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Some Applications of the Perturbation Determinant in Finite von Neumann Algebras

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Abstract. In the finite von Neumann algebra setting, we introduce the concept of a perturbation determinant associated with a pair of self-adjoint elements H_0 and H in the algebra and relate it to the concept of the de la Harpe–Skandalis homotopy invariant determinant associated with piecewise C^1-paths of operators joining H_0 and H. We obtain an analog of Krein's formula that relates the perturbation determinant and the spectral shift function and, based on this relation, we derive subsequently (i) the Birman–Solomyak formula for a general non-linear perturbation, (ii) a universality of a spectral averaging, and (iii) a generalization of the Dixmier–Fuglede–Kadison differentiation formula.

1 Introduction

In his seminal paper, M. G. Krein [21] introduced the spectral shift function (the ξ-function, for short) associated with a pair (H_0, H) of self-adjoint operators via the boundary values of the perturbation determinant

$$\xi(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \arg\left(\det{}_{H/H_0}(\lambda + i\varepsilon)\right) \quad \text{for a.e. } \lambda \in \mathbb{R},$$

with the branch of the argument of $\det_{H/H_0}(\,\cdot\,)$ fixed by the condition

$$\arg\left(\det{}_{H/H_0}(iy)\right) \to 0 \quad \text{as } y \to +\infty.$$

In the initial setup of the theory it is assumed that the difference $H - H_0$ is a trace class operator and, hence, the perturbation determinant is well defined by

$$\det{}_{H/H_0}(z) = \det\left((H - zI)(H_0 - zI)^{-1}\right), \quad z \in \rho(H_0),$$

with $\det(\,\cdot\,)$ the Fredholm determinant.

In [9] and recently in [2–4], the concept of a spectral shift function was extended to the case of a semi-finite von Neumann algebra. These new developments resulted in establishing a remarkable connection between the spectral shift function and the

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analytically defined spectral flow due to J. Phillips [25, 26]. However, no analog of the important concept of a perturbation determinant has been involved in the development of the ξ-function theory in the operator algebraic setting.

In this paper, we introduce a finite von Neumann algebra analog of the perturbation determinant $\det_{H/H_0}$ associated with a pair of self-adjoint elements H_0 and H. We define $\det_{H/H_0}$ to be an analytic function on the complex plane with the cut(s) along the union K of the convex hulls of the spectra of H_0 and H that is uniquely determined by its absolute value

$$|\det_{H/H_0}(z)| = \Delta((H - zI)(H_0 - zI)^{-1}), \quad z \in \mathbb{C} \setminus K,$$

with $\Delta(\,\cdot\,)$ the Fuglede–Kadison determinant [8, 12], and by the requirement that

$$\lim_{y \to +\infty} \det_{H/H_0}(\mathrm{i}y) = 1$$

(see Lemma 2.8 which sheds light on the operator theoretic descent of the perturbation determinant).

We prove that the value of the perturbation determinant at a non-real z can be recognized as the (homotopy invariant) de la Harpe–Skandalis determinant [17] associated with the "z-translation" of a piecewise C^1-path of self-adjoint elements $[0, 1] \ni t \mapsto H_t \in \mathcal{A}$ connecting the end points H_0 and $H_1 = H$ given by

$$\det_{H/H_0}(z) = \exp\Big(\int_0^1 \tau[\dot{H}_t(H_t - zI)^{-1}]\, dt\Big), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.1}$$

(Due to the homotopy invariance, the result does not depend on the particular choice of the path $t \mapsto H_t$, but on the end points only.)

We also obtain that the perturbation determinant admits an exponential representation in terms of the ξ-function

$$\det_{H/H_0}(z) = \exp\Big(\int_{\mathbb{R}} \frac{\xi(\lambda)}{\lambda - z}\, d\lambda\Big), \tag{1.2}$$

where the ξ-function is given by the Lifshits "naive" formula [4, 6, 23],

$$\xi(\lambda) = n_{H_0}(\lambda) - n_H(\lambda), \quad \lambda \in \mathbb{R},$$
$$\mathbb{R} \ni \lambda \mapsto n_A(\lambda) = \tau[E_A((-\infty, \lambda))], \quad A \in \mathcal{A}.$$

Based on (1.2), we derive an analog of Krein's original representation for the spectral shift function via the perturbation determinant

$$\frac{\xi(\lambda)+\xi(\lambda+0)}{2} = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\arg(\Delta_{H/H_0}(\lambda+\mathrm{i}\varepsilon)), \tag{1.3}$$

which, along with (1.1), implies the "path"-representation

$$\frac{\xi(\lambda)+\xi(\lambda+0)}{2} = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_0^1 \operatorname{Im}(\tau[\dot H_t(H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}])\,dt. \tag{1.4}$$

Another source of path representations for the spectral shift function is the aforementioned intrinsic connection between the ξ-function and the spectral flow [2, 3]. For instance, in the case of a finite tracial state τ, the ξ-function can be identified with the spectral flow

$$\xi(\lambda) = \frac{1}{2}\int_0^1 \tau[\dot J_t]\,dt,$$

along a linear path of self-adjoint operators $[0,1]\ni t\mapsto J_t$ connecting the end points $J_0 = 2E_{H_0}((-\infty,0))-I$ and $J_1 = 2E_{H_1}((-\infty,0))-I$ (see [5, §5.1]).

We remark that in contrast to the classical case of the trace class perturbation theory, the limits in (1.3) and (1.4) exist for *all* $\lambda\in\mathbb{R}$ if the algebra $\mathcal{A}$ is of finite type. We also notice that as distinct from the classical case when the Fredholm perturbation determinant is analytic on the resolvent set $\rho(H_0)$ of H_0, in the general algebraic setting $\det_{H/H_0}$ does not admit an analytic continuation to a domain larger than $\mathbb{C}\setminus K$ even if $\rho(H_0)\cap K \neq \varnothing$ (see Remark 2.5). One can also consult [12], where the impossibility of development of a general "signed" determinant theory going beyond the concept of the positive Fuglede–Kadison determinant is discussed. (However, for special classes of unitaries as well as dissipative operators some fragments of the "signed" determinant theory are still available [20]).

As an application of path-representation (1.4) for the ξ-function, we prove an analog of the Birman–Solomyak spectral averaging formula for a non-linear perturbation (Theorem 3.1) and use it to establish a universality of the spectral averaging (Theorem 3.5). For C^1-paths of self-adjoint elements in $\mathcal{A}$, we extend the Dixmier–Fuglede–Kadison differentiation rule [11, 12]

$$\frac{d}{dt}(\tau[f(H_t)]) = \tau[f'(H_t)\dot H_t] \tag{1.5}$$

by relaxing the usual analyticity requirement posed on the function f in a neighborhood of $\bigcup_t \sigma(H_t)$ to the one that f is absolutely continuous with the derivative of bounded variation. For those f's we prove that the function $t\mapsto\tau[f(H_t)]$ is differentiable almost everywhere and (1.5) holds t-a.e. (Theorem 4.1).

As a consequence, in Corollary 4.3 we obtain an extension of the spectral averaging formula (3.1). One of the prerequisites for the proof, a trace formula, can be found in Appendix A. We remark that it is the Birman–Solomyak formula that plays a fundamental role in the spectral shift function theory in the semi-finite setting recently developed in [3]. In particular, it is shown in [3] that the spectral shift function is a 1-form rather than a function of a spectral parameter. We also refer to [31, Chapter 8] where the path-dependent nature of the spectral shift function in the standard I_∞ setting was discussed.

Throughout this paper we assume that $\mathcal{A}$ is a finite von Neumann algebra and τ a normal faithful tracial state on $\mathcal{A}$.

2 The Perturbation Determinant

2.1 The Spectrum Distribution Function, the Abstract Lyapunov Exponent, and the Determinant Function

Given a self-adjoint element H in $\mathcal{A}$, let n_H denote *the spectrum distribution function*

$$n_H(\lambda) = \tau[E_H((-\infty, \lambda))], \quad \lambda \in \mathbb{R},$$

associated with H. Introduce the *abstract Lyapunov exponent* γ_H by setting

$$\gamma_H(z) = \log(\Delta(H - zI)), \quad z \in \mathbb{C},$$

with the natural convention that $\log(0) = -\infty$. Here log denotes the principal branch of the logarithm with the cut along the negative semi-axis, E_H the spectral measure of H, and Δ the Fuglede–Kadison determinant defined by (*cf.* [12, 16])

$$\Delta(A) = \lim_{\varepsilon \downarrow 0} \exp(\tau[\log((A^*A + \varepsilon I)^{1/2})]), \quad A \in \mathcal{A}.$$

The abstract Lyapunov exponent and the spectrum distribution function are related by a variant of the Thouless formula [30] (see [8, Theorem 3.12] and the discussion following).

