ON A CONJECTURE OF CONWAY

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Problem 7 of Section VI of H. T. Croft's "Research Problems" (August, 1967 edition) is by J. H. Conway:

A is a finite set of integers $\{a_i\}$. $A + A$ denotes $\{a_i + a_j\}$, $A - A$ denotes $\{a_i - a_j\}$. Prove that $A - A$ always has more members than $A + A$, unless A is symmetrical about 0.

First note that 'about 0' is redundant and erroneous. If A is symmetric (about any point, which is in general a half-integer) it is easy to see that $|A - A| = |A + A|$ (modulus signs denote cardinals) from the following geometric representation. In Figure 1, the members of $\{a_i\}$ are represented as points of the real line in the usual way, both on the upper and lower lines of the figure. $|A + A|$ is then the number of (distinct) intersections of the joins of all the points of the upper line to each of the points of the lower line with the centre line. Similarly Figure 2 represents $|A - A|$, the points of the lower line being $\{-a_i\}$. It is clear that $A - A$ includes 0 and is symmetrical about it. Moreover, note

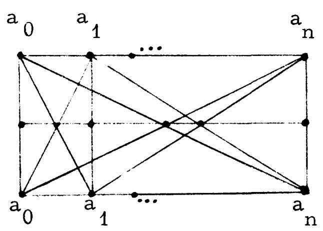

Figure 1

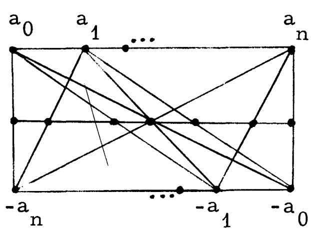

Figure 2

that if A is symmetrical about any point, Figures 1 and 2 are identical, apart from a possible affine transformation, and $|A + A| = |A - A|$.

However, the condition of symmetry of A is not necessary to achieve equality between $|A + A|$ and $|A - A|$, as may be seen from the example

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$A = \{0,1,3,4,5,8\}$, $A + A = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,16\}$, $A - A = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 7, \pm 8\}$, so that

$$|A + A| = 15 = |A - A|.$$

Moreover, the conjecture is false, as may be seen from the example

$A = \{1,2,3,5,8,9,13,15,16\}$, where

$A + A = \{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22$

$23,24,25,26,28,29,30,31,32\}$,

$A - A = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 10, \pm 11, \pm 12, \pm 13, \pm 14, \pm 15\}$,

so that $|A - A| = 29 < 30 = |A + A|$.

Conway also asks for extensions to measurable sets, but the conjecture will also be false for these. Consider the set $B = \{[a_i, a_i + \delta]\}$ where $|\delta| < \frac{1}{2}$ and the a_i are as in the counter-example above. Then $|B + B| = 60\delta$, $|B - B| = 58\delta$, where modulus signs now represent measure.

A translation of the problem into yet another form shows again that symmetry implies equality, and also suggests further problems. One notes that $|\{a_i - a_j\}| = |\{a_i + (a_n - a_j)\}|$. (In fact, one may assume that the a_i are $0, 1$, and some of the points of $[0, 1]$.) For convenience, take $a_0 \leq \ldots a_i \leq \ldots \leq a_n$ and set $a_n - a_j = a'_j$; we may restate the problem as: show that $|A + A| \leq |A + A'|$ where $A' = \{a'_j\}$. It is clear that, if A is symmetric, then $A = A'$ and equality is attained. However the following questions arise.

(1) Given a set of n numbers A, determine that set of n numbers B for which $|A + B|$ is a minimum.

(2) Given a set of n numbers A, and an integer k, determine that set of k numbers C for which $|A + C|$ is a minimum.

Problem (2) can be easily solved for $k = 1, 2$ and is only a little more complicated for $k = 3$. The example $A = \{0,1,2,4,6,7\}$ shows that B in problem (1) is neither A nor A', for $|A + A| = 15$, $|A + A'| = 15$, $|A + B| = 14$ where $B = \{0,1,2,3,4,7\}$.

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