Canad. J. Math. Vol. **66** (3), 2014 pp. 566–595
http://dx.doi.org/10.4153/CJM-2012-063-1

Transfer of Plancherel Measures for Unitary Supercuspidal Representations between p-adic Inner Forms

Kwangho Choiy

Abstract. Let F be a p-adic field of characteristic 0, and let M be an F-Levi subgroup of a connected reductive F-split group such that $\Pi_{i=1}^{r} \mathrm{SL}_{n_i} \subseteq M \subseteq \Pi_{i=1}^{r} \mathrm{GL}_{n_i}$ for positive integers r and n_i. We prove that the Plancherel measure for any unitary supercuspidal representation of $M(F)$ is identically transferred under *the local Jacquet–Langlands type correspondence* between M and its F-inner forms, assuming a working hypothesis that Plancherel measures are invariant on a certain set. This work extends the result of Muić and Savin (2000) for Siegel Levi subgroups of the groups SO_{4n} and Sp_{4n} under the local Jacquet–Langlands correspondence. It can be applied to a simply connected simple F-group of type E_6 or E_7, and a connected reductive F-group of type A_n, B_n, C_n or D_n.

1 Introduction

The theory of Plancherel measures is well known for real groups [1, 22, 29]. Arthur gave an explicit formula of the Plancherel measure in terms of Artin factors [1]. It follows from the local Langlands correspondence for real groups [36] that Plancherel measures are invariant on L-packets, and they are preserved by inner forms. Plancherel measures thus turn out to be identical if they are associated to the same L-parameter.

For p-adic groups, however, the behavior of the Plancherel measure is not completely understood. Although the Lefschetz principle [20] conjectures that what is true for real groups is also true for p-adic groups, the L-packet invariance of the Plancherel measure is currently known only for some cases [2,10,14–17,27,42]. Also, it was proved that the Plancherel measures are preserved by p-adic inner forms for the following cases in characteristic 0. Arthur and Clozel proved the argument for discrete series representations under the local Jacquet-Langlands correspondence between GL_n and its inner forms [3]. The author also verified it for depth-zero supercuspidal representations associated to tempered and tame regular semi-simple elliptic L-parameters with an unramified central character between an unramified group and its inner forms [10]. The approaches in both of these results are based on p-adic harmonic analysis. For the groups SO_{4n} and Sp_{4n}, using a local to global argument, Muić and Savin proved that Plancherel measures for unitary supercuspidal representations are preserved under the local Jacquet–Langlands correspondence between the Siegel Levi subgroup and its inner forms [37]. In a similar way, Gan and Tantono identically

Received by the editors July 31, 2012.
Published electronically February 21, 2013.
AMS subject classification: **22E50**, 11F70, 22E55, 22E35.
Keywords: Plancherel measure, inner form, local to global global argument, cuspidal automorphic representation, Jacquet–Langlands correspondence.

transferred Plancherel measures attached to supercuspidal representations having the same L-parameter from the Levi subgroup $\mathrm{GL}_r \times \mathrm{GSp}_4$ of GSp_{2r+4} to its inner forms [17].

The purpose of this paper is to prove that Plancherel measures attached to unitary supercuspidal representations are preserved *under the local Jacquet–Langlands type correspondence* (defined below), assuming a working hypothesis that Plancherel measures are invariant on a certain set. To be precise, let F denote a p-adic field of characteristic 0, and let M be an F-Levi subgroup of a connected reductive F-split group G such that

$$\Pi_{i=1}^{r} \mathrm{SL}_{n_i} \subseteq M \subseteq \Pi_{i=1}^{r} \mathrm{GL}_{n_i} \tag{1.1}$$

for positive integers r and n_i. Let G' be an F-inner form of G, and let M' be an F-Levi subgroup of G' that is an F-inner form of M. Then M' satisfies the following property

$$\Pi_{i=1}^{r} \mathrm{SL}_{m_i}(D_{d_i}) \subseteq M'(F) \subseteq \Pi_{i=1}^{r} \mathrm{GL}_{m_i}(D_{d_i}).$$

Here D_{d_i} denotes a central division algebra of dimension d_i^2 over F where $n_i = m_i d_i$. Set $\widetilde{M}(F) = \Pi_{i=1}^{r} \mathrm{GL}_{n_i}(F)$ and $\widetilde{M}'(F) = \Pi_{i=1}^{r} \mathrm{GL}_{m_i}(D_{d_i})$. Denote by $\mathcal{E}_u^{\circ}\big(H(F)\big)$ the set of equivalence classes of irreducible unitary supercuspidal representations of $H(F)$ for any algebraic F-group H.

Given $\tau \in \mathcal{E}_u^{\circ}\big(M(F)\big)$, we have $\widetilde{\sigma} \in \mathcal{E}_u^{\circ}\big(\widetilde{M}(F)\big)$ such that τ is isomorphic to an irreducible constituent of the restriction $\widetilde{\sigma}|_{M(F)}$. Denote by $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$ the set of equivalence classes of all irreducible constituents of $\widetilde{\sigma}|_{M(F)}$. Note that the set $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$ does not depend on the choice of $\widetilde{\sigma}$ and is contained in $\mathcal{E}_u^{\circ}\big(M(F)\big)$. On the other hand, we have a unique $\widetilde{\sigma}' \in \mathcal{E}_u^{\circ}\big(\widetilde{M}'(F)\big)$ corresponding to $\widetilde{\sigma}$ under the local Jacquet–Langlands correspondence. Let $\Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ have the corresponding meaning for the F-inner form M'. We say that any two representations $\sigma \in \Pi_{\widetilde{\sigma}}\big(M(F)\big)$ and $\sigma' \in \Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ are *under the local Jacquet–Langlands type (JL-type) correspondence*. This will be defined more generally in Definition 4.5.

Fix a representative $w \in G(F)$ of a Weyl element $\tilde{w}$ such that $\tilde{w}(\theta) \subseteq \Delta$. Here Δ denotes the set of simple roots of A_0 in G, A_0 denotes the split component of a minimal F-Levi subgroup M_0 of G, and θ denotes the subset of Δ such that $M = M_\theta$. We denote by $\mathfrak{a}_{M,\mathbb{C}}^*$ the complex dual of the real Lie algebra of the split component A_M of M. Given an irreducible admissible representation σ of $M(F)$, $\nu \in \mathfrak{a}_{M,\mathbb{C}}^*$, and $\tilde{w}$, in Section 2.2 we define the Plancherel measure as a non-zero complex number $\mu_M(\nu, \sigma, w)$ such that

$$A(\nu, \sigma, w) A\big(w(\nu), w(\sigma), w^{-1}\big) = \mu_M(\nu, \sigma, w)^{-1} \gamma_w(G|P)^2.$$

Let w' and $\mathfrak{a}_{M',\mathbb{C}}^*$ have the corresponding meaning for the F-inner form M' (see Section 2.3 for details).

Working Hypothesis 1.1 (Working Hypothesis 6.1) *Let σ_1' and σ_2' be given in $\Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$. Then we have*

$$\mu_{M'}(\nu', \sigma_1', w') = \mu_{M'}(\nu', \sigma_2', w')$$

for any $\nu' \in \mathfrak{a}_{M',\mathbb{C}}^*$.

When G' is an F-inner form of SL_n and M' is any F-Levi subgroup of G', this hypothesis is a consequence of Proposition 2.4. It is also proved in [10] for the case when the central character of $\widetilde{\sigma}'$ is unramified and the set $\Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ is associated to a tempered and tame regular semi-simple elliptic L-parameter.

Our main result is the following.

Theorem 1.2 (Theorem 6.3) *Let $\sigma \in \mathcal{E}_u^\circ\big(M(F)\big)$ and $\sigma' \in \mathcal{E}_u^\circ\big(M'(F)\big)$ be under the local JL-type correspondence. Assume that Working Hypothesis 6.1 is valid. Then we have*

$$\mu_M(\nu, \sigma, w) = \mu_{M'}(\nu, \sigma', w')$$

for any $\nu \in \mathfrak{a}^*_{M,\mathbb{C}} \simeq \mathfrak{a}^*_{M',\mathbb{C}}$.

As a corollary to Theorem 1.2, we obtain the invariance of Plancherel measures on the set $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$.

Our approach to prove Theorem 1.2 is a general version of the local to global argument by Muić and Savin in [37]. First, given local data F, G, G', M, M', $\widetilde{M}$ and $\widetilde{M}'$ above, the following theorem allows us to construct a number field $\mathbf{F}$ and $\mathbf{F}$-groups $\mathbf{G}$, $\mathbf{G}'$, $\mathbf{M}$, $\mathbf{M}'$, $\widetilde{\mathbf{M}}$, and $\widetilde{\mathbf{M}}'$ with prescribed local behavior.

