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BROWN-HALMOS TYPE THEOREMS OF WEIGHTED TOEPLITZ OPERATORS

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ABSTRACT. The spectra of the Toeplitz operators on the weighted Hardy space $H^2(Wd\theta/2\pi)$ and the Hardy space $H^p(d\theta/2\pi)$, and the singular integral operators on the Lebesgue space $L^2(d\theta/2\pi)$ are studied. For example, the theorems of Brown-Halmos type and Hartman-Wintner type are studied.

1. **Introduction.** Let m be the normalized Lebesgue measure on the unit circle T and let W be a non-negative integrable function on T which does not vanish identically. Suppose $1 \le p \le \infty$. Let $L^p(W) = L^p(Wdm)$ and $L^p(W) = L^p$ when $W \equiv 1$. Let $H^p(W)$ denote the closure in $L^p(W)$ of the set $\mathcal{P}$ of all analytic polynomials when $p \neq \infty$. We will write $H^p(W) = H^p$ when $W \equiv 1$, and then this is a usual Hardy space. H^∞ denotes the weak $*$ closure of $\mathcal{P}$ in L^∞. P denotes the projection from the set $\mathcal{C}$ of all trigonometric polynomials to $\mathcal{P}$. For $1 < p < \infty$, P can be extended to a bounded map of $L^p(W)$ onto $H^p(W)$ if and only if W satisfies the condition

$$(A_p) \qquad \sup_I \left(\frac{1}{|I|}\int_I W\,dm\right)\left(\frac{1}{|I|}\int_I W^{-\frac{1}{p-1}}\,dm\right)^{p-1} < \infty$$

where the supremum is over all intervals I of T. This is the well known theorem of Hunt, Muckenhoupt and Wheeden [7], which is a generalization of the theorem of Helson and Szegő [6].

In this paper, we assume that the weight W satisfies the condition (A_p). For ϕ in L^∞, the Toeplitz operator T^W_ϕ is defined as a bounded map on $H^p(W)$ by

$$T^W_\phi f = P(\phi f).$$

For α and β in L^∞, the singular integral operator $S^W_{\alpha\beta}$ is defined as a bounded map on $L^p(W)$ by

$$S^W_{\alpha\beta} f = \alpha P f + \beta(I - P) f$$

where I is an identity operator. If $W \equiv 1$, we will write $T^W_\phi = T_\phi$ and $S^W_{\alpha\beta} = S_{\alpha\beta}$. Almost all results in this paper will be essentially shown using the following theorems. They are called the *theorems of Widom, Devinatz and Rochberg* (*cf.* [1], [10] and [9]).

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THEOREM A. *Suppose* $1 < p < \infty$ *and* $W = |h|^p$ *satisfies the condition* (A_p), *where* h *is an outer function in* H^p. *Then the following conditions on* ϕ *and* W *are equivalent.*

(1) T_ϕ^W *is an invertible operator on* $H^p(W)$.

(2) $\phi = k(\overline{h_0}/h_0)(h/\bar{h})$, *where* k *is an invertible function in* H^∞ *and* h_0 *is an outer function in* H^p *with* $|h_0|^p$ *satisfying the condition* (A_p).

(3) $\phi = \gamma \exp(U - i\tilde{V})$, *where* γ *is constant with* $|\gamma| = 1, U$ *is a bounded real function,* V *is a real function in* L^1 *and* $W \exp(\frac{p}{2}V)$ *satisfies* (A_p).

THEOREM B. *Suppose* $1 < p < \infty$ *and* $W = |h|^p$ *satisfies the condition* (A_p), *where* h *is an outer function in* H^p. $S_{\alpha\beta}^W$ *is invertible on* $L^p(W)$ *if and only if both* α *and* β *are invertible in* L^∞ *and* $\alpha/\beta = \gamma \exp(U - i\tilde{V})$, *where* γ *is constant with* $|\gamma| = 1, U$ *is a bounded real function,* V *is a real function in* L^1 *and* $W \exp(\frac{p}{2}V)$ *satisfies* (A_p).

THEOREM C. *Suppose* T_ϕ *and* $S_{\alpha\beta}$ *are on* L^2, *where* ϕ, α *and* β *are invertible functions in* L^∞.

(1) T_ϕ *is invertible if and only if* ϕ *has the form:* $\phi = |\phi|e^{it}$ *where* t *is a real function in* L^1 *such that*

$$\|t\|' = \inf\{\|t - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty \text{ and } a \in R\} < \pi/2$$

(2) $S_{\alpha\beta}$ *is invertible if and only if* α/β *has the form:* $\alpha/\beta = |\alpha/\beta|e^{it}$ *where* t *is the same to that of (1). Hence* $S_{\alpha\beta}$ *is invertible if and only if* $T_{\alpha/\beta}$ *is invertible.*

In this paper, we are interested in $\sigma(T_\phi^W)$ and $\sigma(S_{\alpha\beta}^W)$, that is, the spectra of T_ϕ^W and $S_{\alpha\beta}^W$.

For $\alpha = \alpha_1 + i\alpha_2 \in \mathbf{C}$ and $\beta = \beta_1 + i\beta_2 \in \mathbf{C}$, put $\langle \alpha, \beta \rangle = \alpha_1\beta_1 + \alpha_2\beta_2$ and $\theta(\alpha, \beta) = \arccos(\langle \alpha, \beta \rangle / |\alpha|\,|\beta|)$ for $\alpha \neq 0$ and $\beta \neq 0$. Set

$$\ell_\alpha^+ = \{z \in \mathbf{C} \,;\, \langle z, \alpha \rangle \geq 1\} \quad \text{and} \quad \ell_\alpha^- = \{z \in \mathbf{C} \,;\, \langle z, \alpha \rangle \leq 1\}$$

and $E_{\alpha\beta}^{ij}$ denotes $\ell_\alpha^i \cap \ell_\beta^j$ where $i = +$ or $-$ and $j = +$ or $-$. For each pair (α, β),

$$\mathbf{C} = E_{\alpha\beta}^{++} \cup E_{\alpha\beta}^{+-} \cup E_{\alpha\beta}^{-+} \cup E_{\alpha\beta}^{--}$$

and if $\ell = -i$ and $m = -j$, then

$$\overline{(E^{\ell m})^c} = \overline{\mathbf{C} \setminus E^{\ell m}} \supset E_{\alpha\beta}^{ij}.$$

