This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to G. D. Findlay, Department of Mathematics, McGill University, Montreal, P. Q.

SOME REMARKS ON A PROBLEM OF E. L. WHITNEY

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We shall obtain a little more general formula than the one given in the problem P44.

Put

$$Q_m(x) = \frac{1}{x^m} + \frac{(-1)^m}{(x+1)^m} \tag{1}$$

First of all we shall prove the identity

$$\frac{1}{x^m (x+1)^m} = \Sigma_{i=1}^m C_{mi} Q_i(x), \tag{2}$$

where C_{mi} are constants whose values will also be determined.

Let $\frac{1}{x^m (x+1)^m} = \Sigma_{i=1}^m \frac{A_i}{x^i} + \Sigma_{i=1}^m \frac{A_i'}{(x+1)^i}$ be the partial fraction decomposition. By the substitution $x = -y - 1$, we have $\frac{1}{y^m (y+1)^m} = \Sigma_{i=1}^m \frac{(-1)^i A_i}{(y+1)^i} + \Sigma_{i=1}^m \frac{(-1)^i A_i'}{y^i}$, and

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therefore

$$A_i' = (-1)^i A_i ,$$

because of the uniqueness of the decomposition into partial fractions. Thus we have the above-mentioned formula (2), where $C_{mi} = A_i$.

Since

$$(1 + x)^{-m} = x^m \left(\Sigma_{i=1}^m A_i x^{-i} + \Sigma_{i=1}^m A_i' (1 + x)^{-i} \right)$$

$$= \Sigma_{i=1}^m A_i x^{m-i} + x^m \Sigma_{i=1}^m (-1)^i A_i (1 + x)^{-i} ,$$

A_i is the coefficient of x^{m-i} in the Taylor series expansion of $(1 + x)^{-m}$. (The series is convergent for $|x| < 1$.) Thus

$$C_{mi} = A_i = \binom{-m}{m-i} = (-1)^{m-i} \binom{2m - i - 1}{m - i} . \tag{3}$$

Now back to $Q_m(x)$. From (1) it is easy to see that

$$\Sigma_{n=1}^{\infty} Q_{2q-1}(n) = 1 \quad \text{for odd } m = 2q - 1. \tag{4}$$

Setting $S_{2q} = \Sigma_{n=1}^{\infty} \frac{1}{n^{2q}}$ for even $m = 2q$, we have

$$Q_{2q}(n) = 2S_{2q} - 1. \tag{5}$$

It is known that

$$S_{2q} = \frac{(2\pi)^{2q}}{2\,(2q)!} (-1)^{q+1} B_{2q} , \tag{6}$$

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where B_{2q} is the Bernoulli number for $q = 1, 2, 3, \ldots$.

Put $S(p) = \Sigma_{n=1}^{\infty} \dfrac{1}{n^p (n+1)^p}$, whose convergence is obvious for $p = 1, 2, 3, \ldots$. Then

$$S(p) = \Sigma_{n=1}^{\infty} \Sigma_{i=1}^{p} C_{pi} Q_i(n) \qquad \text{by (2)}$$

$$= \Sigma_{q=1}^{[\frac{p+1}{2}]} C_{p,2q-1} + \Sigma_{q=1}^{[p/2]} C_{p,2q} (2 S_{2q} - 1) \quad \text{by (4) and (5)}$$

$$= \Sigma_{q=1}^{[\frac{p+1}{2}]} (-1)^{2q} C_{p,2q-1} + \Sigma_{q=1}^{[p/2]} (-1)^{2q+1} C_{p,2q}$$

$$+ \Sigma_{q=1}^{[p/2]} 2 C_{p,2q} S_{2q}$$

$$= \sum_{q=1}^{p} (-1)^{q+1} C_{p,q} + \sum_{q=1}^{[p/2]} 2 C_{p,2q} S_{2q}$$

$$= T + U,$$

where

$$T = \Sigma_{q=1}^{p} (-1)^{q+1} C_{pq} = \Sigma_{q=1}^{p} (-1)^{q-1} \binom{-p}{p-q} \qquad \text{by (3)}$$

$$= \Sigma_{r=0}^{p-1} (-1)^{p-1} \binom{-p}{r} (-1)^r$$

$$= \binom{-p-1}{p-1} \qquad (*)$$

$$= (-1)^{p-1} \binom{2p-1}{p-1}$$

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$$= \frac{(-1)^{p-1}}{p!\ (p-1)!}\ (2p-1)! \tag{7}$$

and

$$U = \Sigma_{q=1}^{[p/2]} 2\,(-1)^{p-2q} \begin{pmatrix} 2p-2q-1 \\ p-2q \end{pmatrix} \frac{(-1)^{q-1}\,(2\pi)^{2q}}{2\,(2q)!} B_{2q}$$

... by (6)

$$= \Sigma_{q=1}^{[p/2]} (-1)^{p+q-1} \frac{(2p-2q-1)!\ p!\ (2\pi)^{2q}}{(p-1)!\ p!\ (p-2q)!\ (2q)!} B_{2q}$$

$$= \frac{(-1)^{p-1}}{(p-1)!\ p!} \Sigma_{q=1}^{[p/2]} (-1)^{q} \begin{pmatrix} p \\ 2q \end{pmatrix} (2p-2q-1)!\ (2\pi)^{2q} B_{2q}$$

... (8)

Now we combine (7) and (8) and define

$$B_o = 1\ .$$

Then

$$S(p) = \frac{(-1)^{p-1}}{(p-1)!\ p!} \Sigma_{q=0}^{[p/2]} (-1)^{q} \begin{pmatrix} p \\ 2q \end{pmatrix} (2p-2q-1)!\,(2\pi)^{2q} B_{2q}.$$

Also we may write the formula as follows:

$$S(p) = (-1)^{p}\, 2\ \Sigma_{q=0}^{[p/2]} \begin{pmatrix} 2p-2q-1 \\ p-2q \end{pmatrix} S_{2q}\ ,$$

with the convention that

$$S_o = -\frac{1}{2}\ .$$

As special cases, using that $S_2 = \pi^2/6$, $S_4 = \pi^4/90$, $S_6 = \pi^6/945$, ...,

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$S(1) = 1,$

$S(2) = -3 + \pi^2/3,$

$S(3) = 10 - \pi^2,$

$S(4) = -35 + 10\pi^2/3 + \pi^4/45,$

$S(5) = 126 - (35/3)\pi^2 - \pi^4/9,$

$S(6) = -462 + 42\pi^2 + (7/15)\pi^4 + (2/945)\pi^6,$

$S(7) = 1716 - 154\pi^2 - (28/15)\pi^4 - (2/135)\pi^6,$

etc.

The proof of (*).

Since

$$(1 - x)^{-p} = \Sigma_{r=0}^{\infty} \binom{-p}{r} (-1)^r x^r,$$

the coefficient of x^{p-1} in the expansion of $(1-x)^{-p-1}$ which is obtained by $(1 - x)^{-p}(1 + x + x^2 + \ldots)$ is $\Sigma_{r=0}^{p-1} (-1)^r \binom{-p}{r}$.

Therefore

$$T = \binom{-p-1}{p-1}$$

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