Theorem 1.3 (Theorem 5.4) *Let F be a p-adic field of characteristic 0, let G be a connected reductive quasi-split F-group and let G' be its F-inner form. Then there exist a number field $\mathbf{F}$, a non-empty finite set S of finite places of $\mathbf{F}$, a connected reductive quasi-split $\mathbf{F}$-group $\mathbf{G}$, and its $\mathbf{F}$-inner form $\mathbf{G}'$ such that*

(a) *for all $v \in S$, $\mathbf{F}_v \simeq F$, $\mathbf{G}_v \simeq G$, and $\mathbf{G}'_v \simeq G'$ over $\mathbf{F}_v$;*
(b) *for all $v \notin S$ including all the archimedean places, $\mathbf{G}_v \simeq \mathbf{G}'_v$ over $\mathbf{F}_v$;*

where $\mathbf{G}_v$ and $\mathbf{G}'_v$ denote $\mathbf{G} \times_{\mathbf{F}} \mathbf{F}_v$ and $\mathbf{G}' \times_{\mathbf{F}} \mathbf{F}_v$, respectively.

This theorem is proved by local and global cohomological results in [30, 38].

Second, we find a finite set $V \supseteq S$ and two cuspidal automorphic representations as described in the following proposition.

Proposition 1.4 (Proposition 6.9) *Let $\mathbb{A}$ be the ring of adeles of $\mathbf{F}$. Suppose $\sigma \in \mathcal{E}_u^\circ\big(M(F)\big)$ and $\sigma' \in \mathcal{E}_u^\circ\big(M'(F)\big)$ are under the local JL-type correspondence. Then there exist a finite set V of places of $\mathbf{F}$ containing S and all archimedean places, and two cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of $\mathbf{M}(\mathbb{A})$ and $\pi' = \bigotimes_v \pi'_v$ of $\mathbf{M}'(\mathbb{A})$ such that*

(a) *for all $v \in S$, $\pi_v \simeq \sigma$ and $\pi'_v \in \Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$,*
(b) *for all $v \in V - S$, π_v and π'_v are irreducible constituents of the restriction of an irreducible representation of $\widetilde{\mathbf{M}}(\mathbf{F}_v)$ to $\mathbf{M}(\mathbf{F}_v)$ (note that, for $v \notin S$, $\mathbf{M}_v \simeq \mathbf{M}'_v$ and $\widetilde{\mathbf{M}}_v \simeq \widetilde{\mathbf{M}}'_v$ over $\mathbf{F}_v$),*
(c) *for all $v \notin V$, π_v and π'_v are isomorphic and unramified with respect to $\mathbf{M}(\mathcal{O}_v)$. Here $\mathcal{O}_v$ is the ring of integers of $\mathbf{F}_v$.*

We prove Proposition 1.4 by using the result of Henniart [24], the global Jacquet–Langlands correspondence [6], and the result of Hiraga and Saito [25].

Next, we obtain equalities (6.2) and (6.3) below, which consist of a product of Plancherel measures at places in V and a quotient of the local Langlands L-functions for unramified representations. The property (c) of Proposition 1.4 allows us to cancel all local factors appearing outside V. For the places in $V - S$, we use the results of Keys and Shahidi in [27] and of Arthur [1]. We thus retain only Plancherel measures appearing inside S as in equation (6.4). The hypothesis in Theorem 1.2 is needed here to identify all the Plancherel measures attached to $\pi'_v \in \Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ for $v \in S$. Finally, from the fact that Plancherel measures are holomorphic and non-negative on the unitary axis, we deduce Theorem 1.2.

We remark that our construction in Theorem 1.3 can be applied to any F-Levi subgroup M of any connected reductive group over a p-adic field of characteristic 0, including the case when $M \simeq \mathrm{GL}_n$ in [37] (*cf.* Remark 5.6). Further, Proposition 1.4 extends the global Jacquet–Langlands correspondence for GL_n to a connected reductive F-split group M satisfying condition (1.1).

As applications of Theorem 1.2, we transfer the reducibility of the induced representations and the edges of complementary series from unitary supercuspidal representations of maximal F-Levi subgroups under the local JL-type correspondence. We also prove that the reducibility and the edges are invariant on the set $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$. Further, our work can be applied to a simply connected simple F-group of type E_6 or E_7, and a connected reductive F-group of type A_n, B_n, C_n or D_n.

This paper is organized as follows. In Section 2, we recall basic notions, terminologies and known results. After reviewing the local and global Jacquet–Langlands correspondences in Section 3, we define the local JL-type correspondence in Section 4. In Section 5, we set up a generalized local to global argument. We prove in Section 6 that Plancherel measures are preserved under the local JL-type correspondence, then we present some applications in Section 7. In Section 8, we generalize the result in Section 6 to any Levi subgroup under some assumptions. Appendix A gives a few examples of an F-Levi subgroup M and its F-inner form satisfying condition (1.1).

2 Preliminaries

We recall basic notions, terminologies and known results. We mainly refer to [19, 21, 30, 42, 44, 49].

2.1 Notation and Conventions

Throughout this paper, F denotes a p-adic field of characteristic 0 and $\mathbf{F}$ denotes a number field, unless otherwise stated. Fix algebraic closures $\bar{F}$ and $\bar{\mathbf{F}}$. We shall use the ordinary capital letters G, M, *etc.*, for groups defined over a local field and the boldfaced capital letters $\mathbf{G}$, $\mathbf{M}$, *etc.*, for groups defined over a global field.

By abuse of terminology, we identify the set of isomorphism classes with the set of representatives. Given a connected reductive group G over F, we use the following notation: $\mathrm{Irr}\big(G(F)\big)$ denotes the set of isomorphism classes of admissible representations of $G(F)$; $\mathcal{E}^2\big(G(F)\big)$ denotes the set of essentially square-integrable representations in $\mathrm{Irr}\big(G(F)\big)$; $\mathrm{Irr}_u\big(G(F)\big)$ denotes the set of unitary representations in $\mathrm{Irr}\big(G(F)\big)$; $\mathcal{E}_u^\circ\big(G(F)\big)$ denotes the set of unitary supercuspidal representations in $\mathrm{Irr}_u\big(G(F)\big)$; and $\mathcal{E}_u^2\big(G(F)\big)$ denotes the set of discrete series (square-integrable) representations in $\mathrm{Irr}_u\big(G(F)\big)$.

Denote by D a central division algebra over F. We often write D_d to emphasize its dimension d^2 over F. We denote by $\mathrm{GL}_m(D)$ the group of all invertible elements of $m \times m$ matrices over D, and by $\mathrm{SL}_m(D)$ the subgroup of elements in $\mathrm{GL}_m(D)$ whose reduced norm is 1 (see [38, Sections 1.4 and 2.3]).

2.2 Plancherel Measures

Let G be a connected reductive group over a p-adic field F of characteristic 0. Fix a minimal F-parabolic subgroup P_0 of G with Levi component M_0 and unipotent radical N_0. Let A_0 be the split component of M_0, that is, the maximal F-split torus in the center of M_0. Let Δ be the set of simple roots of A_0 in N_0.

Let P be a standard (that is, containing P_0) F-parabolic subgroup of G. Write $P = MN$ with its Levi component $M = M_\theta \supseteq M_0$ generated by a subset $\theta \subseteq \Delta$ and its unipotent radical $N \subseteq N_0$. Let A_M be the split component of M. Denote by $W_M = W(G, A_M) := N_G(A_M)/Z_G(A_M)$ the Weyl group of A_M in G, where $N_G(A_M)$ and $Z_G(A_M)$ are respectively the normalizer and the centralizer of A_M in G. For convenience, we write $A_0 = A_{M_0}$ and $W_G = W_{M_0}$.

Denote by $X^*(M)_F$ the group of F-rational characters of M. We denote by $\mathfrak{a}_M := \mathrm{Hom}\big(X^*(M)_F, \mathbb{R}\big) = \mathrm{Hom}\big(X^*(A_M)_F, \mathbb{R}\big)$ the real Lie algebra of A_M. We set the complex dual

$$\mathfrak{a}^*_{M,\mathbb{C}} := X^*(M)_F \otimes_{\mathbb{Z}} \mathbb{C}.$$

We define the homomorphism $H_M \colon M(F) \to \mathfrak{a}_M$ by

$$q^{\langle \chi, H_M(m)\rangle} = |\chi(m)|_F$$

for all $\chi \in X^*(M)_F$ and $m \in M(F)$. Here $|\cdot|_F$ denotes the normalized absolute value on F. Note that one can extend H_M to $G(F)$ using the Iwasawa decomposition.