For any bounded subset E in $\mathbf{C}$, there exists a pair (α, β) such that $E_{\alpha\beta}^{ij} \supseteq E$ for some (i, j). In fact, there are a lot of such pairs (α, β). Now we can define a set which contains E and is important in this paper. When $|\theta(\alpha, \beta)| = \pi - 2t$ and $0 \leq t \underset{\neq}{<} \pi/2$, put

$$h^t(E) = \cap\{\overline{(E_{\alpha\beta}^{\ell m})^c} \,;\, E_{\alpha\beta}^{ij} \supseteq E \text{ and } \ell = -i, m = -j\}$$

for a subset E in $\mathbf{C}$. If $t < s$, then $h^t(E) \subseteq h^s(E)$. If $t = 0$, then $h^0(E)$ is the closed convex hull of E. For example, if $E = [a, b]$ then

$$h^t(E) = \triangle(c, r) \cap \triangle(\bar{c}, r)$$

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$c = \frac{a+b}{2} - i\frac{a-b}{2}\cot 2t$ and $r = -\frac{a-b}{2\sin 2t}$ where $\triangle(c,r)$ denotes the circle of center c and radius r. If $E = \triangle(0,1)$, then $h^t(E) = \triangle(0, 1/\cos t)$. When T_ϕ is a Toeplitz operator on H^2, Brown and Halmos (*cf.* [2, Corollary 7.19]) showed that $\sigma(T_\phi) \subseteq h^0\big(\mathcal{R}(\phi)\big)$ where $\mathcal{R}(\phi)$ is the essential range of ϕ. In this paper we show this type results for Toeplitz operators on $H^2(W)$ and H^p and for singular integral operators on L^2. When ϕ is a real function and T_ϕ is a Toeplitz operator on H^2, Hartman and Wintner (*cf.* [2, Theorem 7.20]) showed that $\sigma(T_\phi) = h^0\big(\mathcal{R}(\phi)\big)$. In this paper, for real symbols we try to describe the spectra of Toeplitz operators on $H^2(W)$ and H^p, and singular integral operators on L^2. When ϕ is a continuous function, $\sigma(T_\phi)$ is described using $R(\phi)$ and the winding number of the curve determined by ϕ (*cf.* [2, Corollary 7.28]). In this case it is known that $\sigma(T_\phi^W) = \sigma(T_\phi^p) = \sigma(T_\phi)$ for arbitrary weight W satisfying the condition (A_2), and for any p with $1 < p < \infty$, T_ϕ^p denotes the Toeplitz operator on H^p. In this paper, we study symbols ϕ such that $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary weight W.

Now we collect the notations which will be used in this paper. R is the set of all real numbers and X_R denotes the set of the real parts of all elements in X. $[X]^{c\ell}$ denotes the closure of X. D is the open unit disc. C is the set of all continuous functions on T. If v is a real function in L^1, then $\tilde{v}$ denotes the harmonic conjugate function with $v(0) = 0$.

2. **Toeplitz operators on $H^2(W)$.** In this section, we fix arbitrary weight W satisfying the condition (A_2) or equivalently, a Helson-Szegő weight W. We call W a Helson-Szegő weight when $W = e^{u+\tilde{v}}$, u and v are functions in L^∞_R and $\|v\|_\infty < \pi/2$. For a Helson-Szegő weight $W = e^{u+\tilde{v}}$, put

$$t_W = \|v\|' = \inf\{\|v - \tilde{s} - a\|_\infty \,;\, s \in L^\infty_R, a \in R\}.$$

When $W \equiv 1$, (1) of Theorem 1 is a theorem of Brown and Halmos (*cf.* [2, Corollary 7.19]) and (2) and (3) of Theorem 1 is a theorem of Hartman and Wintner (*cf.* [2, Theorem 7.20]). When ϕ is a piecewise continuous function, $\sigma(T_\phi^W)$ is described when W is arbitrary weight [11]. The symbol ϕ in Corollary 2 and (3) of Corollary 3 is not necessarily piecewise continuous. It is known that $\sigma(T_\phi^W) \neq \sigma(T_\phi)$ for some weight W and some piecewise continuous symbol ϕ (*cf.* [4]). In Theorem 2, we determine weight W such that $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary symbol ϕ in L^∞ and study symbols ϕ such that $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary weight W. Spitkovsky [13] showed that the set of all weights W for which $\sigma(T_\phi^W) = \sigma(T_\phi)$ for all ϕ in L^∞ does not depend on p. (1) of Corollary 3 is related with a particular (corresponding to $p = 2$) case of [3, Theorem 6.1 and Corollary 6.2]. For if $\log W \in VMO$ then $\log W = u + \tilde{v}$ for some real functions u and v in C. (2) of Corollary 3 shows the known result [11] such that if ϕ is continuous, then $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary weight W.

THEOREM 1. *Let ϕ be a function in L^∞, let W be a Helson-Szegő weight and $t = t_W$.*
(1) $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^W) \subseteq h^t\big(\mathcal{R}(\phi)\big)$.
(2) if ϕ is real valued, $a = \operatorname{essinf}\phi$ and $b = \operatorname{esssup}\phi$, then

$$\mathcal{R}(\phi) \subseteq \sigma(T_\phi^W) \subseteq \triangle(c,r) \cap \triangle(\overline{c},r)$$

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where $c = \frac{a+b}{2} - i\frac{a-b}{2}\cos 2t$ *and* $r = -\frac{a-b}{2\sin 2t}$.