Lemma 2.1 *The function $\gamma_H(z) = \log\big(\Delta(H - zI)\big)$, $z \in \mathbb{C}$, is subharmonic and admits the representation*

$$\gamma_H(z) = \int_{\mathbb{R}} \log|z - \mu| \, dn_H(\mu), \quad z \in \mathbb{C}. \tag{2.1}$$

Moreover,

$$\frac{1}{2\pi}\Big(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{z}^2}\Big)\gamma_H = dn_H,$$

where the Laplacian is taken in the sense of distributions.

Next, we introduce *the determinant function* d_H, associated with a self-adjoint element H in $\mathcal{A}$ by

$$d_H(z) = \exp(\tau[\log(zI - H)]), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.2}$$

The theorem below shows that the function d_H admits an analytic continuation as a Herglotz function to the complex plane with the cut along the convex hull of the spectrum of H. (For the notion of Herglotz functions and their basic properties we refer to [15, Section 2] and the references therein).

Theorem 2.2 *Let H be a self-adjoint element in $\mathcal{A}$. Then*

$$d_H(z) = \exp\Big(\int_{\mathbb{R}} \log(z - \lambda) \, dn_H(\lambda)\Big), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.3}$$

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and

$$\operatorname{Im}(z) \cdot \operatorname{Im}(d_H(z)) > 0, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.4}$$

that is, d_H is a Herglotz function.

Moreover, the determinant function d_H initially defined by (2.2) *away from the real axis admits an analytic continuation to the complex plane with the cut along the interval* $K = \operatorname{conv\,hull}(\sigma(H))$. *For this analytic continuation (denoted by the same symbol) the Herglotz representation*

$$d_H(z) = z - \tau(H) + \int_K \frac{d\mu(\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus K \tag{2.5}$$

holds, with μ an absolutely continuous measure of total mass

$$\int_K d\mu(\lambda) = \frac{\tau(H^2) - (\tau(H))^2}{2}. \tag{2.6}$$

Proof Applying the Spectral Theorem, one obtains that

$$d_H(z) = \exp(w(z)), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where w is the Herglotz function given by

$$w(z) = \int_{\mathbb{R}} \log(z - \lambda) \, dn_H(\lambda), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.7}$$

and, therefore, d_H is analytic in $\mathbb{C} \setminus \mathbb{R}$. Since the measure dn_H is a probability measure, from (2.7) it follows that

$$0 < \operatorname{Im}(w(z)) < \pi, \quad \operatorname{Im}(z) > 0,$$
$$-\pi < \operatorname{Im}(w(z)) < 0, \quad \operatorname{Im}(z) < 0,$$

and, therefore, (2.4) holds.

Since the normal boundary values $w(\lambda \pm i0)$ of the function w meet the requirements $w(\lambda + i0) = w(\lambda - i0)$, for $\lambda > \sup(\sigma(H))$, and $w(\lambda + i0) = w(\lambda - i0) + 2\pi i$, for $\lambda < \inf(\sigma(H))$, the function $d_H(z) = \exp(w(z))$ admits an analytic continuation (denoted by d_H) from $\mathbb{C} \setminus \mathbb{R}$ to the domain $\mathbb{C} \setminus K$.

Next, since K is a bounded set and the Herglotz function d_H is analytic in $\mathbb{C} \setminus K$ with $\operatorname{Im}(d_H(\lambda + i0)) = 0$, $\lambda \in \mathbb{R} \setminus K$, the determinant function admits the Herglotz representation

$$d_H(z) = Az + B + \int_K \frac{d\mu(\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.8}$$

for some $A \geq 0$, $B \in \mathbb{R}$, and some finite measure μ supported on the finite interval K. In particular, the asymptotic representation

$$d_H(z) = Az + B - \frac{1}{z} \int_K d\mu(\lambda) + \mathcal{O}\Big(\frac{1}{z^2}\Big), \quad \text{as } z \to \infty, \tag{2.9}$$

holds. On the other hand, from exponential representation (2.3), one concludes that

$$(2.10)\quad d_H(z) = z - \int_{\mathbb{R}} \lambda dn_H(\lambda) - \frac{1}{2z}\Big[\int_{\mathbb{R}} \lambda^2\, dn_H(\lambda) - \Big(\int_{\mathbb{R}} \lambda\, dn_H(\lambda)\Big)^2\Big] + \mathcal{O}\Big(\frac{1}{z^2}\Big),$$

as $z \to \infty$. Comparing asymptotic expansions (2.9) and (2.10) yields

$$(2.11)\qquad A = 1,$$

$$(2.12)\qquad B = -\int \lambda\, dn_H(\lambda) = -\tau(H),$$

$$(2.13)\qquad \int_K d\mu(\lambda) = \frac{1}{2}\Big[\int_{\mathbb{R}} \lambda^2\, dn_H(\lambda) - \Big(\int \lambda\, dn_H(\lambda)\Big)^2\Big] = \frac{1}{2}\Big(\tau(H^2) - (\tau(H))^2\Big),$$

thereby proving representations (2.5) and (2.6).

Using representation (2.1), one derives the estimate

$$\sup_{\substack{\lambda\in K,\\ 0\le\varepsilon\le 1}} \gamma_H(\lambda + \mathrm{i}\varepsilon) \le \frac{1}{2}\log((\operatorname{diam} K)^2 + 1),$$

and, therefore,

$$\operatorname{Im}(d_H(\lambda + \mathrm{i}\varepsilon)) \le |d_H(\lambda + \mathrm{i}\varepsilon)| \le \sqrt{(\operatorname{diam} K)^2 + 1}, \quad \lambda \in K,$$

which, by Fatou's theorem, ensures the absolute continuity of the measure μ. Combining (2.8), (2.11), (2.12), and (2.13) completes the proof. ■

The next result allows one to represent the spectrum distribution function and the abstract Lyapunov exponent via the normal boundary values of the determinant function on $\mathbb{R}$.

Theorem 2.3 *For all $\lambda \in \mathbb{R}$, both of the limits*

$$\lim_{\varepsilon\downarrow 0} \arg d_H(\lambda + \mathrm{i}\varepsilon) \quad \textit{and} \quad \lim_{\varepsilon\downarrow 0} \log |d_H(\lambda + \mathrm{i}\varepsilon)|$$

exist, and

$$(2.14)\qquad \frac{n_H(\lambda) + n_H(\lambda + 0)}{2} = \pm\frac{1}{\pi}\lim_{\varepsilon\downarrow 0} \arg(d_H(\lambda \pm \mathrm{i}\varepsilon)),$$

$$(2.15)\qquad \gamma_H(\lambda) = \lim_{\varepsilon\downarrow 0} \log |d_H(\lambda \pm \mathrm{i}\varepsilon)|.$$

Proof Integrating by parts yields

$$\int_{\mathbb{R}} \operatorname{Im}(\log(\lambda + \mathrm{i}\varepsilon - \mu))\, dn_H(\mu) = \int_{\mathbb{R}} \frac{\varepsilon}{(\lambda-\mu)^2+\varepsilon^2} n_H(\mu)\, d\mu$$
$$= \pi(\phi_\varepsilon * n_H)(\lambda),$$

where $\phi_\varepsilon(\lambda)$ is an approximate identity,

$$\phi_\varepsilon(\lambda) = \varepsilon^{-1}\phi(\varepsilon^{-1}\lambda), \qquad \text{with } \varphi(\lambda) = \frac{1}{\pi}\frac{1}{1+\lambda^2}.$$

Hence,

$$\lim_{\varepsilon\downarrow 0} \arg(d_H(\lambda + \mathrm{i}\varepsilon)) = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} \operatorname{Im}(\log(\lambda + \mathrm{i}\varepsilon - \mu))\, dn_H(\mu)$$
$$= \pi \lim_{\varepsilon\downarrow 0}(\phi_\varepsilon * n_H)(\lambda) = \pi\frac{n_H(\lambda) + n(\lambda+0)}{2},$$

proving (2.14). Taking into account that by Lemma 2.1

$$\lim_{\varepsilon\downarrow 0} \log|d_H(\lambda \pm \mathrm{i}\varepsilon)| = \log(\Delta(\lambda I - H)) = \gamma_H(\lambda),$$

one gets (2.15), completing the proof. ■

Corollary 2.4 *For all $\lambda \in \mathbb{R}$, the limits $d_H(\lambda \pm \mathrm{i}0) = \lim_{\varepsilon\downarrow 0} d_H(\lambda \pm \mathrm{i}\varepsilon)$ exist and*

$$d_H(\lambda \pm \mathrm{i}0) = \exp\Big(\pm\pi\mathrm{i}\frac{n_H(\lambda)+n_H(\lambda+0)}{2} + \gamma_H(\lambda)\Big), \quad \lambda \in \mathbb{R}. \tag{2.16}$$