For $\sigma \in \mathrm{Irr}\big(M(F)\big)$ and $\nu \in \mathfrak{a}^*_{M,\mathbb{C}}$, we denote by $I(\nu, \sigma)$ the normalized induced representation

$$I(\nu, \sigma) = \mathrm{Ind}_{P(F)}^{G(F)}(\sigma \otimes q^{\langle \nu, H_M(\,)\rangle} \otimes \mathbb{1}).$$

Here $\mathbb{1}$ is the trivial representation of $N(F)$. The space $V(\nu, \sigma)$ of $I(\nu, \sigma)$ consists of locally constant functions f from $G(F)$ into the representation space $\mathcal{H}(\sigma)$ of σ such that

$$f(mng) = \sigma(m) q^{\langle \nu + \rho_P, H_M(m)\rangle} f(g),$$

for $m \in M(F)$, $n \in N(F)$ and $g \in G(F)$. Here ρ_P denotes the half sum of all positive roots in N. We often write $i_{G,M}(\nu, \sigma)$ for $I(\nu, \sigma)$ in order to specify groups.

We fix a representative $w \in G(F)$ of $\tilde{w} \in W_G$ such that $\tilde{w}(\theta) \subseteq \Delta$. Set $N_{\tilde{w}} := N_0 \cap w N^- w^{-1}$. We fix a Haar measure dn on $N_{\tilde{w}}$. Given $f \in V(\nu, \sigma)$, for $g \in G(F)$, the standard intertwining operator is defined as

$$A(\nu, \sigma, \tilde{w}) f(g) = \int_{N_{\tilde{w}}(F)} f(w^{-1} n g)\, dn.$$

We set

$$\gamma_{\tilde{w}}(G|M) = \int_{\bar{N}_{\tilde{w}}(F)} q^{\langle 2\rho_P, H_M(\bar{n})\rangle} d\bar{n}, \tag{2.1}$$

where $d\bar{n}$ is a normalized Haar measure on $\bar{N}_{\tilde{w}}(F)$ and $\bar{N}_{\tilde{w}} := w^{-1}N_{\tilde{w}}w = N^- \cap w^{-1}N_0 w$.

Definition 2.1 Given $\nu \in \mathfrak{a}^*_{M,\mathbb{C}}$, $\sigma \in \mathrm{Irr}_u\big(M(F)\big)$ and $\tilde{w}(\theta) \subseteq \Delta$, we define *the Plancherel measure* attached to ν, σ and $\tilde{w}$ as a non-zero complex number $\mu_M(\nu, \sigma, w)$ such that

$$A(\nu, \sigma, w)A\big(w(\nu), w(\sigma), w^{-1}\big) = \mu_M(\nu, \sigma, w)^{-1}\gamma_w(G|P)^2.$$

Remark 2.2 The Plancherel $\mu_M(\nu, \sigma, w)$ depends only on ν, σ and $\tilde{w}$. It is independent of the choices of any Haar measure and of any representative w of $\tilde{w}$ [42, p. 280]. Further, as a function $\nu \mapsto \mu_M(\nu, \sigma, w)$ on $\mathfrak{a}^*_{M,\mathbb{C}}$, it extends to a meromorphic function on all of $\mathfrak{a}^*_{M,\mathbb{C}}$. Moreover, it is non-negative and holomorphic on $\sqrt{-1}\mathfrak{a}^*_M$ [21, Theorem 20].

We review two useful properties of the Plancherel measure. Let $\Phi(P, A_M)$ denote the set of reduced roots of P with respect to A_M. For $\alpha \in \Phi(P, A_M)$, $A_\alpha := (\ker\alpha \cap A_M)^\circ$ denotes the identity component of $(\ker\alpha \cap A_M)$ regarding α as an element in $\mathfrak{a}^*_M$. Set $M_\alpha := Z_G(A_\alpha)$ and $P_\alpha := M_\alpha \cap P$. Note that M_α contains M and P_α is a maximal F-parabolic subgroup of M_α with its Levi decomposition MN_α and the split component A_M. The Plancherel measure $\mu_\alpha(\nu, \sigma, w)$ and the function $\gamma_\alpha(P_\alpha|M)$ can be defined by replacing G with P_α in Definition 2.1 and equation (2.1), respectively. The following is the product formula of the rank-one Plancherel measures [21, Theorem 24].

Proposition 2.3 (Product Formula)

$$\gamma_{\tilde{w}}(G|M)^{-2}\mu_M(\nu, \sigma, w) = \prod_{\alpha\in\Phi(P,A_M)} \gamma_\alpha(P_\alpha|M)^{-2}\mu_\alpha(\nu, \sigma, w).$$

Next, suppose that $\widetilde{G}$ is a connected reductive group over F such that

$$G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}} \subseteq G \subseteq \widetilde{G},$$

where the subscript $_{\mathrm{der}}$ means the derived group. Let $\widetilde{M}$ denote an F-Levi subgroup of $\widetilde{G}$ such that $M = \widetilde{M} \cap G$. Given $\sigma \in \mathrm{Irr}\big(M(F)\big)$, we have an irreducible representation $\widetilde{\sigma} \in \mathrm{Irr}\big(\widetilde{M}(F)\big)$ whose restriction $\widetilde{\sigma}|_{M(F)}$ contains σ (see [48, Proposition 2.2]). Since $A_M \subseteq A_{\widetilde{M}}$, we have a canonical surjective homomorphism $\mathfrak{a}^*_{\widetilde{M},\mathbb{C}} \twoheadrightarrow \mathfrak{a}^*_{M,\mathbb{C}}$.

We write $\widetilde{\nu}$ for any pre-image in $\mathfrak{a}^*_{\widetilde{M},\mathbb{C}}$ of $\nu \in \mathfrak{a}^*_{M,\mathbb{C}}$. The following is the compatibility of the Plancherel measure with the restriction.

Proposition 2.4 (Compatibility with Restriction)

$$\mu_M(\nu,\sigma,w)=\mu_{\widetilde{M}}(\widetilde{\nu},\widetilde{\sigma},w).$$

Proof Since G and $\widetilde{G}$ have the same derived group, we have

$$\gamma_{\tilde{w}}(G|M)^{-2}=\gamma_{\tilde{w}}(\widetilde{G}|\widetilde{M})^{-2}. \qquad \blacksquare$$

Hence, the proposition follows from the following property (*cf.* [43, p. 293]):

$$A(\nu,\sigma,\tilde{w})=A(\widetilde{\nu},\widetilde{\sigma},\tilde{w})|_{i_{G,M}(\nu,\sigma)}.$$

Remark 2.5 Remark 2.2, Propositions 2.3 and 2.4 reduce the study of Plancherel measures to the case of irreducible unitary admissible representations of maximal F-Levi subgroups of a semi-simple group G.

2.3 Inner Forms

Let G and G' be connected reductive groups over a p-adic field F of characteristic 0. We say that G and G' are F-*inner forms* with respect to an $\bar{F}$-isomorphism $\varphi\colon G'\to G$ if $\varphi\circ\tau(\varphi)^{-1}$ is an inner automorphism ($g\mapsto xgx^{-1}$) defined over $\bar{F}$ for all $\tau\in\mathrm{Gal}(\bar{F}/F)$ (see [44, p. 851]). Denote by $Z(G)$ the center of G. Set $G^{\mathrm{ad}}:=G/Z(G)$ and $H^i(F,G):=H^i\big(\mathrm{Gal}(\bar{F}/F),G(\bar{F})\big)$. We note [31, p. 280] that there is a bijection between $H^1(F,G^{\mathrm{ad}})$ and the set of isomorphism classes of F-inner forms of G, obtained by sending the isomorphism class of a pair (G',φ) to the class of the 1-cocycle $\tau\mapsto\varphi\circ\tau(\varphi)^{-1}$. We note that a $\mathrm{Gal}(\bar{F}/F)$-stable $G^{\mathrm{ad}}(\bar{F})$-orbit of φ gives the same isomorphism class of a pair (G',φ). We often omit the reference to φ when there is no danger of confusion. We note that this notion holds for any field F with an algebraic separable closure $\bar{F}$.