(3) Suppose $W = e^{u+\tilde{v}}$ *and* $\lambda \in [a,b] \cap \mathcal{R}(\phi)^c$ *in (2). Then* $\frac{\phi-\lambda}{|\phi-\lambda|} = e^{i\ell}$ *and* $\ell = \pi(1-\chi_E)$ *for some measurable set E in T with* $0 < m(E) < 1$. $\lambda \in \sigma(T_\phi^W)$ *if and only if*

$$\|\pi\chi_E - v\|' \geq \pi/2.$$

PROOF. In (1) and (2), it is well known that $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^W)$. Suppose $W = e^{u+\tilde{v}}$, u and v are functions in L_R^∞ and $\|v\|_\infty < \pi/2$, and $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$. Then $W = |g|^2$.

(1) By Theorem A in Introduction, for $\lambda \in \mathbf{C}$, $T_{\phi-\lambda}^W$ is invertible if and only if

$$T_{\frac{\phi-\lambda}{|\phi-\lambda|}\frac{\bar{g}}{g}} \quad \text{is invertible.}$$

Suppose $|\theta(\alpha,\beta)| = \pi - 2t$ and $\mathcal{R}(\phi) \subseteq E_{\alpha\beta}^{ij}$. If $\lambda \in (E_{\alpha\beta}^{\ell m})^0$ with $\ell = -i, m = -j$, then T_ϕ^W is invertible. In fact, then $(\phi-\lambda)/|\phi-\lambda| = e^{is_\lambda}$ where $0 \leq s_\lambda \leq \pi - 2t - 2\varepsilon$ a.e. or $-\pi + 2t + 2\varepsilon \leq s_\lambda \leq 0$ a.e. for some $\varepsilon > 0$. Hence $|s_\lambda - \frac{\pi}{2} + t + \varepsilon| \leq \frac{\pi}{2} - t - \varepsilon$ a.e. or $|s_\lambda + \frac{\pi}{2} - t - \varepsilon| \leq \frac{\pi}{2} - t - \varepsilon$ a.e. Hence

$$\frac{\phi-\lambda}{|\phi-\lambda|}\frac{\bar{g}}{g} = e^{i(s_\lambda+v-\tilde{u})}$$

and

$$\|s_\lambda + v - \tilde{u}\|' \leq \frac{\pi}{2} - \varepsilon.$$

Thus $T_{\frac{\phi-\lambda}{|\phi-\lambda|}\frac{\bar{g}}{g}}$ is invertible by Theorem C and hence $T_{\phi-\lambda}^W$ is invertible. If $\lambda \notin h^t\big(\mathcal{R}(\phi)\big)$, then by definition $\lambda \in \cup\{(E_{\alpha\beta}^{\ell m})^0 \,;\, E_{\alpha\beta}^{ij} \supseteq \mathcal{R}(\phi)$ and $\ell = -i, m = -j\}$ and $|\theta(\alpha,\beta)| = \pi - 2t$. By what was just proved, $\lambda \notin \sigma(T_\phi^W)$. (2) By (1), $\sigma(T_\phi^W) \subseteq h^t\big(\mathcal{R}(\phi)\big) \subseteq h^t([a,b])$ for $t = t_W$. It is elementary to see that $h^t([a,b]) \subseteq \triangle(c,r) \cap \triangle(\bar{c},r)$. (3) The first part is clear. The second statement is a result of Theorems A and C.

COROLLARY 1. *Suppose* $\phi = a\chi_E + b\chi_{E^c}$ *where a and b are real numbers,* $a \neq b$ *and* $0 < m(E) < 1$. *Let* $W = e^{u+\tilde{v}}$, *then* $\sigma(T_\phi^W) \supseteq [a,b]$ *if and only if* $\|\pi\chi_E - v\|' \geq \pi/2$.

COROLLARY 2. *Let E be a measurable set with* $0 < m(E) < 1$. *Suppose W and* ϕ *satisfy the following (i) and (ii):*

(i) $W = e^{u+\tilde{v}}$ *where* $u \in L_R^\infty$, $\tilde{v} = d(\chi_E - \chi_{E^c}) + q$, $q \in C_R$ *and d is a constant with* $0 < d < \pi/2$.

(ii) $\phi = a\chi_E + b\chi_{E^c}$ *where a and b are real numbers.*

Then $t_W = d$,

$$\sigma(T_\phi^W) = \Big\{\lambda \in \mathbf{C} \,;\, \arg\frac{a-\lambda}{b-\lambda} = \pi - 2d\Big\}.$$

and

$$h^d\big(\mathcal{R}(\phi)\big) = \Big\{\lambda \in \mathbf{C} \,;\, \arg\frac{a-\lambda}{b-\lambda} = \pi - 2d \text{ or } -\pi + 2d\Big\}.$$

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PROOF. Put $v_0 = \frac{\pi}{2}(\chi_E - \chi_{E^c})$, then $h^2 = e^{\tilde{v}_0 - iv_0}$ and $|h|^2/h^2 = e^{iv_0} = i(\chi_E - \chi_{E^c})$. If $\|\chi_E - \chi_{E^c}\|' < 1$, then $|h|^2 = e^{\tilde{v}_0}$ is a Helson-Szegő weight and so $\| |h|^2/h^2 + zH^\infty \| < 1$ (see [3, Chapter IV, Theorem 3.1]). On the other hand, $\| |h|^2/h^2 + zH^\infty \| = \| i(\chi_E - \chi_{E^c}) + zH^\infty \| = 1$. This contradiction shows that $\|\chi_E - \chi_{E^c}\|' = 1$. Thus

$$\begin{aligned} t_W &= \inf\{\|d(\chi_E - \chi_{E^c}) - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty, a \in R\} \\ &= d \inf\{\|\chi_E - \chi_{E^c} - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty, a \in R\} \\ &= d. \end{aligned}$$