Proof As far as the boundary values from the upper half-plane are concerned, one simply needs to restate Theorem 2.3 in the form

$$d_H(\lambda + \mathrm{i}0) = \exp\Big(\pi\mathrm{i}\frac{n_H(\lambda)+n_H(\lambda+0)}{2} + \gamma_H(\lambda)\Big), \quad \lambda \in \mathbb{R}.$$

To complete the proof, one remarks that

$$d_H(\lambda - \mathrm{i}0) = \overline{d_H(\lambda + \mathrm{i}0)}, \quad \lambda \in \mathbb{R},$$

for d_H is a Herglotz function. ■

Remark 2.5 As an immediate consequence of representation (2.16), one obtains that the Radon–Nykodim derivative of the absolute continuous measure μ referred to in Theorem 2.2 is given by

$$\frac{d\mu}{d\lambda}(\lambda) = \frac{\sin(\pi n_H(\lambda))}{\pi}\, e^{\gamma_H(\lambda)} \quad \text{for a.e. } \lambda \in K.$$

We also note that since the inequality $0 < n_H(\lambda) < 1$ holds for all λ in the interior of the cut $K = \operatorname{conv\,hull}(\sigma(H))$, from (2.16) it follows that the boundary values of the determinant function d_H on the upper rim of the cut K are different from those on the lower one. In other words, the cut K is unremovable, and therefore, the domain $\mathbb{C} \setminus K$ is the maximal planar domain where the Herglotz function d_H is analytic.

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The concluding result in this section relates the determinant function to the de la Harpe–Skandalis determinant [17].

***Theorem 2.6* (A path representation)** *Suppose that* $[0,1] \ni t \to H_t \in \mathcal{A}$ *is a continuous piecewise* C^1*-path of self-adjoint elements joining* $H_0 = I$ *and* $H_1 = H$. *Then*

$$d_H(z) = (z-1)\exp\Big(\int_0^1 \tau\big[\dot{H}_t(H_t - zI)^{-1}\big]\, dt\Big), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Proof Let $t_0 = 0 < t_1 < \cdots < t_{N-1} < t_N = 1$ be a partition of the interval $[0,1]$ such that $[t_k, t_{k+1}] \ni t \to H_t$ is a C^1-path, $k = 1, \dots, N-1$, for some $N \in \mathbb{N}$. By the Fuglede–Kadison–Dixmier differentiation formula applied to the logarithmic function log, one obtains

$$\frac{d}{dt}\tau[\log(zI - H_t)] = \tau[\dot{H}_t(H_t - zI)^{-1}],$$
$$t \in [t_k, t_{k+1}], \quad k = 0, \dots, N-1.$$

By the Newton–Leibniz rule,

$$\begin{aligned} (2.17)\quad & \tau[\log(zI - H)] - \tau[\log(z-1)I] \\ &= \sum_{k=0}^{N-1}(\tau[\log(zI - H_{t_{k+1}})] - \tau[\log(zI - H_{t_k})]) \\ &= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \tau[\dot{H}_t(H_t - zI)^{-1}]\, dt = \int_0^1 \tau[\dot{H}_t(H_t - zI)^{-1}]\, dt. \end{aligned}$$

One proves the claim by exponentiating on both sides in (2.17) and then taking into account that $d_H(z) = \exp(\tau[\log(zI - H)])$. ■

Corollary 2.7 *Under the hypotheses of Theorem* 2.6, *the representation*

$$\frac{n_H(\lambda) + n_H(\lambda+0)}{2} = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\operatorname{Im}\Big(\int_0^1 \tau[\dot{H}_t(H_t - (\lambda + i\varepsilon)I)^{-1}]\, dt\Big) + \delta(\lambda)$$

holds, with

$$\delta(\lambda) = \begin{cases} 1, & \lambda < 1, \\ \frac{1}{2}, & \lambda = 1, \\ 0, & \lambda > 1. \end{cases}$$

2.2 The Perturbation Determinant

Given a pair (H_0, H) of self-adjoint elements in $\mathcal{A}$, we introduce $\det_{H/H_0}$, *the perturbation determinant*, as the analytic function on the complex plane with the cut K given by

$$(2.18)\qquad K = \operatorname{conv\,hull}(\sigma(H)) \cup \operatorname{conv\,hull}(\sigma(H_0)),$$

uniquely determined by the properties

(i) $|\det_{H/H_0}(z)| = \Delta((H - zI)(H_0 - zI)^{-1}), \quad z \in \mathbb{C} \setminus K,$
(ii) $\lim_{y\to+\infty} \det_{H/H_0}(\mathrm{i}y) = 1,$

with Δ the Fuglede–Kadison determinant.

The following lemma shows that the function $\det_{H/H_0}$ is well defined.

Lemma 2.8 *Let d_H and d_{H_0} be the determinant functions associated with the operators H and H_0, respectively. Then the function*

$$d(z) = \frac{d_H(z)}{d_{H_0}(z)}, \quad z \in \mathbb{C} \setminus K,$$

satisfies the properties (i) *and* (ii) *above.*

Proof Since the determinant function d_{H_0} does not vanish outside of the convex hull of $\sigma(H_0)$, from Lemma 2.2 it follows that the function d is analytic on the domain $\mathbb{C} \setminus K$. Moreover, using the multiplicativity property of the Fuglede–Kadison determinant, one obtains that

$$|d_H(z)| = \frac{|d_H(z)|}{|d_{H_0}(z)|} = \frac{\Delta(H - zI)}{\Delta(H_0 - zI)} = \Delta((H - zI)(H_0 - zI)^{-1}), \quad z \in \mathbb{C} \setminus K,$$

which, along with utilizing asymptotic expansion (2.9) for the determinant functions d_H and d_{H_0}, yields $\lim_{y\to+\infty} \det_{H/H_0}(\mathrm{i}y) = 1$, completing the proof. ∎

Remark 2.9 The perturbation determinant provided by Lemma 2.8 satisfies the "traditional" chain rule. That is, for any self-adjoint elements H_0, H_1, and H in $\mathcal{A}$ we have

$$\det_{H/H_0}(z) = \det_{H/H_1}(z) \cdot \det_{H_1/H_0}(z),$$

$z \in \mathbb{C} \setminus (\operatorname{conv\,hull}(\sigma(H)) \cup \operatorname{conv\,hull}(\sigma(H_0)) \cup \operatorname{conv\,hull}(\sigma(H_1))).$

Our next result provides several exponential representations for the perturbation determinant that may also be considered the alternative definitions of the analytic function $\det_{H/H_0}$.

It is convenient to introduce the following hypothesis.

Hypothesis 2.10 *Assume that H_0 and H are self-adjoint elements in $\mathcal{A}$. Suppose that ξ is the spectral shift function associated with the pair (H_0, H), that is,*

$$\xi(\lambda) = n_{H_0}(\lambda) - n_H(\lambda), \quad \lambda \in \mathbb{R},$$

and γ is the difference of the abstract Lyapunov exponents associated with H_0 and H, that is,

$$\gamma(\lambda) = \gamma_{H_0}(\lambda) - \gamma_H(\lambda) \quad \textit{for a.e. } \lambda \in \mathbb{R}.$$

Assume, in addition, that $[0, 1] \ni t \to H_t \in \mathcal{A}$ is a continuous piecewise C^1-path of self-adjoint elements joining H_0 with $H_1 = H$.