Suppose that G is *quasi-split* over F. Let G' be an F-inner form of G with respect to an $\bar{F}$-isomorphism $\varphi\colon G'\to G$ (G' can be G itself). Let P_0, M_0, N_0, A_0, P_0', M_0', N_0', A_0' and Δ' be as in Section 2.2. We denote by $P'=M'N'$ a standard F-parabolic subgroup of G'. Then from [19, Section 11.2] we can choose an element φ_1 in a $\mathrm{Gal}(\bar{F}/F)$-stable $G^{\mathrm{ad}}(\bar{F})$-orbit of φ such that: $\varphi_1(A_0')\subseteq A_0$; $\varphi_1(P')=P=MN$ is a standard F-parabolic subgroup of G; and $\varphi_1(M')=M$. As we discussed above, we identify (M',φ_1) with (M',φ). Hence, M' is the F-inner form of quasi-split group M over F with respect to the $\bar{F}$-isomorphism $\varphi\colon M'\to M$. In this case, we often say that M and M' are corresponding. In fact, the split components A_M and $A_{M'}$ are isomorphic over F via the $\bar{F}$-isomorphism φ. We thus have $\mathfrak{a}^*_{M,\mathbb{C}}\simeq\mathfrak{a}^*_{M',\mathbb{C}}$ as $\mathbb{C}$-vector spaces and identify $\nu\in\mathfrak{a}^*_{M,\mathbb{C}}$ with $\nu'\in\mathfrak{a}^*_{M',\mathbb{C}}$ through the isomorphism. Denote by $W_M=W(G,A_M)$ and $W_{M'}=W(G',A_{M'})$ the Weyl groups as defined in Section 2.2. Let $\tilde{w}'\in W_{G'}$ be given such that $\tilde{w}'(\theta')\subseteq\Delta'$. Then the image $\tilde{w}:=\varphi(\tilde{w}')$ in W_G satisfies the property that $\tilde{w}(\theta)\subseteq\Delta$ (see [19, Section 11.2]).

We recall the Kottwitz isomorphism. Set $A(G):=\pi_0\big(Z(\hat{G})^{\Gamma_F}\big)^D$. Here $\hat{G}$ denotes the connected Langlands dual group (L-group) of G, $Z(\hat{G})$ denotes the center of $\hat{G}$,

$\Gamma_F = \mathrm{Gal}(\bar{F}/F)$, $\pi_0(\,\cdot\,)$ denotes the group of connected components, and $(\,\cdot\,)^D$ denotes the Pontryagin dual, that is, $\mathrm{Hom}(\,\cdot\,, \mathbb{R}/\mathbb{Z})$. We note that $A(G)$ is a finite abelian group when $G = G^{\mathrm{ad}}$. To see this, since the center of a simply-connected semisimple complex Lie group is finite, $Z(\widehat{G^{\mathrm{ad}}}) = Z(\hat{G}_{\mathrm{sc}})$ turns out to be a finite abelian group. Here $\hat{G}_{\mathrm{sc}}$ denotes the simply connected cover of the derived group of $\hat{G}$.

Proposition 2.6 (Kottwitz, [30, Theorem 1.2]) *Let G be a connected reductive group over a p-adic field F of characteristic* 0*. One has the following bijection*

$$H^1(F, G) \simeq A(G).$$

We note that if the adjoint group G^{ad} of G is simply connected (*e.g.*, G_2, F_4, or E_8), there is no non-quasi-split F-inner form due to Proposition 2.6.

Example 2.7 Let G be either GL_n or SL_n over a p-adic field F of characteristic 0. Then the set of isomorphism classes of F-inner forms of G is in bijection with the subgroup $\mathrm{Br}(F)_n$ of n-torsion elements in the Brauer group $\mathrm{Br}(F)$. Indeed, we have

$$H^1(F, \mathrm{PGL}_n) \simeq A(\mathrm{PGL}_n) \simeq \mu_n(\mathbb{C})^D.$$

By Hilbert's theorem 90, $\mu_n(\mathbb{C})^D \simeq H^2(F, \mu_n) \simeq \ker\big(\mathrm{Br}(F) \xrightarrow{n} \mathrm{Br}(F)\big) = \mathrm{Br}(F)_n$. Here μ_n is the algebraic group of n-th root of unity.

Example 2.8 Let $\mathbf{G}$ be either GL_n or SL_n over a number field $\mathbf{F}$. Then the set of isomorphism classes of $\mathbf{F}$-inner forms of $\mathbf{G}$ is also in bijection with $\mathrm{Br}(\mathbf{F})_n$. Indeed, since $H^1(\mathbf{F}, \mathrm{GL}_n) = 1$ (in fact, this is true for any perfect field [38, Lemma 2.2, p. 70]), we have

$$H^1(\mathbf{F}, \mathrm{PGL}_n) \hookrightarrow H^2(\mathbf{F}, \mu_n) \simeq \mu_n(\mathbb{C})^D.$$

The injection turns out to be surjective due to [38, Theorem 6.20].

3 Local and Global Jacquet–Langlands Correspondences for GL_n

In this section, we recall the Jacquet–Langlands correspondence for GL_n over a p-adic field F of characteristic 0 and a number field $\mathbf{F}$. We mainly refer to [6].

The local and global Jacquet–Langlands correspondences were initially found by Jacquet and Langlands [26] for the case GL_2 in any characteristic. The local generalization to GL_n in zero characteristic was established by Rogawski [39] and independently by Deligne, Kazhdan and Vigneras [12]. Badulescu completed the proof of the local correspondence for GL_n in positive characteristic [5]. On the other hand, the global generalization of the correspondence to GL_n was proved only for a number field by Badulescu [6]. For some particular cases in zero characteristic, Flath treated the local and global correspondences for GL_3 [13], and Snowden presented a new purely local proof of the local Jacquet–Langlands correspondence for GL_2 [47].

3.1 Local Jacquet–Langlands Correspondence

Let G be GL_n over a p-adic field F of characteristic 0 and let G' be an F-inner form of G. Then $G'(F)$ is of the form of $\mathrm{GL}_m(D)$, where D denotes a central division algebra of dimension d^2 over F and $n = md$.

For semisimple elements $g \in G(F)$ and $g' \in G'(F)$, we write $g \leftrightarrow g'$ if both are regular (*i.e.*, all roots in $\bar{F}$ of the characteristic polynomial are distinct) and have the same characteristic polynomial. We write $G(F)^{\mathrm{reg}}$ for the set of regular semisimple elements in $G(F)$. We denote by $C_c^\infty\big(G(F)\big)$ the Hecke algebra of locally constant functions on $G(F)$ with compact support. Fix a Haar measure dg on $G(F)$. For any $\rho \in \mathrm{Irr}\big(G(F)\big)$, there is a unique locally constant function Θ_ρ on $G(F)^{\mathrm{reg}}$ which is invariant under conjugation by $G(F)$ such that

$$tr\rho(f) = \int_{G(F)^{\mathrm{reg}}} \Theta_\rho(g) f(g)\, dg$$

for all $f \in C_c^\infty\big(G(F)\big)$. We refer to [23, p. 96] and [12, b., p. 33] for details. The same is true for the F-inner form G'. We state the local Jacquet–Langlands correspondence as follows.

Proposition 3.1 ([12, B.2.a], [39, Theorem 5.8], and [6, Theorem 2.2]) *There is a unique bijection* $\mathbf{C}\colon \mathcal{E}^2\big(G(F)\big) \longrightarrow \mathcal{E}^2\big(G'(F)\big)$ *such that: for all* $\sigma \in \mathcal{E}^2\big(G(F)\big)$, *we have*

$$\Theta_\sigma(g) = (-1)^{n-m}\Theta_{\mathbf{C}(\sigma)}(g')$$

for all $g \leftrightarrow g'$.

Remark 3.2 ([12, Introduction d.4)]) For any $\sigma \in \mathcal{E}^2\big(G(F)\big)$ and quasi-character η on $F^\times$, we have $\mathbf{C}\big(\sigma \otimes (\eta \circ \det)\big) = \mathbf{C}(\sigma) \otimes (\eta \circ \mathrm{Nrd})$, where $\mathrm{Nrd}\colon\ \mathrm{GL}_m(D) \to F^\times$ is the reduced norm (*cf.* [8, Section 53.5]).

Example 3.3 Denote by St_G (resp. $\mathrm{St}_{G'}$) the Steinberg representation of $G(F)$ (resp. $G'(F)$). Since $\Theta_{\mathrm{St}_G}(g) = (-1)^{n-m}\Theta_{\mathrm{St}_{G'}}(g')$ for all $g \leftrightarrow g'$ [21, Section 15], we have $\mathbf{C}(\mathrm{St}_G) = \mathrm{St}_{G'}$.

We denote by $R(G)$ the Grothendieck group of admissible representations of finite length of $G(F)$. So, $R(G)$ is a free abelian group with basis $\mathrm{Irr}\big(G(F)\big)$. Let $R(G')$ be the Grothendieck group for the F-inner form G'. In what follows, we extend the correspondence $\mathbf{C}$ to a $\mathbb{Z}$-morphism from $R(G)$ to $R(G')$. We refer to [6, Section 2.7].