Put $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$, then $\bar{g}/g = e^{i(\tilde{u}-v)} = \exp i\{\tilde{u} - d(\chi_E - \chi_{E^c}) - q\}$. If $\lambda \neq a$ and $\lambda \neq b$, then

$$\begin{aligned} \frac{\phi - \lambda}{|\phi - \lambda|} &= \frac{a - \lambda}{|a - \lambda|}\chi_E + \frac{b - \lambda}{|b - \lambda|}\chi_{E^c} \\ &= \exp i\{a(\lambda)\chi_E + b(\lambda)\chi_{E^c}\} \end{aligned}$$

where $a(\lambda) = \arg(a - \lambda)$ and $b(\lambda) = \arg(b - \lambda)$. Thus $(\phi - \lambda)\bar{g}/|\phi - \lambda|g = \exp i\{a(\lambda)\chi_E + b(\lambda)\chi_{E^c} + \tilde{u} - d(\chi_E - \chi_{E^c}) - q\}$. Since $q \in C_R$, by the first part of the proof,

$$\begin{aligned} &\inf\{\|a(\lambda)\chi_E + b(\lambda)\chi_{E^c} - d(\chi_E - \chi_{E^c}) + \tilde{u} - q - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty, a \in R\} \\ &\quad = \left|\frac{a(\lambda) - b(\lambda)}{2} - d\right| \inf\{\|\chi_E - \chi_{E^c} - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty, a \in R\} \\ &\quad = \left|\frac{a(\lambda) - b(\lambda)}{2} - d\right| = \frac{1}{2}\left|\arg\frac{a - \lambda}{b - \lambda} - 2d\right|. \end{aligned}$$

Thus, by (1) of Theorem C $\lambda \notin \sigma(T_\phi^W)$ if and only if $\left|\arg\frac{a-\lambda}{b-\lambda} - 2d\right| \neq \pi$. If $\arg\frac{a-\lambda}{b-\lambda} > 0$, then $\left|\arg\frac{a-\lambda}{b-\lambda} - 2d\right| \neq \pi$ because $d > 0$, and hence $\sigma(T_\phi^W) = \{\lambda \notin \mathbf{C} \,;\, \arg\frac{a-\lambda}{b-\lambda} = \pi - 2d\}$. The description of $h^d(\mathcal{R}(\phi))$ is a result of (2) of Theorem 1.

THEOREM 2. *Let ϕ be a function in L^∞ and let W be a Helson-Szegő weight.*

(1) $t_W = 0$ if and only if $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary symbol ϕ in L^∞.

(2) $\sigma(T_\phi) \supseteq \sigma(T_\phi^W)$ for arbitrary Helson-Szegő weight W if and only if for any $\lambda \notin \sigma(T_\phi)$, $\frac{\phi-\lambda}{|\phi-\lambda|} = e^{i\ell}$ and $\|\ell\|' = 0$.

PROOF. (1) Suppose $W = e^{u+\tilde{v}}$, $t_W = 0$ and $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$. If $\lambda \notin \sigma(T_\phi)$, then by Theorem C $\frac{\phi-\lambda}{|\phi-\lambda|} = e^{i\ell}$ and $\|\ell\|' < \pi/2$. Hence

$$\frac{\phi - \lambda}{|\phi - \lambda|}\frac{\bar{g}}{g} = \exp i(\ell + \tilde{u} - v)$$

and since $t_W = 0$,

$$\begin{aligned} &\inf\{\|\ell + \tilde{u} - v - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty \text{ and } a \in R\} \\ &\qquad = \inf\{\|\ell - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty \text{ and } a \in R\} \\ &\qquad < \frac{\pi}{2}. \end{aligned}$$

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Thus $\lambda \notin \sigma(T_\phi^W)$ by Theorems A and C. Similarly we can show that if $\lambda \notin \sigma(T_\phi^W)$ then $\lambda \in \sigma(T_\phi)$. Suppose $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary symbol ϕ in $Ł^\infty$. If $t = t_W$ is nonzero and $W = e^{u+\tilde{v}}$ is a Helson-Szegő weight, then T_ϕ is invertible where $\phi = e^{-ikv}$ and $k = \pi/2t - 1$. For $\inf\{\|kv - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty$ and $a \in R\} = kt = \pi/2 - 1$. On the other hand, T_ϕ^W is not invertible. For

$$\frac{\phi}{|\phi|}\frac{\bar{g}}{g} = \exp i\{\tilde{u} - (k+1)v\}$$

and

$$\inf\{\|\tilde{u} - (k+1)v - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty \text{ and } a \in R\} = (k+1)t = \frac{\pi}{2}$$

where $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$.

(2) Suppose for any $\lambda \notin \sigma(T_\phi)$, $\frac{\phi-\lambda}{|\phi-\lambda|} = e^{i\ell}$ and $\inf\{\|\ell - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty$ and $a \in R\} = 0$. We will show that $\sigma(T_\phi) \supseteq \sigma(T_\phi^W)$ for arbitrary Helson-Szegő weight W. If $\lambda \notin \sigma(T_\phi)$, $W = e^{u+\tilde{v}}$ is a Helson-Szegő weight and $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$, then

$$\frac{\phi-\lambda}{|\phi-\lambda|}\frac{\bar{g}}{g} = e^{i(\ell+\tilde{u}-v)}$$

and $\inf\{\|\ell+\tilde{u}-v-\tilde{s}-a\|_\infty \,;\, s \in L_R^\infty, a \in R\} < \pi/2$ by the hypothesis. This implies that $\sigma(T_\phi^W) \not\ni \lambda$. Conversely suppose that $\sigma(T_\phi) \supseteq \sigma(T_\phi^W)$ for arbitrary Helson-Szegő weight W. If $\lambda \notin \sigma(T_\phi)$, then $\frac{\phi-\lambda}{|\phi-\lambda|} = e^{i\ell}$ and $b = \inf\{\|\ell - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty, a \in R\} < \pi/2$. If $b \neq 0$, put $W = e^{k\tilde{\ell}}$ and $g^2 = e^{k\tilde{\ell}-ik\ell}$ where $k = \frac{\pi}{2b} - 1$, then W is a Helson-Szegő weight. However T_ϕ^W is not invertible and so $\lambda \in \sigma(T_\phi^W)$. This contradiction implies that $b = 0$.

COROLLARY 3. *Let ϕ be a function in L^∞.*

(1) If $W = e^{u+\tilde{v}}$, u and v are real functions in L^∞ and C respectively, then $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary symbol ϕ in L^∞.