Theorem 2.11 *Assume Hypothesis* 2.10. *Then*

(i) *(Krein's exponential representation)*

$$\det_{H/H_0}(z) = \exp\Big(\int_K \frac{\xi(\lambda)}{\lambda - z}\, d\lambda \Big), \quad z \in \mathbb{C} \setminus K; \tag{2.19}$$

(ii)

$$\det_{H/H_0}(z) = \exp\Big(\operatorname{sign}(\operatorname{Im}(z)) \frac{\mathrm{i}}{\pi} \int_{\mathbb{R}} \frac{\gamma(\lambda)}{\lambda - z}\, d\lambda \Big), \quad z \in \mathbb{C} \setminus \mathbb{R}; \tag{2.20}$$

(iii) *(A path representation)*

$$\det_{H/H_0}(z) = \exp\Big(\int_0^1 \tau\big[\dot{H}_t (H_t - zI)^{-1}\big]\, dt \Big), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Proof (i) By Lemma 2.8 and Theorem 2.2,

$$\det_{H/H_0}(z) = \frac{d_H(z)}{d_{H_0}(z)} = \exp\Big(- \int_K \log(z - \lambda) d\xi(\lambda) \Big),$$

which proves (2.19) by integrating by parts.

(ii) From Lemma 2.1 it follows that the functions πn_{H_0} and γ_{H_0} are harmonic conjugates to each other. Since $\xi \in L^\infty(\mathbb{R})$ and $\xi(\lambda) = 0$ for $\lambda \in \mathbb{R} \setminus K$, one concludes that $\xi \in L^2(\mathbb{R})$. Therefore, $\gamma \in L^2(\mathbb{R})$ since the Hilbert transform $\mathbf{H}$ is bounded in $L^2(\mathbb{R})$, so that the integral on the right-hand side of (2.20) is well defined. One gets

$$\begin{aligned}
\int_K \frac{\xi(\lambda)}{\lambda - z} d\lambda &= (\xi, (\,\cdot\, - \overline{z})^{-1})_{L^2(\mathbb{R})} = \frac{1}{\pi}\big(\mathbf{H}\gamma, (\,\cdot\, - \overline{z})^{-1}\big)_{L^2(\mathbb{R})} \\
&= -\frac{1}{\pi}\big(\gamma, \mathbf{H}(\,\cdot\, - \overline{z})^{-1}\big)_{L^2(\mathbb{R})} \\
&= -\operatorname{sign}(\operatorname{Im}(z)) \frac{1}{\pi}\big(\gamma, \mathrm{i}(\,\cdot\, - \overline{z})^{-1}\big)_{L^2(\mathbb{R})} \\
&= \operatorname{sign}(\operatorname{Im}(z)) \frac{\mathrm{i}}{\pi}\big(\gamma, (\,\cdot\, - \overline{z})^{-1}\big)_{L^2(\mathbb{R})} \\
&= \operatorname{sign}(\operatorname{Im}(z)) \frac{\mathrm{i}}{\pi} \int_R \frac{\gamma(\lambda)}{\lambda - z}\, d\lambda,
\end{aligned}$$

which proves (2.20). Here we used the fact that the operator $\mathbf{H}$ is anti-self-adjoint in $L^2(\mathbb{R})$ and that the function $(\,\cdot\, - \overline{z})^{-1}$ is an eigenfunction of $\mathbf{H}$ with the eigenvalue $\mathrm{i} \cdot \operatorname{sign}(\operatorname{Im}(z))$.

(iii) is a direct consequence of Theorem 2.6 and Lemma 2.8. ∎

Below we derive an analog of Krein's exponential representation for the spectral shift function via the perturbation determinant as well as via a path-integral.

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Theorem 2.12 *Assume Hypothesis* 2.10. *Then for all $\lambda \in \mathbb{R}$, the normal boundary values* $\lim_{\varepsilon\downarrow 0} \arg(\det_{H/H_0}(\lambda + i\varepsilon))$ *of the perturbation determinant* $\det_{H/H_0}$ *exist and*

$$\frac{\xi(\lambda) + \xi(\lambda + 0)}{2} = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \arg(\det_{H/H_0}(\lambda + i\varepsilon)), \quad \lambda \in \mathbb{R}. \tag{2.21}$$

In particular,

$$\frac{\xi(\lambda) + \xi(\lambda + 0)}{2} = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_0^1 \operatorname{Im}\left(\tau[\dot{H}_t(H_t - (\lambda + i\varepsilon)I)^{-1}]\, dt\right), \quad \lambda \in \mathbb{R}, \tag{2.22}$$

whenever $[0, 1] \ni t \to H_t \in \mathcal{A}$ *is a continuous piecewise C^1-path of self-adjoint elements joining H_0 with $H_1 = H$.*

Proof Representation (2.21) directly follows from Theorem 2.3 and the definition of the perturbation determinant. In turn, (2.22) is a consequence of representation (2.21) and Theorem 2.11(iii). ∎

Remark 2.13 We remark that representation (2.21) on the set of λ of full measure is also guaranteed by the standard properties of the Poisson kernel.

Remark 2.14 Let $\mathcal{A}$ be a semi-finite von Neumann algebra, τ a normal faithful semi-finite trace on it, $L_1(\mathcal{A}, \tau)$ the noncommutative L_1-space associated with $(\mathcal{A}, \tau)$, and $[0, 1] \ni t \to H_t \in L_1(\mathcal{A}, \tau) \cap \mathcal{A}$ a path continuous and piecewise-C^1 in the norm $\|\cdot\|_1 + \|\cdot\|$, with $\|A\|_1 = \tau(|A|)$ for $A \in L_1(\mathcal{A}, \tau)$ and $\|\cdot\|$ the operator norm. Then for a.e. $\lambda \in \mathbb{R}$, the ξ-function $\xi(\lambda, H_0, H)$ associated with the pair of operators H_0 and $H = H_1$ [4, 9] admits the representation

$$\xi(\lambda, H, H_0) = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_0^1 \operatorname{Im}\left(\tau\left[\dot{H}_t(H_t - (\lambda + i\varepsilon)I)^{-1}\right] dt\right). \tag{2.23}$$

Representation (2.23) can be proved by expressing $\xi(\lambda, H, H_0)$ in terms of the operator arguments of H_0 and H (see [29, Corollary 2.4]) and by using the identity

$$\frac{d}{dt}\tau\left[\log(H_t - (\lambda + i\varepsilon)I) - \log(-(\lambda + i\varepsilon)I)\right] = \tau\left[\dot{H}_t(H_t - (\lambda + i\varepsilon)I)^{-1}\right]$$

(see [29, Lemma 2.5]).

Remark 2.15 We remark that from Theorem 2.3 it follows that the boundary values $\det_{H/H_0}(\lambda \pm i0) = \lim_{\varepsilon\downarrow 0} \det_{H/H_0}(\lambda \pm i\varepsilon)$ can be computed explicitly

$$\det_{H/H_0}(\lambda \pm i0) = e^{\pm\pi i\frac{\xi(\lambda)+\xi(\lambda+0)}{2}} \Delta\left((H - \lambda I)(H_0 - \lambda I)^{-1}\right),$$

for all $\lambda \in \mathbb{R}$ such that at least one of the abstract Lyapunov exponents $\gamma_H(\lambda)$ or $\gamma_{H_0}(\lambda)$ is finite, and therefore, for a.e. $\lambda \in \mathbb{R}$,

$$\lim_{\varepsilon\downarrow 0} \det_{H/H_0}(\lambda \pm i\varepsilon) = e^{\pm\pi i\xi(\lambda)} \Delta\left((H - \lambda I)(H_0 - \lambda I)^{-1}\right).$$

In particular, the perturbation determinant $\det_{H/H_0}$ admits an analytic continuation from the domain $\mathbb{C} \setminus K$ to a (possibly) larger multiconnected planar domain

$$\mathcal{D} = (\mathbb{C} \setminus K) \cup \{\lambda \in \rho(H_0) \cap \rho(H) \mid \xi(\lambda) = 0\}.$$

Moreover,

$$\det_{H/H_0}(\lambda) = \Delta\big((H - \lambda I)(H_0 - \lambda I)^{-1}\big), \quad \lambda \in \mathcal{D} \cap \mathbb{R}.$$

This observation shows that, in contrast to the case of the determinant function d_H (see Remark 2.5), the initial cut along the interval K given by (2.18) might eventually be shrunk to a "smaller" set

$$\widetilde{K} = K \setminus \{\lambda \in \rho(H_0) \cap \rho(H) \mid \xi(\lambda) = 0\}.$$

We conclude this section with characterization of the zero set of the ξ-function intersected with the resolvent sets of H_0 and H which allows one to understand better the topological nature of the boundary $\widetilde{K}$ of the analyticity domain of the perturbation determinant.