Let $\mathcal{B}$ be the collection of all normalized (twisted by $\delta_P^{1/2}$) induced representation $i_{G,L}\sigma$, where L is a standard Levi subgroup of G and $\sigma \in \mathcal{E}^2\big(L(F)\big)$. Let $\mathcal{B}'$ have the corresponding meaning for the F-inner form G'. We notice that any element $\Sigma \in \mathcal{B}$ (resp. $\Sigma' \in \mathcal{B}'$) has a unique irreducible quotient by the Langlands classification (see [32, Theorem 1.2.5]). We denote it by $\mathrm{Lg}(\Sigma)$ (resp. $\mathrm{Lg}(\Sigma')$). We note that the set $\mathcal{B}$ is a basis of $R(G)$ and the map $\Sigma \mapsto \mathrm{Lg}(\Sigma)$ is a bijection from $\mathcal{B}$ onto $\mathrm{Irr}\big(G(F)\big)$. The same is true for the F-inner form G'.

Given a basis element $\Sigma' = i_{G',L'}\sigma' \in \mathcal{B}'$ with a standard F-Levi subgroup L' of G', we set $\Lambda(\Sigma') := i_{G,L}\mathbf{C}^{-1}(\sigma')$. Here L is the standard Levi subgroup of G

corresponding to L' (see Section 2.3). From Proposition 3.1, we notice that $\Lambda(\Sigma')$ lies in $\mathcal{B}$. Thus, Λ defines a map from $\mathcal{B}'$ into $\mathcal{B}$, which is clearly injective. Further, since $\Lambda(\Sigma')$ has a unique irreducible quotient (denoted by $\mathrm{Lg}\big(\Lambda(\Sigma')\big)$), Λ induces a map from $\mathrm{Irr}\big(G'(F)\big)$ into $\mathrm{Irr}\big(G(F)\big)$ by sending $\mathrm{Lg}(\Sigma') \mapsto \mathrm{Lg}\big(\Lambda(\Sigma')\big)$.

Definition 3.4 We define a $\mathbb{Z}$-morphism $\mathbf{LJ}\colon R(G) \to R(G')$ by $\mathbf{LJ}\big(\Lambda(\Sigma')\big) = \Sigma'$ and $\mathbf{LJ}(\Sigma) = 0$ if Σ is not in the image of Λ.

Remark 3.5 There exists an irreducible unitary representation of $G'(F)$ which is not in the image of Λ [6, Lemma 3.11]. Moreover, the map $\mathbf{LJ}$ sends an irreducible unitary representation of $G(F)$ to either 0 or an irreducible unitary representation of $G'(F)$ [6, Proposition 3.9].

We have the following correspondence between the Grothendieck groups.

Theorem 3.6 ([6, Theorem 2.7]) *There is a unique map* $\mathbf{LJ}\colon R(G) \to R(G')$ *such that: for all* $\sigma \in R(G)$, *we have*

$$\Theta_\sigma(g) = (-1)^{n-m}\Theta_{\mathbf{LJ}(\sigma)}(g')$$

for all $g \leftrightarrow g'$. *Furthermore,* $\mathbf{LJ}$ *is a surjective group homomorphism.*

Example 3.7 Suppose $G = \mathrm{GL}_2$. Denote by $\mathbb{1}_G$ the trivial representation of $G(F)$. Fix a Borel subgroup $B = TU$ of G. Note that $i_{G,T}(\delta_B^{-1/2}) = \mathbb{1}_G + \mathrm{St}_G$ as elements of $R(G)$. Since there is no F-Levi subgroup of G' corresponding T, we have $\mathbf{LJ}\big(i_{G,T}(\delta_B^{-1/2})\big) = 0$. Thus, Example 3.3 yields

$$\mathbf{LJ}(\mathrm{St}_G) = \mathrm{St}_{G'} \quad \text{and} \quad \mathbf{LJ}(\mathbb{1}_G) = -\mathrm{St}_{G'}.$$

Definition 3.8 We say that $\sigma \in R(G)$ is *d-compatible* if $\mathbf{LJ}(\sigma) \neq 0$.

Remark 3.9 It follows from Proposition 3.1 that $\sigma \in \mathcal{E}^2\big(G(F)\big)$ is always d-compatible.

We have the following correspondence for d-compatible irreducible unitary representations.

Proposition 3.10 ([6, Proposition 3.9]) *If u is a d-compatible irreducible unitary representation of $G(F)$, then there exists a unique irreducible unitary representation u' of $G'(F)$ and a unique sign $\epsilon \in \{-1, 1\}$ such that*

$$\Theta_u(g) = \epsilon\Theta_{u'}(g')$$

for all $g \leftrightarrow g'$.

Definition 3.11 By sending $u \mapsto u'$, we define a map $|\mathbf{LJ}|$ from the set of irreducible unitary d-compatible representations of $G(F)$ to the set of irreducible unitary representations of $G'(F)$.

https://doi.org/10.4153/CJM-2012-063-1 Published online by Cambridge University Press

Remark 3.12 The restriction of $|\mathbf{LJ}|$ to the set $\mathcal{E}^2_u\big(G(F)\big)$ equals $\mathbf{C}$.

Example 3.13 For the case that $G = \mathrm{GL}_2$, the representation $i_{G,T}(\delta_B^{-1/2})$ is not d-compatible due to Example 3.7. We also note that $|\mathbf{LJ}|(\mathrm{St}_G) = |\mathbf{LJ}|(\mathbb{1}_G) = \mathrm{St}_{G'}$.

Remark 3.14 All statements in this section admit an obvious generalization to the case that G is a product of a general linear groups.

3.2 Global Jacquet–Langlands Correspondence

Let $\mathbf{G}$ be GL_n over a number field $\mathbf{F}$ and let $\mathbf{G}'$ be an $\mathbf{F}$-inner form of $\mathbf{G}$. Then $\mathbf{G}'(F)$ is of the form of $\mathrm{GL}_m(\mathbf{D})$, where $\mathbf{D}$ denotes a central division algebra of dimension d^2 over $\mathbf{F}$ and $n = md$.

Set $\mathbf{A} := M_m(\mathbf{D})$, the $m \times m$ matrix algebra over $\mathbf{D}$. For each place v of $\mathbf{F}$, we have $\mathbf{A}_v = \mathbf{A} \otimes_{\mathbf{F}} \mathbf{F}_v \simeq M_{m_v}(\mathbf{D}_v)$ for some positive integer m_v and some central division algebra $\mathbf{D}_v$ of dimension d_v^2 over $\mathbf{F}_v$ such that $m_v d_v = n$. If $d_v = 1$ at some place v, we say that $\mathbf{A}$ is *split* at v. We denote by S the set of places where $\mathbf{A}$ is not split. Then S turns out to be finite. Set $G'_v := G' \times_{\mathbf{F}} \mathbf{F}_v$. We notice that $\mathbf{G}'_v \simeq \mathbf{G}_v \simeq \mathrm{GL}_n$ over $\mathbf{F}_v$ for all $v \notin S$.

Let $\mathbb{A}$ be the ring of adeles of $\mathbf{F}$. We identify $\mathbf{Z}(\mathbf{G}) = \mathbf{Z}(\mathbf{G}') =: \mathbf{Z}$, the centres of $\mathbf{G}$ and $\mathbf{G}'$. Fix a unitary smooth character ω of the quotient $\mathbf{Z}(\mathbf{F})\backslash\mathbf{Z}(\mathbb{A})$. Let $L^2\big(\mathbf{Z}(\mathbb{A})\mathbf{G}'(\mathbf{F})\backslash\mathbf{G}'(\mathbb{A});\omega\big)$ be the space of classes of functions $f\colon \mathbf{G}'(\mathbb{A}) \to \mathbb{C}$ which are left invariant under $\mathbf{G}'(\mathbf{F})$, transform under $\mathbf{Z}(\mathbb{A})$ by ω and are square-integrable modulo $\mathbf{Z}(\mathbb{A})\mathbf{G}'(\mathbf{F})$. Consider the representation R'_ω of $\mathbf{G}'(\mathbb{A})$ by the right translation in the space $L^2\big(\mathbf{Z}(\mathbb{A})\mathbf{G}'(\mathbf{F})\backslash\mathbf{G}'(\mathbb{A});\omega\big)$. Any irreducible subrepresentation of R'_ω is called to be a *discrete series* of $\mathbf{G}'(\mathbb{A})$. Denote by DS' (resp. DS) the set of all discrete series of $\mathbf{G}'(\mathbb{A})$ (resp. $\mathbf{G}(\mathbb{A})$). Every $\pi' \in DS'$ admits the restrict tensor product $\otimes_v \pi'_v$. It turns out that each π'_v is an irreducible unitary admissible representation of $\mathbf{G}'(\mathbf{F}_v)$. For each place v, we use d_v-compatible and $|\mathbf{LJ}|_v$ to emphasize the place v in Definitions 3.8 and 3.11, respectively.