(2) If ϕ is a function in C or H^∞, then $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary Helson-Szegő weight W.

(3) If $\phi = a\chi_E + b\chi_{E^c}$, $0 < m(E) < 1$ and $a, b \in \mathbf{C}$ with $a \neq b$, then there exists a Helson-Szegő weight W such that $\sigma(T_\phi^W) \subsetneq \sigma(T_\phi)$.

PROOF. Since $t_W = 0$ because $v \in C_R$, (1) of Theorem 2 implies (1). Suppose ϕ is a function in C and $\lambda \notin \sigma(T_\phi^{W'})$ for a Helson-Szegő weight $W' = e^{u+\tilde{v}}$. Since $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^{W'})$,

$$\frac{\phi-\lambda}{|\phi-\lambda|}\frac{\bar{g}}{g} = z^m e^{i\ell} e^{i(\tilde{u}-v)}$$

where m is an integer, $\ell \in C_R$ and $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$. By Theorems A and C, we can show $m = 0$. As $W' \equiv 1$, (2) of Theorem 2 implies that $\sigma(T_\phi) \supseteq \sigma(T_\phi^W)$ for arbitrary Helson-Szegő weight W. The converse is trivial. Suppose ϕ is a function in H^∞ and $\lambda \notin \sigma(T_\phi^{W'})$ for a Helson-Szegő weight $W' = e^{u+\tilde{v}}$. Since $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^{W'})$, $\phi - \lambda$ is invertible in L^∞

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and so $\phi - \lambda = qh$ where q is inner and h is invertible in H^∞. Since $h = e^{\ell + i\tilde{\ell}}$ and $\ell = \log|h| \in L^\infty$,

$$\frac{\phi - \lambda}{|\phi - \lambda|}\frac{\bar{g}}{g} = qe^{i\tilde{\ell}}e^{i(\tilde{u}-v)}$$

where $g^2 = e^{u+\tilde{v}+i(\tilde{u}-v)}$. By Theorems A and C, we can show that q is constant. As in case $\phi \in C$, we can show $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary Helson-Szegő weight W. This completes the proof of (2). Suppose $\phi = a\chi_E + b\chi_{E^c}$, $0 < m(E) < 1$ and $a, b \in \mathbf{C}$ with $a \neq b$. To prove (3), without loss of generality, we may assume that a and b are real numbers. By a theorem of Hartman and Wintner (*cf.* [2, Theorem 7.20]), $\sigma(T_\phi) = [a, b]$. If $\lambda \notin [a, b]$,

$$\frac{\phi - \lambda}{|\phi - \lambda|} = \exp i\{a(\lambda)\chi_E + b(\lambda)\chi_{E^c}\}$$

where $a(\lambda) = \arg(a - \lambda)$ and $b(\lambda) = \arg(b - \lambda)$. By the proof of Corollary 1,

$$\inf\{\|a(\lambda)\chi_E + b(\lambda)\chi_{E^c} - \tilde{s} - a\|_\infty \,;\, s \in L_R^\infty \text{ and } a \in R\} = \frac{1}{2}\left|\arg\frac{a-\lambda}{b-\lambda}\right| \neq 0$$

and hence by (2) of Theorem 2, there exists a Helson-Szegő weight W such that $\sigma(T_\phi^W) \subsetneq \sigma(T_\phi)$.

3**. Toeplitz operators on** H^p**.** For $1 < p < \infty$, T_ϕ^p denotes a Toeplitz operator on H^p. We will write $T_\phi^2 = T_\phi$. By a theorem of Widom, Devinatz and Rochberg (*cf.* [8]), we know the invertibility of T_ϕ^p and by a theorem of Widom (*cf.* [2, Corollary 7.46]), $\sigma(T_\phi^p)$ is connected. If $1 < q < 2 < p < \infty$, then $A_q \subset A_2 \subset A_p$. It is more difficult to describe $\sigma(T_\phi^q)$ than $\sigma(T_\phi^p)$. In this paper, we study only $\sigma(T_\phi^p)$. When $p = 2$, (1) of Theorem 3 is a theorem of Brown and Halmos and (2) is a theorem of Hartman and Wintner. (3) of Theorem 3 is known in [10] for arbitrary $1 < p < \infty$. Our proof is different from it.

THEOREM 3. *Suppose* $p \geq 2$ *and* $t = (p-2)\pi/2p$.
(1) If ϕ *is a function in* L^∞, *then* $\sigma(T_\phi^p) \subseteq h^t\big(\mathcal{R}(\phi)\big)$.
(2) If ϕ *is a real function in* L^∞, $a = \operatorname{essinf}\phi$ *and* $b = \operatorname{esssup}\phi$, *then*

$$[a, b] \subseteq \sigma(T_\phi^p) \subseteq \triangle(c, r) \cap \triangle(\bar{c}, r)$$

where $c = \frac{a+b}{2} - i\frac{a-b}{2}\cot 2t$ *and* $r = -\frac{a-b}{2\sin 2t}$. *In particular, if* $p = 2$, *then* $t = 0$ *and hence* $\sigma(T_\phi^p) = [a, b]$.
(3) If ϕ *is a function in* C, *then* $\sigma(T_\phi^p) = \sigma(T_\phi)$.

PROOF. (1) If $\lambda \notin h^t\big(\mathcal{R}(\phi)\big)$, then by definition $\lambda \in \cup\{(E_{\alpha\beta}^{\ell m})^0 \,;\, E_{\alpha\beta}^{ij} \supseteq \mathcal{R}(\phi)$ and $\ell = -i, m = -j\}$ and $|\theta(\alpha, \beta)| = \pi - 2t$. Hence $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$ where $0 \leq s_\lambda \leq \pi - 2t - 2\varepsilon$ a.e. or $-\pi + 2t + 2\varepsilon \leq s_\lambda \leq 0$ a.e. for some $\varepsilon > 0$. Put $v_\lambda = s_\lambda - \frac{\pi}{2} + t + \varepsilon$ or $v_\lambda = s_\lambda + \frac{\pi}{2} - t - \varepsilon$, then $\|v_\lambda\|_\infty \leq \frac{\pi}{2} - t - \varepsilon$. Put $g^2 = e^{-\tilde{v}_\lambda + iv_\lambda}$, then g^2 is an outer function and $|g|^2 = e^{-\tilde{v}_\lambda}$. Then $\|\frac{p}{2}v_\lambda\|_\infty < \frac{\pi}{2}$ because $\|v_\lambda\|_\infty < \frac{\pi}{2} - \frac{(p-2)\pi}{2p}$. Hence $|g|^p$ satisfies (A_2) condition and so $|g|^p$ satisfies (A_p) condition by (*cf.* [3, Lemma 6.8]) because $p > 2$.