Theorem 2.16 *Assume that $[0,1] \ni t \mapsto H_t$ is a continuous path of self-adjoint operators in $\mathcal{A}$ connecting the end points H_0 and $H_1 = H$ and $\lambda \in \mathbb{R}$ such that*

$$\lambda \in \rho(H_t) \quad \textit{for all } t \in [0,1]. \tag{2.24}$$

Then

$$\xi(\lambda) = 0. \tag{2.25}$$

Proof We split the proof into three steps.

Step 1. Assume that the path $[0,1] \ni t \mapsto H_t$ is of class C^1. Applying Theorem 2.11(iii), under hypothesis (2.24) one gets that

$$\xi(\lambda) = \int_0^1 \operatorname{Im}\big(\tau[(H_t - \lambda I)^{-1}\dot{H}_t]\big)\, dt.$$

Taking into account that $\operatorname{Im}(\tau[AB]) = 0$ whenever A and B are self-adjoint operators in $\mathcal{A}$, we obtain that $\xi(\lambda) = 0$, proving (2.25).

Step 2. Assume that $\|H_1 - H_0\| < \operatorname{dist}(\sigma(H_0), \lambda)$. Then (2.25) holds by Step 1 applied to the smooth path $[0,1] \ni t \mapsto H_0 + t(H_1 - H_0)$.

Step 3 (the general case). Since by hypothesis the path $[0,1] \ni t \mapsto H_t$ is continuous, there exists a natural number N such that

$$\big\| H_{\frac{k+1}{N}} - H_{\frac{k}{N}} \big\| < \inf_{t \in [0,1]} \operatorname{dist}(\sigma(H_t), \lambda), \quad k = 0, 1, \dots, N-1.$$

The spectral shift functions associated with the pairs $(H_{\frac{k}{N}}, H_{\frac{k+1}{N}})$, $k = 0, 1, \dots, N-1$, vanish at the point λ (by Step 2), and, therefore,

$$n_{H_{\frac{k+1}{N}}}(\lambda) = n_{H_{\frac{k}{N}}}(\lambda), \quad k = 0, 1, \dots, N-1.$$

Hence (2.25) holds, completing the proof of the theorem. ∎

In the case when $\mathcal{A}$ is a finite-type factor the converse is also true.

Theorem 2.17 *Assume that H_0 and H are self-adjoint elements in a finite-type factor $\mathcal{A}$ and $\xi(\lambda) = 0$ for some $\lambda \in \rho(H_0) \cap \rho(H) \cap \mathbb{R}$. Then there exists a continuous path $[0,1] \ni t \mapsto H_t$ of self-adjoint operators connecting the end points H_0 and $H_1 = H$ such that $\lambda \in \rho(H_t)$ for all $t \in [0,1]$.*

Proof Without loss of generality, assume that $\lambda = 0$. Since for every boundedly invertible self-adjoint operator H in $\mathcal{A}$, the continuous path of boundedly invertible self-adjoint operators given by $[0,1] \ni s \mapsto J_s = s(H - \operatorname{sign}(H)) + \operatorname{sign}(H)$, connects H and the signature operator $\operatorname{sign}(H) = (-E_H(\mathbb{R}_-)) \oplus E_H((0,\infty))$, it is sufficient to prove the assertion in the particular case of $H_j = \operatorname{sign}(H_j)$, $j = 0, 1$. Assume that this is the case.

Since $\xi(0) = 0$, and hence $n_H(0) = n_{H_0}(0)$, the projections $E_{H_0}(\mathbb{R}_-)$ and $E_H(\mathbb{R}_-)$ are equivalent relative to $\mathcal{A}$. Therefore, there exists a unitary operator $U \in \mathcal{A}$ such that $E_H(\mathbb{R}_-) = UE_{H_0}(\mathbb{R}_-)U^{-1}$ (see [11, III.2.3, Proposition 6]), and hence,

$$\operatorname{sign}(H) = U\ \operatorname{sign}(H_0)\, U^{-1}.$$

Let $U = e^{iA}$ for some self-adjoint bounded operator A with spectrum in the interval $[0, 2\pi]$ such that the point 2π is not an eigenvalue of A. Then

$$[0,1] \ni t \mapsto S_t = e^{iAt}[\operatorname{sign}(H_0)]e^{-iAt}$$

is a continuous path of self-adjoint boundedly invertible operators connecting the operators $\operatorname{sign}(H_0)$ and $\operatorname{sign}(H_1)$. ■

The following example shows that the assumption in Theorem 2.17 that $\mathcal{A}$ is a factor cannot be relaxed.

Example 2.18 Let $\mathcal{A}$ be the commutative von Neumann algebra of the 2×2 diagonal matrices, $\tau(A) = \frac{1}{2}\operatorname{tr}(A)$, with tr the standard matrix trace,

$$H_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly, the spectral shift function ξ associated with the pair (H_0, H) vanishes on $\mathbb{R}$, in particular, $\xi(0) = 0$. However, for any continuous path $[0,1] \ni t \to H_t \in \mathcal{A}$ connecting the end points H_0 and $H = H_1$, the diagonal matrix H_t is not invertible for at least one value of the parameter $t \in (0,1)$.

3 A Spectral Averaging

The main goal of this section is to obtain a finite von Neumann algebra variant of the spectral averaging formula associated with arbitrary C^1-paths of self-adjoint elements H_t in $\mathcal{A}$ which yields, in particular, the absolute continuity of a certain spectral measure averaged with respect to the parameter.

Theorem 3.1 *Assume that $[0,1] \ni t \to H_t$ is a C^1-path of self-adjoint elements connecting the end points H_0 and $H_1 = H$. Let ξ be the spectral shift function associated with the pair (H_0, H). Then for any Borel set $\delta \subset \mathbb{R}$, the representation*

$$\int_0^1 \tau[E_{H_t}(\delta)\dot{H}_t]\,dt = \int_\delta \xi(\lambda)\,d\lambda \tag{3.1}$$

holds.

Proof From Theorem 2.11(iii) it follows that for all $\alpha < \beta$ in $\mathbb{R}$,

$$\begin{aligned}\int_\alpha^\beta \xi(\lambda)d\lambda &= \int_\alpha^\beta \lim_{\varepsilon\downarrow 0}\int_0^1 \operatorname{Im}\,\tau\big[\dot{H}_t(H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}\big]\,dtd\lambda \\ &= \int_\alpha^\beta \lim_{\varepsilon\downarrow 0}\int_0^1 \tau\big[\dot{H}_t\ \operatorname{Im}\big((H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}\big)\big]\,dtd\lambda.\end{aligned} \tag{3.2}$$

Note that

$$\int_\alpha^\beta \tau[\dot{H}_t\ \operatorname{Im}(H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}]\,d\lambda = \tau\Big[\dot{H}_t\ \operatorname{Im}\int_\alpha^\beta (H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}\,d\lambda\Big].$$

Since

$$\int_\alpha^\beta (H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}d\lambda = \log(H_t-(\alpha+\mathrm{i}\varepsilon)I) - \log(H_t-(\beta+\mathrm{i}\varepsilon)I),$$

we have that for any $\varepsilon > 0$,

$$\Big\|\operatorname{Im}\int_\alpha^\beta (H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}\,d\lambda\Big\| \le 2\pi.$$

By this observation, using Lebesgue's dominated convergence theorem and the Fubini theorem, we can write the right-hand side of (3.2) as

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_\alpha^\beta\int_0^1 \tau\big[\dot{H}_t\ \operatorname{Im}\,(H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}\big]\,dtd\lambda \\ &\qquad= \lim_{\varepsilon\downarrow 0}\int_0^1\int_\alpha^\beta \tau\big[\dot{H}_t\ \operatorname{Im}\,(H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}\big]\,d\lambda dt.\end{aligned}$$

Applying one more time Lebesgue's dominated convergence theorem, we get

$$\int_\alpha^\beta \xi(\lambda)\,d\lambda = \int_0^1 \lim_{\varepsilon\downarrow 0}\int_\alpha^\beta \tau\big[\dot{H}_t\,\operatorname{Im}(H_t-(\lambda+\mathrm{i}\varepsilon)I)^{-1}\big]\,d\lambda\,dt.$$

Since τ is continuous in the strong operator topology, it follows from Stone's formula that

$$\int_\alpha^\beta \xi(\lambda)\,d\lambda = \frac{1}{2}\int_0^1 \tau\big[\dot{H}_t\big(E_{H_t}([\alpha,\beta]) + E_{H_t}((\alpha,\beta))\big)\big]\,dt. \tag{3.3}$$