Definition 3.15 Let $\pi = \otimes_v \pi_v$ be in DS. We say that π is *D-compatible* if π_v is d_v-compatible for all places $v \in S$.

For all $v \notin S$, we abuse the notation $|\mathbf{LJ}|_v$ for *the identity map* from $\mathrm{Irr}_u\big(\mathbf{G}(\mathbf{F}_v)\big)$ to $\mathrm{Irr}_u\big(\mathbf{G}'(\mathbf{F}_v)\big)$. The global Jacquet–Langlands correspondence is as follows.

Theorem 3.16 ([6, Theorem 5.1]) *There exists a unique injective map $\Phi\colon DS' \to DS$ such that: for all $\pi' = \otimes_v \pi'_v \in DS'$, we have*

$$|\mathbf{LJ}|_v\big(\Phi(\pi')_v\big) = \pi'_v.$$

Moreover, the image of Φ is exactly the set of D-compatible discrete series of $\mathbf{G}(\mathbb{A})$.

We note that there was an assumption on the set S in [6] and it has been removed in [4]. Let $L^2_c\big(\mathbf{Z}(\mathbb{A})\mathbf{G}'(\mathbf{F})\backslash\mathbf{G}'(\mathbb{A});\omega\big)$ denote the subspace of all the cusp forms in $L^2\big(\mathbf{Z}(\mathbb{A})\mathbf{G}'(\mathbf{F})\backslash\mathbf{G}'(\mathbb{A});\omega\big)$. It turns out that $L^2_c\big(\mathbf{Z}(\mathbb{A})\mathbf{G}'(\mathbf{F})\backslash\mathbf{G}'(\mathbb{A});\omega\big)$ is stable under R'_ω. Any irreducible subrepresentation of R'_ω in $L^2_c\big(\mathbf{Z}(\mathbb{A})\mathbf{G}'(\mathbf{F})\backslash\mathbf{G}'(\mathbb{A});\omega\big)$ is called to be a *cuspidal automorphic representation* of $\mathbf{G}'(\mathbb{A})$. The following proposition describes the behavior of cuspidal automorphic representations under the map Φ.

Proposition 3.17 ([6, Proposition 5.5 and Corollary A.8]) *Let $\pi \in DS$ be a cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$. If π is D-compatible, then $\Phi^{-1}(\pi)$ is a cuspidal automorphic representation of $\mathbf{G}'(\mathbb{A})$.*

Remark 3.18 There exists a cuspidal automorphic representation of $\mathbf{G}'(\mathbb{A})$ whose image through the map Φ is not cuspidal (see [6, Proposition 5.5.(b)]).

Remark 3.19 All statements in this section admit an obvious generalization to the case that $\mathbf{G}$ is a product of a general linear groups.

4 Local Jacquet–Langlands Type Correspondence

In this section, we define the local Jacquet–Langlands type (JL-type) correspondence which is a general version of the local Jacquet–Langlands correspondence.

4.1 Restriction of Representations

We recall the results of Tadić in [48]. Throughout Section 4.1, G and $\widetilde{G}$ denote connected reductive groups over a p-adic field F of characteristic 0, such that

$$G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}} \subseteq G \subseteq \widetilde{G},$$

where the subscript $_{\mathrm{der}}$ means the derived group.

Proposition 4.1 ([48, Lemma 2.1 and Proposition 2.2]) *For any $\sigma \in \mathrm{Irr}\big(G(F)\big)$, there exists $\widetilde{\sigma} \in \mathrm{Irr}\big(\widetilde{G}(F)\big)$ such that σ is isomorphic to an irreducible constituent of the restriction $\widetilde{\sigma}|_{G(F)}$ of $\widetilde{\sigma}$ to $G(F)$.*

Given $\sigma \in \mathrm{Irr}\big(G(F)\big)$, we denote by $\Pi_{\widetilde{\sigma}}\big(G(F)\big)$ the set of equivalence classes of all irreducible constituents of $\widetilde{\sigma}|_{G(F)}$.

Remark 4.2 ([48, Proposition 2.7]) Any member in $\Pi_{\widetilde{\sigma}}\big(G(F)\big)$ is supercuspidal, essentially square-integrable, discrete series or tempered if and only if $\widetilde{\sigma}$ is.

Remark 4.3 ([48, Corollary 2.5]) If $\widetilde{\sigma}_1 \in \mathrm{Irr}\big(\widetilde{G}(F)\big)$ is another choice of $\widetilde{\sigma}$ in Proposition 4.1, then there exist a quasi-character η on $\widetilde{G}(F)/G(F)$ such that $\widetilde{\sigma}_1 \simeq \widetilde{\sigma} \otimes \eta$. It turns out that the set $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$ is finite and independent of the choice of $\widetilde{\sigma}$.

Remark 4.4 Let $G = \mathrm{SL}_n$ over F or its F-inner form, and let $\widetilde{G} = \mathrm{GL}_n$ over F or its F-inner form. It then turns out [18, 25] that any L-packet is of the form $\Pi_{\widetilde{\sigma}}\big(G(F)\big)$ for some $\widetilde{\sigma} \in \mathrm{Irr}\big(\widetilde{G}(F)\big)$.

4.2 Definition of the Local Jacquet–Langlands Type Correspondence

Let $\widetilde{G}$ be $\prod_{i=1}^{r} \mathrm{GL}_{n_i}$ over a p-adic field F of characteristic 0 and let $\widetilde{G}'$ be an F-inner form of $\widetilde{G}$. Then $G'(F)$ is of the form of $\prod_{i=1}^{r} \mathrm{GL}_{m_i}(D_{d_i})$, where D_{d_i} denotes a central division algebra of dimension d_i^2 over F and $n_i = m_i d_i$. Let G be a connected reductive F-split group such that

$$G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}} \subseteq G \subseteq \widetilde{G}. \tag{4.1}$$

Let G' be an F-inner form of G. It follows that $G'_{\mathrm{der}} = \widetilde{G}'_{\mathrm{der}} \subseteq G' \subseteq \widetilde{G}'$.

Definition 4.5 Given $\sigma \in \mathrm{Irr}\big(G(F)\big)$ and $\sigma' \in \mathrm{Irr}(G'(F))$, we say that σ and σ' are *under the local Jacquet–Langlands type (JL-type) correspondence* if there exist $\widetilde{\sigma} \in \mathrm{Irr}\big(\widetilde{G}(F)\big)$ and $\widetilde{\sigma}' \in \mathrm{Irr}\big(\widetilde{G}'(F)\big)$ such that

(a) σ and σ' are isomorphic to irreducible constituents of the restrictions $\widetilde{\sigma}|_{G(F)}$ and $\widetilde{\sigma}'|_{G'(F)}$, respectively,

(b) $\mathbf{LJ}(\tilde{\sigma}) = \tilde{\sigma}'$.

Example 4.6 Steinberg representations St_G and $\mathrm{St}_{G'}$ are always under the local JL-type correspondence. This follows from Example 3.3 and the fact that the restrictions $\mathrm{St}_{\widetilde{G}}|_{G(F)}$ and $\mathrm{St}_{\widetilde{G}'}|_{G'(F)}$ are again Steinberg representations.

Remark 4.7 Given $\widetilde{\sigma} \in \mathcal{E}^2\big(\widetilde{G}(F)\big)$ and $\eta \in \big(\widetilde{G}(F)/G(F)\big)^D$, set $\widetilde{\sigma}' := \mathbf{C}(\widetilde{\sigma} \otimes \eta) \in \mathcal{E}^2\big(\widetilde{G}'(F)\big)$ (see Proposition 3.1). Then any $\sigma \in \Pi_{\widetilde{\sigma}}\big(G(F)\big)$ and $\sigma' \in \Pi_{\widetilde{\sigma}'}\big(G'(F)\big)$ are under the local JL-type correspondence.

Remark 4.8 The local JL-type correspondence can be regarded as a correspondence between L-packets of $G(F)$ and $G'(F)$ (*cf.* [25, Chapters 11–15]).

5 A Local to Global Argument

In this section, we set up a local to global argument which will be used in Section 6. This generalizes the method of Muić and Savin in [37].

5.1 Construction of Global Data from Local Data

Given local data, we first construct global data with prescribed local behavior.