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Since $(\phi-\lambda)/|\phi-\lambda| = \alpha(\bar{g}/g)$ for some constant α with $|\alpha| = 1$, Theorem A implies (1).

(2) We may assume that ϕ is not constant. By Theorem A, $\mathcal{R}(\phi) \subseteq \sigma(T^p_\phi)$. Suppose $\lambda \in [a,b]$ and $\lambda \notin \mathcal{R}(\phi)$, then $(\phi-\lambda)/|\phi-\lambda| = 2\chi_E - 1$ for some measurable set E in T. If $\lambda \notin \sigma(T^p_\phi)$, then by Theorem A, there exists an outer function h_0 in H^p such that $2\chi_E - 1 = \bar{h}_0/h_0$. This implies that h_0^2 is a real function in H^1 because $p \geq 2$. It is well known that only one real function in H^1 is constant. Hence h_0 is constant and this contradicts that ϕ is not constant. Thus $[a,b] \subseteq \sigma(T^p_\phi)$. Now (1) implies (2).

(3) If $\lambda \notin \mathcal{R}(\phi)$, then $(\phi-\lambda)/|\phi-\lambda|$ is a continuous function and hence

$$\frac{\phi-\lambda}{|\phi-\lambda|} = z^\ell e^{iv}$$

where ℓ is an integer and v is a real function in C. Put $g^2 = e^{-\tilde{v}+iv}$, then $|g|^2 = e^{-\tilde{v}}$. Since v is continuous, for any $\varepsilon > 0$, $\tilde{v} = s + \tilde{t}$ where both s and t are in C and $\|t\|_\infty < \varepsilon$. Suppose $\ell = 0$. If $\varepsilon < \pi/p$, then $|g|^p = |g^2|^{\frac{p}{2}} = \exp(-\frac{p}{2}\tilde{v}) = \exp(-\frac{p}{2}s - \frac{p}{2}\tilde{t})$ and $\|\frac{p}{2}t\|_\infty < \frac{\pi}{2}$. Hence $|g|^p$ satisfies (A_2) condition and so (A_p). By Theorem A, $T^p_{\phi-\lambda}$ is invertible and so $\lambda \notin \sigma(T^p_\phi)$. Suppose $\ell \neq 0$. If $T^p_{\phi-\lambda}$ is invertible, then by Theorem A

$$\frac{\phi-\lambda}{|\phi-\lambda|} = z^\ell e^{iv} = \frac{|k|}{k}\frac{|h|^2}{h^2}$$

where k and k^{-1} are in H^∞, and h is an outer function in H^p with $|h|^p$ satisfying (A_p) condition. Since $z^\ell |g|^2/g^2 = |kh^2|/kh^2$, $z^\ell f \geq 0$ a.e. where $f = kh^2/g^2$. If $\ell > 0$, $z^\ell f$ is a nonnegative function in $H^{1/2}$ and hence it is constant. This contradicts that z^ℓ is zero on the origin. If $\ell < 0$, $z^\ell|1+\bar{z}^\ell|/(1+\bar{z}^\ell)^2 \geq 0$ and so $(1+\bar{z}^\ell)^2 f \geq 0$ a.e. Thus $(1+\bar{z}^\ell)^2 f$ is a nonnegative function in $H^{1/2}$ and so $f = c(1+\bar{z}^\ell)^2$ for some constant $c > 0$. This contradicts that $f^{-1} \in H^{1/2}$.

4. **Singular integral operators on L^2.** By Theorems A, B and C, we can expect that $\sigma(S_{\alpha\beta})$ is strongly related with $\sigma(T_\alpha)$ and $\sigma(T_\beta)$. (1) of Theorem 4 is an analogy of a theorem of Brown and Halmos, and (2) of Theorem 4 is an analogy of a theorem of Hartman and Wintner.

THEOREM 4. *Suppose α and β are functions in L^∞.*

(1) $\mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta}) \subseteq h^t\big(\mathcal{R}(\alpha) \cup \mathcal{R}(\beta)\big)$ where $t = \pi/4$.

(2) If α and β are real functions in L^∞,

$$\Big\{h\big(\mathcal{R}(\alpha)\big) \cap h\big(\mathcal{R}(\beta)\big)^c\Big\} \cup \Big\{h\big(\mathcal{R}(\alpha)\big)^c \cap h\big(\mathcal{R}(\beta)\big)\Big\} \subseteq \sigma(S_{\alpha\beta}) \subseteq \triangle(c,r) \cap \triangle(\bar{c},r)$$

where $a = \min\{\operatorname{essinf}\alpha, \operatorname{essinf}\beta\}$, $b = \max\{\operatorname{esssup}\alpha, \operatorname{esssup}\beta\}$, $c = \frac{a+b}{2} - i\frac{a-b}{2}$ and $r = -\frac{a-b}{2}$.

(3) If β is in C,

$$\sigma(T_\alpha) \cap \{\lambda \in \mathbf{C}\, ;\, i_t(\beta,\lambda) = 0\} \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta}) \subseteq \sigma(T_\alpha) \cup \sigma(T_\beta).$$

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(4) If both α and β are in C, then $\sigma(S_{\alpha\beta}) = \{\sigma(T_\alpha) \cup \sigma(T_\beta)\} \setminus \{\lambda \in \mathbf{C} \,;\, i_t(\alpha,\lambda) = i_t(\beta,\lambda) \neq 0\}$.