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Taking the limit $\alpha \to \beta$ in (3.3), we get

$$\int_0^1 \tau\big[\dot{H}_t E_{H_t}(\{\beta\})\big]\, dt = 0.$$

Similarly, $\int_0^1 \tau[\dot{H}_t E_{H_t}(\{\alpha\})]\, dt = 0$, which proves the equality

$$\int_0^1 \tau\big[E_{H_t}(\delta)\dot{H}_t\big]\, dt = \int_\delta \xi(\lambda)\, d\lambda$$

for δ being an arbitrary (open, closed, semi-open) interval and, hence, for any Borel set $\delta \subset \mathbb{R}$. ■

Remark 3.2 Let $\mathcal{A}$ be a semi-finite von Neumann algebra, τ a normal faithful semi-finite trace on it, and $[0,1] \ni t \to H_t \in L_1(\mathcal{A},\tau) \cap \mathcal{A}$ a path of class C^1 in the norm $\|\cdot\|_1 + \|\cdot\|$, with $H = H_1$. Then for any Borel set $\delta \subset \mathbb{R}$, the representation

$$\int_0^1 \tau\big[E_{H_t}(\delta)\dot{H}_t\big]\, dt = \int_\delta \xi(\lambda, H, H_0)\, d\lambda \tag{3.4}$$

holds. The proof of (3.4) is an appropriate modification of the one of (3.1). In particular, a semi-finite analog of (3.2) is discussed in Remark 2.14.

We will also need a particular case of Theorem 3.1 stated in a slightly different form.

Corollary 3.3 *Under hypotheses of Theorem* 3.1*, for any* $\lambda \in \mathbb{R}$*,*

$$\int_0^1 \tau\big[E_{H_t}((-\infty,\lambda))\dot{H}_t\big]\, dt$$
$$= \tau\big[E_H((-\infty,\lambda))(H-\lambda I)\big] - \tau\big[E_{H_0}((-\infty,\lambda))(H_0-\lambda I)\big].$$

Proof Applying Theorem 3.1 to the Borel set $\delta = (-\infty,\lambda)$, one obtains the representation

$$\int_0^1 \tau\big[E_{H_t}((-\infty,\lambda))\dot{H}_t\big]\, dt = \int_{-\infty}^{\lambda} \xi(\mu)\, d\mu. \tag{3.5}$$

The left-hand side in (3.5), being understood as the Riemann–Stieltjes integral over the finite closed interval $[m,\lambda]$ with $m = \min\{\inf\sigma(H_0), \inf\sigma(H)\}$, can be evaluated by integrating by parts

$$\int_{-\infty}^{\lambda} \xi(\mu)\, d\mu = \int_{[m,\lambda]} \mu\, dn_H(\mu) - \int_{[m,\lambda]} \mu\, dn_{H_0}(\mu) + \lambda(n_{H_0}(\lambda) - n_H(\lambda)).$$

Therefore,

$$\int_{-\infty}^{\lambda} \xi(\mu)\, d\mu = \tau\big[E_H((-\infty,\lambda))(H-\lambda I)\big] - \tau\big[E_{H_0}((-\infty,\lambda))(H_0-\lambda I)\big]. \tag{3.6}$$

Combining (3.5) and (3.6) completes the proof. ■

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Remark 3.4 To the best of our knowledge, the credit for the first paper on spectral averaging belongs to V. A. Javrjan [18, 19]. M. Š. Birman and M. Z. Solomyak [7] derived (3.1) for a path $t \mapsto H_t = H_0 + tV$ under the assumption that the operator V is in the trace class. For nonlinear paths $t \mapsto H_t$ continuously differentiable in the standard trace class norm, the spectral averaging formula (3.1) was obtained in [28] in the case of a non-negative $\dot{H}_t$ and in [14] in the general case. We also refer to [1], where a spectral averaging of a measure and a separate averaging of its singular part in connection with the boundary behavior of inner functions in the unit disk are considered. An extensive list of additional references related to applications of the spectral averaging can be found in [13].

In the operator algebras context, (3.1) was proved in [2] for a linear path $t \mapsto H_t = H_0 + tV$ of self-adjoint operators affiliated with a semi-finite von Neumann algebra $\mathcal{A}$ equipped with normal faithful semi-finite trace τ and V a τ-trace class perturbation. We remark that the proof of (3.1) in [2] was based on the multiple Stieltjes operator integration theory originally developed in [10] and then extended to the case of semi-finite von Neumann algebras in [2].

As a consequence of Theorem 3.1, we establish a universality of the spectral averaging in the case when the path is a linear operator function of the parameter (see [13, 18, 19, 28]).

Theorem 3.5 *Assume that H and V are self-adjoint operators in $\mathcal{A}$ and* $\ker(V) = \{0\}$. *Suppose, in addition, that δ is a Borel subset of $\mathbb{R}$ of finite Lebesgue measure. Then*

$$\lim_{T\to\infty} \int_{-T}^{T} \tau[E_{H+tV}(\delta)V]dt = \tau[\operatorname{sign}(V)] \cdot |\delta|,$$

with $|\cdot|$ the Lebesgue measure.

Proof From Theorem 3.1 it follows that

$$\int_{-T}^{T} \tau[E_{H+tV}(\delta)V]\, dt = \int_{\delta} \xi_T(\lambda)\, d\lambda, \tag{3.7}$$

where $\xi_T(\lambda) = \tau[E_{(H-\lambda I)-TV}(\mathbb{R}_-)] - \tau[E_{(H-\lambda I)+TV}(\mathbb{R}_-)]$. Since

$$E_{(H-\lambda I)\pm TV}(\mathbb{R}_-) = E_{T^{-1}(H-\lambda I)\pm V}(\mathbb{R}_-), \quad T > 0,$$

and $\ker(V) = \{0\}$, one concludes by, e.g., [27, Theorem VIII.24] that

$$\operatorname*{s-lim}_{T\to\infty} E_{(H-\lambda I)\pm TV}(\mathbb{R}_-) = \operatorname*{s-lim}_{T\to\infty} E_{T^{-1}(H-\lambda I)\pm V}(\mathbb{R}_-) = E_{\pm V}(\mathbb{R}_-)$$

and hence,

$$\lim_{T\to\infty} \xi_T(\lambda) = \tau[E_{-V}(\mathbb{R}_-)] - \tau[E_V(\mathbb{R}_-)] = \tau[\operatorname{sign}(V)],$$

for the state τ is continuous in the strong operator topology. Since the bound

$$|\xi_T(\lambda)| \le 2, \quad \lambda \in \mathbb{R},$$

takes place and $|\delta|$ is finite, passing to the limit in (3.7) by Lebesgue's dominated convergence theorem one gets

$$\lim_{T\to\infty}\int_{-T}^{T}\tau\big[E_{H+tV}(\delta)V\big]\,dt=\int_{\delta}\lim_{T\to\infty}\xi_T(\lambda)\,d\lambda=\int_{\delta}\tau[\operatorname{sign}(V)]\,d\lambda$$
$$=\tau[\operatorname{sign}(V)]\cdot|\delta|,$$

completing the proof. ■

4 The Differentiation Rule

We recall that if $[0,1]\ni t\mapsto H_t$ is a continuously differentiable path of operators in $\mathcal{A}$ and $\mathfrak{K}$ is a compact set in the complex plane, including the spectrum of each of the operators H_t, the Dixmier–Fuglede–Kadison differentiation formula yields

$$\frac{d}{dt}(\tau[f(H_t)])=\tau[f'(H_t)\dot{H}_t],\tag{4.1}$$

provided that f is analytic on a neighborhood of the set $\mathfrak{K}$.

The main goal of this section is to extend differentiation rule (4.1) to a larger class of functions by relaxing the analyticity requirement posed on f. The extended rule will follow from the Newton–Leibniz formula provided by the theorem below.

Theorem 4.1 *Assume that $[0,1]\ni t\mapsto H_t$ is a C^1-path of self-adjoint operators in $\mathcal{A}$. Suppose, in addition, that f is absolutely continuous and its derivative f' has a representative of bounded variation on the convex hull of the set $\bigcup_{t\in[0,1]}\sigma(H_t)$. Then the Newton–Leibniz formula*

$$\int_0^1\tau\big[f'(H_t)\dot{H}_t\big]\,dt=\tau[f(H_1)]-\tau[f(H_0)]\tag{4.2}$$

holds.

Before proving Theorem 4.1, we discuss its two immediate consequences.

First, we have the following extension of differentiation rule (4.1).