Lemma 5.1 *Given a p-adic field F of characteristic* 0*, there exists a number field* $\mathbf{F}^0$ *such that $\mathbf{F}^0_{v_0} \simeq F$ for some finite place v_0 of* $\mathbf{F}^0$.

Proof Let F be a finite extension of $\mathbb{Q}_p$ for some prime number p. By [33, Corollary, p. 44], we have a number field $\mathbf{F}^0$ such that: $\mathbf{F}^0$ is dense in F; $F = \mathbf{F}^0 \cdot \mathbb{Q}_p$; and $[F : \mathbb{Q}_p] = [\mathbf{F}^0 : \mathbb{Q}]$. Since $[\mathbf{F}^0 : \mathbb{Q}] = \sum_{\mathfrak{p}|p} [\mathbf{F}^0_{\mathfrak{p}} : \mathbb{Q}_p]$, there exists a unique prime $\mathfrak{p}$ of $\mathbf{F}^0$ lying over p, and thus $\mathbf{F}^0 \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbf{F}^0_{\mathfrak{p}} \simeq F$. By taking $v_0 = \mathfrak{p}$, we complete the proof. ∎

Lemma 5.2 *For any prime p, there exist infinitely many odd primes q such that p splits completely over $\mathbb{Q}(\sqrt{q^*})$, where $q^* = (-1)^{\frac{q-1}{2}} q$.*

Proof This is an easy consequence of Dirichlet's density theorem. ∎

Proposition 5.3 *Let F be a p-adic field of characteristic 0. For any positive integer l, there exist a number field $\mathbf{F}$ and a finite set T of finite places of $\mathbf{F}$, with the cardinality l, such that $\mathbf{F}_v \simeq F$ for all $v \in T$.*

Proof Given F, we fix a number field $\mathbf{F}^0$ and a place v_0 as defined in Lemma 5.1, so that $\mathbf{F}^0_{v_0} \simeq F$. From Lemma 5.2, we have infinitely many odd primes q such that v_0 splits completely over $\mathbf{F}^0(\sqrt{q^*})$. Denote by T_0 the set of all such primes q. For any positive integer l, we choose a positive integer r such that $2^r \geq l$. Pick a subset $\{q_1, \dots, q_r\} \subseteq T_0$. Set $\mathbf{F} := \mathbf{F}^0(\sqrt{q_1^*}, \dots, \sqrt{q_r^*})$. We note that $[\mathbf{F} : \mathbf{F}^0] = 2^r$. Since v_0 splits completely over $\mathbf{F}^0(\sqrt{q_i^*})$ for each i, we have $\mathbf{F}_v \simeq F$ for all $v|v_0$. So, the cardinality $|\{v \mid \mathbf{F}_v \simeq F\}| \geq 2^r \geq l$. By taking T to be any subset of $\{v|\mathbf{F}_v \simeq F\}$ with cardinality l, we complete the proof. ∎

The following theorem allows us to construct a number field and connected reductive groups with prescribed local behavior.

Theorem 5.4 *Let G be a connected reductive quasi-split group over a p-adic field F of characteristic 0, and let G' be an F-inner form of G with respect to an $\bar{F}$-isomorphism $\varphi\colon G' \to G$. Then there exist a number field $\mathbf{F}$, a non-empty finite set S of finite places of $\mathbf{F}$, a connected reductive group $\mathbf{G}$ defined over $\mathbf{F}$ and its $\mathbf{F}$-inner form $\mathbf{G}'$ such that*

(a) *for all $v \in S$, $\mathbf{F}_v \simeq F$, $\mathbf{G}_v \simeq G$, and $\mathbf{G}'_v \simeq G'$ over $\mathbf{F}_v$,*
(b) *for all $v \notin S$ including all the archimedean places, $\mathbf{G}_v \simeq \mathbf{G}'_v$ over $\mathbf{F}_v$,*

where $\mathbf{G}_v$ and $\mathbf{G}'_v$ denote $\mathbf{G} \times_{\mathbf{F}} \mathbf{F}_v$ and $\mathbf{G}' \times_{\mathbf{F}} \mathbf{F}_v$, respectively.

Proof Let l denote a sufficiently large multiple of the cardinality $|A(G^{\mathrm{ad}})|$ of $A(G^{\mathrm{ad}})$ (see Section 2.3 for the definition of $A(G^{\mathrm{ad}})$). From Proposition 5.3, we obtain a number field $\mathbf{F}$ and a finite set T of places with the cardinality l such that $\mathbf{F}_v \simeq F$ for all $v \in T$. We note that the set $A(G^{\mathrm{ad}})$ is a finite abelian group.

Next we choose a connected reductive quasi-split group $\mathbf{G}$ defined over $\mathbf{F}$. It follows that $\mathbf{G}_v := \mathbf{G} \times_{\mathbf{F}} \mathbf{F}_v \simeq G$ over $\mathbf{F}_v$ for all $v \in T$. We note that $A(\mathbf{G}^{\mathrm{ad}})$ is a finite abelian group, and there is a surjective homomorphism $\bigoplus_v A(\mathbf{G}_v^{\mathrm{ad}}) \to A(\mathbf{G}^{\mathrm{ad}})$ for any place v of $\mathbf{F}$ (see [30, 2.3]). Here $\bigoplus_v$ denotes the subset of the direct product consisting of (x_v) such that $x_v = 1$ for all but a finite number of v. Since the integer $l = |T|$ is taken to be sufficiently large, we can assume that the cardinality $|A(\mathbf{G}^{\mathrm{ad}})|$ is smaller than l due to the surjective homomorphism. Choose a subset S of T such that the cardinality $|S|$ equals a multiple of $|A(\mathbf{G}^{\mathrm{ad}})|$.

We recall the following lemma from [30, Theorem 2.2 and Proposition 2.6] and [38, Theorem 6.22].

Lemma 5.5 *Let $\bar{\mathbb{A}}$ denote the adele ring of $\bar{\mathbf{F}}$. Then there is an exact sequence with a* *commutative diagram:*

$$\begin{array}{ccccccc} 1 \longrightarrow H^1\big(\mathbf{F}, \mathbf{G}^{\mathrm{ad}}(\bar{\mathbf{F}})\big) & \xrightarrow{\iota_{\mathbf{G}}} & H^1\big(\mathbf{F}, \mathbf{G}^{\mathrm{ad}}(\bar{\mathbb{A}})\big) & \xrightarrow{\beta_{\mathbf{G}}} & A(\mathbf{G}^{\mathrm{ad}}) & \longrightarrow & 1 \\ & & \approx\big\downarrow & & \Sigma\big\uparrow & & \\ & & \bigoplus_v H^1\big(\mathbf{F}_v, \mathbf{G}_v^{\mathrm{ad}}(\bar{\mathbf{F}}_v)\big) & \longrightarrow & \bigoplus_v A(\mathbf{G}_v^{\mathrm{ad}}) & & \end{array}$$

The bottom map is given by local maps defined in Proposition 2.6, which are isomorphisms for all finite places of $\mathbf{F}$. Thus, the commutative diagram implies that the morphism $\beta_{\mathbf{G}}$ is equal to sum of local 1-cocycles.

Since G' is an F-inner form of G with respect to an $\bar{F}$-isomorphism $\varphi\colon G' \to G$, we have a 1-cocycle

$$\varphi_{\tau_v} \in H^1\big(\mathbf{F}_v, \mathbf{G}_v^{\mathrm{ad}}(\bar{\mathbf{F}}_v)\big)$$

such that $\varphi_{\tau_v} := \varphi \circ \tau_v(\varphi)^{-1}$ for $\tau_v \in \mathrm{Gal}(\bar{\mathbf{F}}_v/\mathbf{F}_v)$. Let

$$a_\tau := (a_{\tau_v}) \in H^1\big(\mathbf{F}, \mathbf{G}^{\mathrm{ad}}(\bar{\mathbb{A}})\big)$$

be a nontrivial 1-cocycle such that $a_{\tau_v} = \varphi_{\tau_v}$ for all $v \in S$ and $a_{\tau_v} = 1$ for all $v \notin S$. Since $A(\mathbf{G}^{\mathrm{ad}})$ is a finite abelian group and $|S|$ is a multiple of $|A(\mathbf{G}^{\mathrm{ad}})|$, we get

$$\beta_{\mathbf{G}}(a_\tau) = \sum_v a_{\tau_v} = |S| \cdot \varphi_{\tau_v} = 1.$$