(5) Suppose both α and β are in C. If β is a real function, then $\sigma(S_{\alpha\beta}) = \sigma(T_\alpha) \cup h\big(\mathcal{R}(\beta)\big)$ and hence if both α and β are real functions, then $\sigma(S_{\alpha\beta}) = h\big(\mathcal{R}(\alpha)\big) \cup h\big(\mathcal{R}(\beta)\big)$.

(6) If α and $\bar{\beta}$ are functions in H^∞, then $\sigma(S_{\alpha\beta}) = [\alpha(D)]^{\mathrm{cl}} \cup [\overline{\beta(D)}]^{\mathrm{cl}}$.

(7) If α and β are functions in H^∞, then $\sigma(S_{\alpha\beta}) = [\alpha(D)]^{\mathrm{cl}} \cup [\beta(D)]^{\mathrm{cl}} \setminus \{\lambda \notin \mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \,;\, T_{q_\lambda \bar{p}_\lambda}$ is invertible$\}$ where q_λ is the inner part of $\alpha - \lambda$ and p_λ is the inner part of $\beta - \lambda$.

(8) If α and β are inner functions, and $\operatorname{sing}\alpha \neq \operatorname{sing}\beta$, then $\sigma(S_{\alpha\beta}) = [D]^{\mathrm{cl}}$, where $\operatorname{sing}\alpha$ and $\operatorname{sing}\beta$ denote the subsets of ∂D on which α and β can not be analytically extended, respectively.

PROOF. (1) By Theorem B, it is clear that $\mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta})$. If $\lambda \notin h^t\big(\mathcal{R}(\alpha) \cup \mathcal{R}(\beta)\big)$, then $(\alpha - \lambda)/|\alpha - \lambda| = e^{is_\lambda}$ and $(\beta - \lambda)/|\beta - \lambda| = e^{it_\lambda}$ where $0 \leq s_\lambda, t_\lambda \leq \frac{\pi}{2} - \varepsilon$ a.e. or $-\frac{\pi}{2} + \varepsilon \leq s_\lambda, t_\lambda \leq 0$ a.e. for some $\varepsilon > 0$. Therefore

$$\frac{\alpha - \lambda}{\beta - \lambda} = \exp(U - i\tilde{V})$$

where $U = \log|\alpha - \lambda| - \log|\beta - \lambda|$ and $\tilde{V} = t_\lambda - s_\lambda$. Then U is bounded and $\exp V = \exp -(t_\lambda - s_\lambda)^{\sim}$ and $\|t_\lambda - s_\lambda\|_\infty \leq \frac{\pi}{2} - \varepsilon$. By Theorem C, $S_{\alpha-\lambda,\beta-\lambda}$ is invertible.

(2) If α and β are real functions and $\lambda \in h\big(\mathcal{R}(\alpha)\big) \cap h\big(\mathcal{R}(\beta)\big)^c$, then $\alpha - \lambda$ is a real function which is not nonnegative or nonpositive, and $\beta - \lambda$ is a nonnegative or nonpositive function which is invertible in L^∞. $(\alpha - \lambda)/(\beta - \lambda)$ is a real function in L^∞ which is not nonnegative or nonpositive. If $S_{\alpha-\lambda,\beta-\lambda}$ is invertible, then by Theorems B and C both $\alpha - \lambda$ and $\beta - \lambda$ are invertible in L^∞, and

$$\frac{\alpha - \lambda}{\beta - \lambda} = \left|\frac{\alpha - \lambda}{\beta - \lambda}\right| e^{it}$$

where $\inf\{\|t - \tilde{s} - a\|_\infty : s \in L^\infty_R \text{ and } a \in R\} < \pi/2$. Let $g = e^{-\tilde{t}+it}$, then g is a real function in H^1. Since only one real function in H^1 is constant, g is constant and so it contradicts that $(\alpha - \lambda)|\beta - \lambda|/(\beta - \lambda)|\alpha - \lambda|$ is nonconstant. This implies that $h\big(\mathcal{R}(\alpha)\big) \cap h\big(\mathcal{R}(\beta)\big)^c \subseteq \sigma(S_{\alpha\beta})$. The same method shows that $h\big(\mathcal{R}(\alpha)\big)^c \cap h\big(\mathcal{R}(\beta)\big) \subseteq \sigma(S_{\alpha\beta})$. Since $\mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq [a,b]$, by (1) $\sigma(S_{\alpha\beta}) \subseteq h^t([a,b])$ where $t = \pi/4$. This implies (2).

(3) Suppose $\lambda \in \sigma(T_\alpha) \cap \{\lambda \in \mathbf{C} \,;\, i_t(\beta,\lambda) = 0\}$. Then $\beta - \lambda = |\beta - \lambda| e^{iv}$ and $v \in C$ because β is continuous. If $S_{\alpha-\lambda,\beta-\lambda}$ is invertible, then by Theorem B

$$\frac{\alpha - \lambda}{\beta - \lambda} = \gamma e^{(U - i\tilde{V})}$$

where γ is constant, U is a bounded real function, V is a real function in L^1 and $\exp V$ satisfies (A_2) condition. Hence

$$\alpha - \lambda = \gamma \exp\{U + \log|\beta - \lambda| - i(\tilde{V} - v)\},$$

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$U + \log|\beta - \lambda|$ is in L^∞ and $e^{V-\tilde{v}}$ satisfies (A_2) condition because $v \in C$. By Theorem A, this implies that $\lambda \notin \sigma(T_\alpha)$. This contradiction shows that $\lambda \in \sigma(S_{\alpha\beta})$ and hence $\sigma(T_\alpha) \cap \{\lambda \in \mathbf{C} \ ; \ i_t(\beta,\lambda) = 0\} \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta})$. If $\lambda \notin \sigma(T_\alpha) \cup \sigma(T_\beta)$, then by Theorem C and [2, Corollary 7.28] $\alpha - \lambda = |\alpha - \lambda| e^{it}$ and $\beta - \lambda = |\beta - \lambda| e^{i\ell}$ where $\inf\{\|t - \tilde{s} - a\|_\infty \ ; s \in L^\infty_R \text{ and } a \in R\} < \pi/2$ and $\ell \in C$. Therefore

$$\frac{\alpha - \lambda}{\beta - \lambda} = \frac{|\alpha - \lambda|}{|\beta - \lambda|} e^{i(t-\ell)}$$

and hence by Theorem C $\lambda \notin \sigma(S_{\alpha\beta})$.