Corollary 4.2 *Assume the hypothesis of Theorem* 4.1. *Then*

$$\frac{d}{dt}(\tau[f(H_t)])=\tau\big[f'(H_t)\dot{H}_t\big]\quad\text{for a.e. } t\in[0,1].$$

Proof It follows from Theorem 4.1 that

$$\int_0^t\tau\big[f'(H_s)\dot{H}_s\big]\,ds=\tau[f(H_t)]-\tau[f(H_0)],\quad t\in[0,1],$$

which along with Lebesgue's theorem on differentiation of the integral of an L^1-function completes the proof. ■

Secondly, we obtain an extension of the spectral averaging formula to the class of absolutely continuous functions with the derivative of bounded variation.

Corollary 4.3 *Assume that $[0,1] \ni t \mapsto H_t$ is a C^1-path of self-adjoint operators in $\mathcal{A}$. Suppose that g is a function of bounded variation on the set*

$$S = \overline{\bigcup_{t\in[0,1]} \sigma(H_t)}.$$

Then

$$\int_0^1 \tau\big[g(H_t)\dot{H}_t\big]\,dt = \int_S g(\lambda)\xi(\lambda)\,d\lambda. \tag{4.3}$$

Proof Let f be an absolutely continuous function given by

$$f(\lambda) = \int_{-\infty}^{\lambda} g(\mu)\chi_S(\mu)\,d\mu,$$

with χ_δ the indicator of the Borel set $\delta \subset \mathbb{R}$. Applying Krein's trace formula, Theorem A.1 (*cf.* [4, 22]), to the right-hand side of (4.2) yields

$$\int_0^1 \tau\big[f'(H_t)\dot{H}_t\big]\,dt = \int_{\mathbb{R}} f'(\lambda)\xi(\lambda)\,d\lambda. \tag{4.4}$$

To prove (4.3), one remarks that by Theorem 2.16 the ξ-function vanishes outside of the set S, that is, $\xi(\lambda) = 0$, $\lambda \in \mathbb{R} \setminus S$, and then (4.3) follows from (4.4). ∎

Remark 4.4 In the case of a linear path in a semi-finite von Neumann algebra, formula (4.3) was obtained in [3, Lemma 2.8] (see also [3, Proposition 3.5, Corollary 3.6] for identifying the left-hand side of (4.5) with an integral of a 1-form.

The proof of Theorem 4.1 proceeds in several steps. First, by an approximation argument, we prove the Newton–Leibniz formula (4.2) for continuously differentiable functions f on $\mathbb{R}$.

Lemma 4.5 *Assume the hypothesis of Theorem* 4.1. *Suppose, in addition, that the function f is continuously differentiable on $\mathbb{R}$. Then the Newton–Leibniz formula* (4.2) *holds.*

Proof Let $I = (a,b)$ be an open interval such that $\bigcup_{t\in[0,1]} \sigma(H_t) \subset I$. Fix $\varepsilon > 0$. Since f' is continuous, by the Weierstrass theorem there exists a polynomial p_ε such that $p_\varepsilon(a) = f(a)$,

$$\max_{x\in I} |f'(x) - p'_\varepsilon(x)| < \varepsilon, \tag{4.5}$$

and, therefore,

$$\max_{x\in I} |f(x) - p_\varepsilon(x)| \le |I|\varepsilon. \tag{4.6}$$

Since p_ε is an entire function, applying the Dixmier–Fuglede–Kadison differentiation formula

$$\frac{d}{dt}\tau[p_\varepsilon(H_t)] = \tau\big[p'_\varepsilon(H_t)\dot{H}_t\big]$$

yields

$$\int_0^1 \tau\big[p'_\varepsilon(H_t)\dot{H}_t\big]\,dt = \tau[p_\varepsilon(H_1)] - \tau[p_\varepsilon(H_0)]. \tag{4.7}$$

From (4.5) one concludes that the inequalities

$$\begin{aligned}\Big|\int_0^1 \tau\big[f'(H_t)\dot{H}_t\big]\,dt - \int_0^1 \tau\big[p'_\varepsilon(H_t)\dot{H}_t\big]\,dt\Big| &\\ \le \int_0^1 \big|\tau\big[(f'(H_t)-p'_\varepsilon(H_t))\dot{H}_t\big]\big|\,dt &\\ \le \max_{t\in[0,1]}\big|\tau\big[(f'(H_t)-p'_\varepsilon(H_t))\dot{H}_t\big]\big| &\\ \le \max_{t\in[0,1]}\|f'(H_t)-p'_\varepsilon(H_t)\|\cdot\max_{t\in[0,1]}\|\dot{H}_t\| \le \varepsilon\max_{t\in[0,1]}\|\dot{H}_t\| &\end{aligned} \tag{4.8}$$

hold. Applying (4.6), one concludes that

$$\Big|\big(\tau[f(H_1)]-\tau[f(H_0)]\big) - \big(\tau[p_\varepsilon(H_1)]-\tau[p_\varepsilon(H_0)]\big)\Big| \le 2\varepsilon|I|. \tag{4.9}$$

Combining (4.8), (4.9), and (4.7) yields the inequality

$$\Big|\int_0^1 \tau\big[f'(H_t)\dot{H}_t\big]\,dt - (\tau[f(H_1)]-\tau[f(H_0)])\Big| \le \varepsilon\big(\max_{t\in[0,1]}\|\dot{H}_t\| + 2|I|\big).$$

Since ε can be chosen arbitrarily small, one gets (4.2). ■

Next, applying the spectral averaging result provides the proof of Theorem 4.1 in its particular case when the derivative f' is a step function with finitely many discontinuity points.

Lemma 4.6 *Assume the hypothesis of Theorem* 4.1. *Suppose, in addition, that the derivative f' is a piecewise constant function with finitely many discontinuity points. Then the Newton–Leibniz formula* (4.2) *holds.*

Proof Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be the discontinuity points of the derivative f'. Then under the hypothesis, for some $\alpha_1, \alpha_2, \dots, \alpha_N$ one has the representations

$$f'(x) = \sum_{k=1}^N \alpha_k \chi_{(-\infty,\lambda_k)}(x) \quad \text{for a.e. } x \in \mathbb{R},$$

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and

$$f(x) = \sum_{k=1}^{N} \alpha_k \chi_{(-\infty,\lambda_k)}(x)(x-\lambda_k) + C, \quad x \in \mathbb{R},$$

where χ_δ denotes the characteristic function of the set δ and C is a constant.

Applying Corollary 3.3, by the Spectral Theorem one concludes that

$$\begin{aligned}\int_0^1 \tau\big[f'(H_t)\dot{H}_t\big]\,dt &= \sum_{k=1}^{N} \alpha_k \int_0^1 \tau\big[E_{H_t}((-\infty,\lambda_k))\dot{H}_t\big]\,dt \\ &= \sum_{k=1}^{N} \alpha_k \tau\big[E_H((-\infty,\lambda_k))(H-\lambda_k I) - E_{H_0}((-\infty,\lambda_k))(H_0-\lambda_k I)\big] \\ &= \tau[f(H_1)] - \tau[f(H_0)],\end{aligned}$$

completing the proof. ■

Finally, combining the results obtained above, we are ready to complete the proof of Theorem 4.1.

