ &\geqslant \|y_n\|^2 + 2\langle x_n - y_n, J(y_n)\rangle + g(\|x_n - y_n\|) \\ &= -\|y_n\|^2 + 2\langle x_n, J(y_n)\rangle + g(\|x_n - y_n\|).\end{aligned}$$

It now follows from the definition of φ that

$$g(\|x_n - y_n\|) \leqslant \varphi(x_n, y_n) \to 0.$$

Therefore $\|x_n - y_n\| \to 0$.

To see the sufficiency part (true indeed in any smooth Banach space), we assume $\|x_n - y_n\| \to 0$ and thus both sequences $\{x_n\}$ and $\{y_n\}$ are bounded. That $\varphi(x_n, y_n) \to 0$ now follows from the following computations:

$$\begin{aligned}\varphi(x_n, y_n) &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, J(y_n)\rangle \\ &\leqslant \|x_n - y_n\|(\|x_n\| + 3\|y_n\|).\end{aligned}$$

(ii) Since for each fixed $y \in X$, $\varphi(\cdot, y)$ is convex, $z \in C$ is a minimiser of $\varphi(\cdot, y)$ over C if and only if there holds the optimality condition:

$$\langle \nabla\varphi(z, y), v - z\rangle \geqslant 0 \quad \forall v \in C \tag{2.6}$$

where $\nabla\varphi(z, y)$ is the gradient of $\varphi(\cdot, y)$ at z. Since it is easily computed that

$$\langle \nabla\varphi(z, y), v - z\rangle = 2\langle v - z, J(z) - J(y)\rangle$$

we obtain (2.4).

(iii) Using the definition of φ, we find that (2.5) is equivalent to the inequality:

$$\langle P_C y - x, J(P_C y) - J(y)\rangle \leqslant 0.$$

This is however the inequality (2.4) with v and z replaced by x and $P_C y$, respectively. ∎

We shall use the notation:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Lemma 2.2. *Let X be a real smooth and uniformly convex Banach space and K be a nonempty closed convex subset of X. Let $\{x_n\}$ be a bounded sequence in X and $u \in X$. Let $q = P_K u$. Assume that $\{x_n\}$ satisfies the conditions*

(i) $\omega_w(x_n) \subset K$ *and*

(ii) $\varphi(x_n, u) \leqslant \varphi(q, u)$ *for all* n.

Then $x_n \to q$.

PROOF: Since X is reflexive and $\{x_n\}$ is bounded, $\omega_w(x_n)$ is nonempty. Noticing the weak lower semi-continuity of $\varphi(\cdot, u)$, we derive from condition (ii) that

$$\varphi(v, u) \leqslant \varphi(q, u) \quad \forall v \in \omega_w(x_n).$$

However, since $\omega_w(x_n) \subset K$ and $q = P_K u$, we must have $v = q$ for all $v \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{q\}$ and $x_n \rightharpoonup q$.

To see $x_n \to q$, we observe that the inequality $\varphi(x_n, u) \leqslant \varphi(q, u)$ in condition (ii) is actually equivalent to the following one

$$\|x_n\|^2 \leqslant \|q\|^2 + 2\langle x_n - q, J(u)\rangle.$$

Since $x_n \rightharpoonup q$, it follows that

$$\limsup_n \|x_n\| \leqslant \|q\|.$$

This and the uniform convexity of X imply that $x_n \to q$. ∎

3. Strong convergence of approximating fixed point sequences

Let C be a nonempty closed convex subset of a smooth and uniformly Banach space X and let $T : C \to C$ be a nonexpansive mapping with a fixed point. Starting an arbitrary initial guess x_0, we can construct an approximating fixed point sequence of T as follows. Take a sequence $\{t_n\}$ in (0,1) so that $t_n \to 0$ as $n \to \infty$. Once x_n has been constructed, we then construct two closed convex subsets C_n and Q_n such that

$$C_n = \overline{\text{co}}\{z \in C : \|z - Tz\| \leqslant t_n\|x_n - Tx_n\|\}$$

and

$$Q_n = \Big\{v \in C : \langle x_n - v, J(x_0) - J(x_n)\rangle \geqslant 0\Big\}.$$

Then we define the $(n+1)$th iterate x_{n+1} to be the projection of x_0 onto the intersection of C_n and Q_n:

$$x_{n+1} = P_{C_n \cap Q_n} x_0. \tag{3.1}$$

Before discussing the convergence of the sequence $\{x_n\}$, we first use induction to verify that $\text{Fix}(T) \subset C_n \cap Q_n$ and x_{n+1} is well-defined. As a matter of fact, it is trivial that $\text{Fix}(T) \subset C_n$ for all n. It is also trivial that $\text{Fix}(T) \subset Q_0 = C$ and thus $x_1 = P_{C_0 \cap Q_0} x_0$ is well-defined. Assume now $\text{Fix}(T) \subset Q_n$ and x_{n+1} is well-defined. We need to prove that $\text{Fix}(T) \subset Q_{n+1}$ and x_{n+2} is well-defined.

Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Proposition 2.1 (ii) we have

$$\langle x_{n+1} - z, J(x_0) - J(x_{n+1})\rangle \geqslant 0 \quad \forall z \in C_n \cap Q_n.$$

As $\mathrm{Fix}(T) \subset C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in \mathrm{Fix}(T)$. This together with the definition of Q_{n+1} implies that $\mathrm{Fix}(T) \subset Q_{n+1}$. Now as the projection of x_0 onto the nonempty closed convex subset $C_{n+1} \cap Q_{n+1}$, x_{n+2} is well-defined.

We now state and prove the main result of this paper.

Theorem 3.1. *Let X be a real smooth and uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the process* (3.1)*. Then $\{x_n\}$ is an approximating fixed point sequence for T and strongly convergent to a fixed point of T.*

Proof: First we observe that $\{x_n\}$ is bounded. As a matter of fact, by the definition of Q_n, we have $x_n = P_{Q_n} x_0$. Hence by Proposition 2.1 (iii)

$$\varphi(y, x_n) + \varphi(x_n, x_0) \leqslant \varphi(y, x_0) \quad \forall y \in Q_n. \tag{3.2}$$

Since $\mathrm{Fix}(T) \subset Q_n$, we get

$$\varphi(x_n, x_0) \leqslant \varphi(p, x_0) \quad \forall p \in \mathrm{Fix}(T). \tag{3.3}$$

This implies the boundedness of $\{x_n\}$. Because x_{n+1} belongs to Q_n, we can substitute it for y in (3.2) to get

$$\varphi(x_{n+1}, x_n) \leqslant \varphi(x_{n+1}, x_0) - \varphi(x_n, x_0). \tag{3.4}$$

This implies that the real sequence $\{\varphi(x_n, x_0)\}$ is increasing (and also bounded) and thus $\lim_n \varphi(x_n, x_0)$ exists. Back to (3.4), we conclude that $\varphi(x_{n+1}, x_n) \to 0$ which implies $\|x_{n+1} - x_n\| \to 0$ by virtue of Proposition 2.1 (i).

We now claim that $\{x_n\}$ is an approximating fixed point sequence of T. Let $\widetilde{C}$ be a bounded closed convex subset of C which contains all the points x_n and Tx_n for all n and let $\eta = \mathrm{diam}\,(\widetilde{C})$. Since $x_{n+1} \in C_n$ and by definition of C_n, we have

$$\left\| x_{n+1} - \sum_{i=1}^{l} \lambda_i z_i \right\| < t_n$$

where $\lambda_i > 0$ satisfying $\sum_{i=1}^{l} \lambda_i = 1$ and each $z_i \in C$ satisfies

$$\|z_i - Tz_i\| \leqslant t_n \|x_n - Tx_n\| \leqslant \eta t_n.$$

By Bruck [2], there exists a continuous strictly increasing function γ (depending only on η) with $\gamma(0) = 0$ and such that

$$\gamma\left(\left\| T\left(\sum_{i=1}^{m} \mu_i v_i\right) - \sum_{i=1}^{m} \mu_i T v_i \right\|\right) \leqslant \max\left(\|v_i - v_j\| - \|Tv_i - Tv_j\| : 1 \leqslant i, j \leqslant m\right)$$

for all integers $m > 1$, all points $\{v_i\}$ in $\widetilde{C}$, and all nonnegative numbers $\{\mu_i\}$ such that $\sum_{i=1}^{m} \mu_i = 1$. It follows that

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leqslant \Big\|x_{n+1} - \sum_{i=1}^{l} \lambda_i z_i\Big\| + \Big\|\sum_{i=1}^{l} \lambda_i (z_i - Tz_i)\Big\| \\
&\quad + \Big\|\sum_{i=1}^{l} \lambda_i T z_i - T\Big(\sum_{i=1}^{l} \lambda_i z_i\Big)\Big\| + \Big\|T\Big(\sum_{i=1}^{l} \lambda_i z_i\Big) - Tx_{n+1}\Big\| \\
&\leqslant (2+\eta)t_n + \gamma^{-1}\Big(\max\big(\|z_i - z_j\| - \|Tz_i - Tz_j\| : 1 \leqslant i,j \leqslant l\big)\Big) \\
&\leqslant (2+\eta)t_n + \gamma^{-1}\Big(\max\big(\|z_i - Tz_i\| + \|z_j - Tz_j\| : 1 \leqslant i,j \leqslant l\big)\Big) \\
&\leqslant (2+\eta)t_n + \gamma^{-1}(2\eta t_n) \to 0.
\end{aligned}$$

Therefore, $\{x_n\}$ is an approximating fixed point sequence.

Finally let us prove that $\{x_n\}$ is strongly convergent to a fixed point of T. By the demiclosedness principle (Lemma 1.1), we have $\omega_w(x_n) \subset \mathrm{Fix}(T)$. Let $q = P_{\mathrm{Fix}(T)} x_0$. By (3.3) we see that $\varphi(x_n, x_0) \leqslant \varphi(q, x_0)$ for all n. Therefore, applying Lemma 2.2 to the nonempty closed convex subset $K := \mathrm{Fix}(T)$, we conclude that $x_n \to q$. ∎

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