(4) If $\lambda \notin \mathcal{R}(\alpha) \cup \mathcal{R}(\beta)$ and $i_t(\alpha,\lambda) \neq i_t(\beta,\lambda)$, then $\alpha - \lambda = |\alpha - \lambda| z^\ell e^{iu}$ and $\beta - \lambda = |\beta - \lambda| z^t e^{iv}$ where u and v are in C, and ℓ and t are integers with $\ell \neq t$. Hence

$$\frac{\alpha - \lambda}{\beta - \lambda} = \frac{|\alpha - \lambda|}{|\beta - \lambda|} z^{\ell - t} e^{i(u-v)}$$

and $\ell - t \neq 0$. By Theorem C, we can show that $\lambda \notin \sigma(S_{\alpha\beta})$. This implies that $\{\sigma(T_\alpha) \cup \sigma(T_\beta)\} \setminus \{\lambda \in \mathbf{C} \ ; i_t(\alpha,\lambda) = i_t(\beta,\lambda) \neq 0\} \subseteq \sigma(S_{\alpha\beta})$. If $\lambda \notin \{\sigma(T_\alpha) \cup \sigma(T_\beta)\} \setminus \{\lambda \in \mathbf{C} \ ; i_t(\alpha,\lambda) = i_t(\beta,\lambda) \neq 0\}$, then $\alpha - \lambda = |\alpha - \lambda| z^\ell e^{iu}$ and $\beta - \lambda = |\beta - \lambda| z^\ell e^{iv}$ where u and v are in C, and ℓ is an integer. Hence $(\alpha - \lambda)/(\beta - \lambda) = (|\alpha - \lambda|/|\beta - \lambda|) e^{i(u-v)}$. By Theorem C, $\lambda \notin \sigma(S_{\alpha\beta})$. This completes the proof of (4). (5) is a result of (4).

(6) If $\lambda \in \alpha(D) \setminus \mathcal{R}(\alpha) \cup \mathcal{R}(\beta)$, then $\alpha - \lambda = qh$ and $\beta - \lambda = \bar{p}\bar{k}$ where q and p are inner, and h and k are invertible in H^∞. Hence $(\alpha - \lambda)/(\beta - \lambda) = qph/\bar{k}$ and so by Theorem C $\lambda \in \sigma(S_{\alpha\beta})$. This shows that $\alpha(D) \setminus \mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta})$. By the same method we can show that $\overline{\beta(D)} \setminus \mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha\beta})$. By (1), $[\alpha(D)]^{\mathrm{cl}} \cup [\overline{\beta(D)}]^{\mathrm{cl}} \subseteq \sigma(S_{\alpha\beta})$. If $\lambda \notin [\alpha(D)]^{\mathrm{cl}} \cup [\overline{\beta(D)}]^{\mathrm{cl}}$, then $\alpha - \lambda = h$ and $\beta - \lambda = \bar{k}$ where both h and k are invertible in H^∞. By Theorem C, $\lambda \notin \sigma(S_{\alpha\beta})$.

(7) If $\lambda \in [\alpha(D)]^{\mathrm{cl}} \setminus \mathcal{R}(\alpha) \cup \mathcal{R}(\beta)$, then $\alpha - \lambda = q_\lambda h_\lambda$ and $\beta - \lambda = p_\lambda k_\lambda$ where both q_λ and p_λ are inner and both h_λ and k_λ are invertible in H^∞. Hence $(\alpha - \lambda)/(\beta - \lambda) = q_\lambda \bar{p}_\lambda h_\lambda / k_\lambda$. If $T_{q_\lambda \bar{p}_\lambda}$ is not invertible, by Theorem C $\lambda \in \sigma(S_{\alpha\beta})$. This implies that $\{[\alpha(D)]^{\mathrm{cl}} \cup [\beta(D)]^{\mathrm{cl}}\} \setminus \{\lambda \notin \mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \ ; \ T_{q_\lambda \bar{p}_\lambda} \text{ is invertible }\} \subseteq \sigma(S_{\alpha\beta})$. If $\lambda \notin [\alpha(D)]^{\mathrm{cl}} \cup [\beta(D)]^{\mathrm{cl}}$, then $\lambda \notin \sigma(S_{\alpha\beta})$ as in (6). If $\lambda \notin \mathcal{R}(\alpha) \cup \mathcal{R}(\beta)$ and $T_{q_\lambda \bar{p}_\lambda}$ is invertible, then by Theorem C $\lambda \notin \sigma(S_{\alpha\beta})$.

(8) $\sigma(S_{\alpha\beta}) \subseteq [D]^{\mathrm{cl}}$ by (7) and so if $\lambda \notin \big(\mathcal{R}(\alpha) \cup \mathcal{R}(\beta)\big) \cap [D]^{\mathrm{cl}}$, then the inner part of $\alpha - \lambda$ is $q_\lambda = (\alpha - \lambda)/(1 - \bar{\lambda}\alpha)$ and the inner part of $\beta - \lambda$ is $p_\lambda = (\beta - \lambda)/(1 - \bar{\lambda}\beta)$. Then $\operatorname{sing} q_\lambda = \operatorname{sing} q \neq \operatorname{sing} p = \operatorname{sing} p_\lambda$. By [6, Theorem 1], $T_{q_\lambda \bar{p}_\lambda}$ is not invertible. By (7), this implies that $\sigma(S_{\alpha\beta}) = [\alpha(D)]^{\mathrm{cl}} \cup [\beta(D)]^{\mathrm{cl}} = [D]^{\mathrm{cl}}$.

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