Proof of Theorem 4.1 Let $I = (a,b)$ be an open interval such that

$$\bigcup_{t\in[0,1]} \sigma(H_t) \subset I.$$

Assume, without loss of generality, that f' coincides with its representative of bounded variation on the closed interval $[a,b]$ and that f' is continuous at the point a. Let $f'(x) = h(x) + g(x)$ be the Lebesgue decomposition, where h is the continuous part of f' and g is a jump function, that is, the measure dg is a pure point Borel measure, so that $g(a) = 0$. Given $\varepsilon > 0$, the function g can be decomposed as

$$g(x) = \tilde{g}(x) + g_\varepsilon(x),$$

where $\tilde{g}$ is piecewise constant on I and $\mathrm{Var}_a^b(g_\varepsilon) < \varepsilon$ on I. Note that under our assumptions $g_\varepsilon(a) = 0$. Applying Lemmas 4.5 and 4.6, one arrives at

$$\begin{aligned}&\int_0^1 \tau\big[f'(H_t)\dot{H}_t\big]\,dt - \big(\tau[f(H_1)] - \tau[f(H_0)]\big) \\ &\qquad = \int_0^1 \tau\big[g_\varepsilon(H_t)\dot{H}_t\big]\,dt - \big(\tau[G_\varepsilon(H_1)] - \tau[G_\varepsilon(H_0)]\big),\end{aligned}$$

where $G_\varepsilon(x) = \int_a^x g_\varepsilon(t)\,dt$. Clearly, the inequality $\sup_{x\in I}|g_\varepsilon(x)| \leq \mathrm{Var}_a^b(g_\varepsilon) \leq \varepsilon$ holds and since $g_\varepsilon(a) = 0$, one also obtains that

$$\max_{x\in I} |G_\varepsilon(x)| \leq \mathrm{Var}_a^b(g_\varepsilon)\cdot|I| \leq \varepsilon|I|.$$

Using these estimates, exactly as in the proof of Lemma 4.5, one gets the inequalities

$$\left|\tau[G_\varepsilon(H_1)] - \tau[G_\varepsilon(H_0)]\right| \le 2\varepsilon|I| \quad \text{and} \quad \left|\int_0^1 \tau\big[g_\varepsilon(H_t)\dot{H}_t\big]\, dt\right| \le \varepsilon \max_{t\in[0,1]} \|\dot{H}_t\|.$$

Therefore,

$$\left|\int_0^1 \tau\big[f'(H_t)\dot{H}_t\big]\, dt - (\tau[f(H_1)] - \tau[f(H_0)])\right| \le \varepsilon\big(2|I| + \max_{t\in[0,1]} \|\dot{H}_t\|\big).$$

Since ε can be chosen arbitrarily small, one arrives at (4.2), completing the proof. ■

A The Trace Formula

In this Appendix we revisit the Lifshits–Krein trace formula

$$\operatorname{tr}[f(H) - f(H_0)] = \int_{\mathbb{R}} \xi(\lambda) df(\lambda)$$

associated with a pair of self-adjoint operators (H_0, H) and some suitable class of functions f, originally proved in [21] in the framework of trace class perturbation theory and further discussed in the context of a semi-finite von Neumann algebra in [2–4, 9].

As the following result shows, in the context of a finite von Neumann algebra the usual smoothness requirement on the function class can be relaxed to the one that f is of bounded variation only.

Theorem A.1 *Let H_0 and H be self-adjoint elements in $\mathcal{A}$ and f a function of bounded variation on $[a, b]$ such that the open interval (a, b) contains the spectra $\sigma(H_0) \cup \sigma(H)$ of the elements H and H_0. Denote by ξ the spectral shift function associated with the pair (H_0, H),*

$$\xi(\lambda) = \Big(\tau\big[E_{H_0}((-\infty, \lambda))\big] - \tau\big[E_H((-\infty, \lambda))\big]\Big). \tag{A.1}$$

Then

(i) $\tau[f(H) - f(H_0)] = \int_{\mathbb{R}} \xi(\lambda + 0)\, df(\lambda)$, *whenever f is continuous from the left;*
(ii) $\tau[f(H) - f(H_0)] = \int_{\mathbb{R}} \xi(\lambda)\, df(\lambda)$, *whenever f is continuous from the right;*
(iii)
$$\tau\big[f(H) - f(H_0)\big] = \int_{\mathbb{R}} \frac{\xi(\lambda) + \xi(\lambda + 0)}{2}\, df(\lambda),$$

whenever $f(x) = \frac{1}{2}\big(f(x - 0) + f(x + 0)\big)$ at every point of discontinuity of f.

Here the integrals are understood in the Lebesgue–Stieltjes sense.

In particular, if f is absolutely continuous on the interval $[a, b]$, then

$$\tau\big[f(H) - f(H_0)\big] = \int_{\mathbb{R}} f'(\lambda)\xi(\lambda)\, d\lambda.$$

Proof Applying the Spectral Theorem leads to the equality

$$\tau\left[f(H)-f(H_0)\right]=-\int_a^b f(\lambda)\,d\xi(\lambda).$$

Since $\xi(\lambda)=0$ for $\lambda\in\mathbb{R}\setminus(a,b)$, integrating by parts (see [24, Theorem 7.5.9]) yields

$$\int_a^b f(\lambda)\,d\xi(\lambda)=\int_{(a,b)}\xi(\lambda+0)\,df(\lambda)=\int_{\mathbb{R}}\xi(\lambda+0)\,df(\lambda)\quad\text{in case (i)}\tag{A.2}$$

and

$$\int_a^b f(\lambda)\,d\xi(\lambda)=\int_{(a,b)}\xi(\lambda)\,df(\lambda)=\int_{\mathbb{R}}\xi(\lambda)\,df(\lambda)\quad\text{in case (ii).}\tag{A.3}$$

Representation (iii) can be obtained by adding equations (A.2) and (A.3) and dividing by 2. ■

Trace formulae (i)–(iii) are particular cases of a more general "interpolation" result that does not require any specific behavior of the function f at its points of discontinuity.

Corollary A.2 *Assume the hypothesis of Theorem* A.1. *Denote by* Λ *the set of discontinuities points of* f. *Then*

$$\tau[f(H)-f(H_0)]=\int_{\mathbb{R}}\widetilde{\xi}_f(\lambda)\,df(\lambda),$$

where

$$\widetilde{\xi}_f(\lambda)=\begin{cases}\xi(\lambda), & \lambda\in\mathbb{R}\setminus\Lambda,\\ \dfrac{f^+(\lambda)-f(\lambda)}{f^+(\lambda)-f^-(\lambda)}\xi(\lambda+0)+\dfrac{f(\lambda)-f^-(\lambda)}{f^+(\lambda)-f^-(\lambda)}\xi(\lambda), & \lambda\in\Lambda,\end{cases}$$

with $f^{\pm}(\lambda)=f(\lambda\pm 0)$.

Proof Introduce the set $\widetilde{\Lambda}=(\sigma_{pp}(H)\cup\sigma_{pp}(H_0))\cap\Lambda$, where $\sigma_{pp}(A)$ stands for the point spectrum of an element $A\in\mathcal{A}$. Applying the Spectral Theorem, it is easy to see that

$$\begin{aligned}&\tau[f(H)-f(H_0)]-\tau[f^+(H)-f^+(H_0)]\\&\quad=\sum_{\lambda\in\widetilde{\Lambda}}[f^+(\lambda)-f(\lambda)](\xi(\lambda+0)-\xi(\lambda))\\&\quad=\sum_{\lambda\in\widetilde{\Lambda}}\frac{f^+(\lambda)-f(\lambda)}{f^+(\lambda)-f^-(\lambda)}(\xi(\lambda+0)-\xi(\lambda))[f^+(\lambda)-f^-(\lambda)]\\&\quad=\int_{\widetilde{\Lambda}}\frac{f^+(\lambda)-f(\lambda)}{f^+(\lambda)-f^-(\lambda)}(\xi(\lambda+0)-\xi(\lambda))\,df(\lambda).\end{aligned}\tag{A.4}$$

Since the ξ-function given by (A.1) is continuous on $\mathbb{R} \setminus (\sigma_{pp}(H) \cup \sigma_{pp}(H_0))$, one can spread integration in (A.4) from the set $\widetilde{\Lambda}$ to the whole "discontinuity" set Λ, thus obtaining that

$$\text{(A.5)} \quad \tau[f(H) - f(H_0)] = \tau[f^+(H) - f^+(H_0)] + \int_\Lambda \frac{f^+(\lambda) - f(\lambda)}{f^+(\lambda) - f^-(\lambda)} (\xi(\lambda + 0) - \xi(\lambda))\, df(\lambda).$$

By Theorem A.1(ii),

$$\text{(A.6)} \quad \tau\left[f^+(H) - f^+(H_0)\right] = \int_{\mathbb{R}} \xi(\lambda)\, df(\lambda) = \int_\Lambda \xi(\lambda)\, df(\lambda) + \int_{\mathbb{R}\setminus\Lambda} \xi(\lambda)\, df(\lambda),$$

and, hence, combining (A.5) and (A.6) yields

$$\begin{aligned} \tau[f(H) - f(H_0)] &= \int_{\mathbb{R}\setminus\Lambda} \xi(\lambda)\, df(\lambda) \\ &\quad + \int_\Lambda \Big(\xi(\lambda) + \frac{f^+(\lambda) - f(\lambda)}{f^+(\lambda) - f^-(\lambda)} (\xi(\lambda + 0) - \xi(\lambda)) \Big)\, df(\lambda) \\ &= \int_{\mathbb{R}} \widetilde{\xi}_f(\lambda)\, df(\lambda), \end{aligned}$$

completing the proof. ■

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