From the exactness in Lemma 5.5, we have a nontrivial 1-cocycle $b_\tau \in H^1\big(\mathbf{F}, \mathbf{G}^{\mathrm{ad}}(\bar{\mathbf{F}})\big)$ such that $\iota_{\mathbf{G}}(b_\tau) = a_\tau$. Thus, we obtain an $\mathbf{F}$-inner form $\mathbf{G}'$ of $\mathbf{G}$ associated to b_τ. From the definition of a_τ in (5.1), it follows that $\mathbf{G}'_v \simeq G'$ over $\mathbf{F}_v$ for all $v \in S$, and $\mathbf{G}_v \simeq \mathbf{G}'_v$ over $\mathbf{F}_v$ for all $v \notin S$. This completes the proof of Theorem 5.4. ■

Remark 5.6 Theorem 5.4 is an analogue of the well-known result:

$$1 \longrightarrow H^2\big(\mathrm{Gal}(\bar{\mathbf{F}}/\mathbf{F}), \mathbf{F}^\times\big) \longrightarrow \bigoplus_v H^2\big(\mathrm{Gal}(\bar{\mathbf{F}}_v/\mathbf{F}_v), \mathbf{F}_v^\times\big) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

To be precise, this exact sequence explains how to obtain a central division algebra over $\mathbf{F}$ from a given central division algebra over F. When an F-group G satisfies condition (4.1), this is a manner to construct an $\mathbf{F}$-group $\mathbf{G}$ and its $\mathbf{F}$-inner form $\mathbf{G}'$ with prescribed local behavior. Theorem 5.4 fully extends this notion to any connected reductive group over F.

Given an irreducible unitary supercuspidal representation, the following proposition tells us how to construct a cuspidal automorphic representation with specified local behavior at a finite set of places.

Proposition 5.7 ([24, Théorème, Appendice 1]) *Let $\mathbf{F}$ be a global field, $\mathbf{G}$ a connected reductive group over $\mathbf{F}$, $Z(\mathbf{G})$ its center, ω a unitary character of $Z(\mathbf{G})(\mathbf{F})\backslash Z(\mathbf{G})(\mathbb{A})$, S a nonempty finite set of finite places of $\mathbf{F}$ and, for $v \in S$, ρ_v an irreducible unitary supercuspidal representation of $\mathbf{G}(\mathbf{F}_v)$ with central character ω_v. Then there exists a cuspidal automorphic representation $\pi = \bigotimes_v \pi_v$ of $\mathbf{G}(\mathbb{A})$ with central character ω such that $\pi_v \simeq \rho_v$ for all $v \in S$.*

Remark 5.8 If the set S is chosen so that each group $\mathbf{G}_v$ is unramified over $\mathbf{F}_v$ at all finite place $v \notin S$, then each representation π_v is unramified for all finite place $v \notin S$ by the choice of each f_v in the proof of [24, Théorème, Appendice 1] (*cf.* [42, Section 5]).

5.2 Local and Global Compatibility in Restriction

In this section, we establish the local and global compatibility in the restriction of representations from a group to its subgroup sharing the same derived group. Let F denote a p-adic field of characteristic 0 with the ring of integers $\mathcal{O}_F$.

Proposition 5.9 *Let G and $\widetilde{G}$ be unramified groups over F such that*

$$G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}} \subseteq G \subseteq \widetilde{G}.$$

Given an unramified representation $\widetilde{\tau}$ of $\widetilde{G}(F)$, its restriction $\widetilde{\tau}|_{G(F)}$ to $G(F)$ has a unique unramified constituent with respect to $G(\mathcal{O}_F)$.

Proof Fix a Borel subgroup $\widetilde{B} = \widetilde{T}U$ of $\widetilde{G}$. Then we have a Borel subgroup $B = \widetilde{B} \cap G = (\widetilde{T} \cap G)U$ of G. Write $T = \widetilde{T} \cap G$. From [9, Proposition 2.6], we have a $\widetilde{G}(F)$-embedding of $\widetilde{\tau}$ into an unramified principal series $i_{\widetilde{G},\widetilde{B}}\widetilde{\chi}$, where $\widetilde{\chi}$ is an unramified character of $\widetilde{T}(F)$. Consider the restriction $(i_{\widetilde{G},\widetilde{B}}\widetilde{\chi})|_{G(F)}$. We note that $\widetilde{G}(F) = \widetilde{T}(F)G(F)$, and $f(\tilde{t}g) = \widetilde{\chi}(\tilde{t})f(g)$ for $\tilde{t} \in \widetilde{T}(F)$ and $g \in G(F)$. It follows that if $f|_{G(F)} = 0$ for $f \in i_{\widetilde{G},\widetilde{B}}\widetilde{\chi}$, then $f = 0$. By sending $f \mapsto f|_{G(F)}$, we thus have a $G(F)$-equivariant embedding

$$i_{\widetilde{G},\widetilde{B}}\widetilde{\chi} \hookrightarrow i_{G,B}\chi.$$

Here χ is the restriction of $\widetilde{\chi}$ to $T(F)$. Since χ is an unramified character of $T(F)$, we note that $i_{G,B}\chi$ has a unique non-trivial spherical vector with respect to $G(\mathcal{O}_F)$ up to scalar. Hence, there is exactly one irreducible constituent of $i_{G,B}\chi$ which contains the unique non-trivial spherical vector. This completes the proof. ∎

Let $\mathbf{F}$ denote a number field, and let $\mathbb{A}$ denote the adele ring of $\mathbf{F}$. Let $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ be connected reductive groups over $\mathbf{F}$ such that

$$\widetilde{\mathbf{G}}_{\mathrm{der}} = \mathbf{G}_{\mathrm{der}} \subseteq \mathbf{G} \subseteq \widetilde{\mathbf{G}}.$$

Let $\widetilde{\pi} = \bigotimes_v \widetilde{\pi}_v$ be an irreducible admissible representation of $\widetilde{\mathbf{G}}(\mathbb{A})$. Proposition 5.9 implies that $\widetilde{\pi}_v|_{\mathbf{G}(\mathbf{F}_v)}$ has a unique unramified constituent (denoted by π_v^0) with respect to $\mathbf{G}(\mathcal{O}_v)$ for almost all places v, where $\mathcal{O}_v$ is the ring of integers of $\mathbf{F}_v$. The following proposition states the local and global compatibility in the restriction which is an analogue of [35, Lemma 1].

Proposition 5.10 *Every irreducible constituent of $\widetilde{\pi}|_{\mathbf{G}(\mathbb{A})}$ is of the form $\pi = \bigotimes_v \pi_v$, where π_v is an irreducible constituent of $\widetilde{\pi}_v|_{\mathbf{G}(\mathbf{F}_v)}$ and $\pi_v \simeq \pi_v^0$ for almost all v.*

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Proof We follow the proof of [35, Lemma 1]. It is clear that any representation of the above form $\pi = \bigotimes_v \pi_v$ is an irreducible constituent of $\widetilde{\pi}|_{\mathbf{G}(\mathbb{A})}$. Conversely, let the constituent π act on V/U with $0 \subseteq U \subseteq V \subseteq X = \bigotimes_v X_v$. To construct the tensor product, we choose a finite set T_0 of places and a non-zero spherical vector x_0 with respect to $\mathbf{G}(\mathcal{O}_v)$ for each $v \notin T_0$. We can find a finite set T of places and a vector $x_T \in X_T := \bigotimes_{v \in T} X_v$ such that T contains T_0 and $x = x_T \otimes (\bigotimes_{v \notin T} x_v^0)$ lies in V but not in U. Let V_T be the smallest subspace of X_T containing x_T and invariant under $\mathbf{G}_T := \Pi_{v \in T} \mathbf{G}(\mathbf{F}_v)$. There is a surjective map

$$V_T \otimes \Big(\bigotimes_{v \notin T} V_v \Big) \longrightarrow V/U.$$

If $v_0 \notin T$, then the kernel contains $V_T \otimes U_{v_0} \otimes (\bigotimes_{v \notin T \cup \{v_0\}} V_v)$. We obtain a surjection $V_T \otimes (\bigotimes_{v \in T} V_v/U_v) \to V/U$ with a kernel of the form $U_T \otimes (\bigotimes_{v \in T} V_v/U_v)$, where U_T lies in V_T. We note from [48, Lemma 2.1] that $\widetilde{\pi}_v|_{\mathbf{G}(\mathbf{F}_v)}$ is a finite direct sum of irreducible constituents of $\mathbf{G}(\mathbf{F}_v)$. So, the representation of $\mathbf{G}_T$ on V_T/U_T is irreducible and is of the form $\bigotimes_{v \in T} \pi_v$, where π_v is an irreducible constituent of $\widetilde{\pi}_v|_{\mathbf{G}(\mathbf{F}_v)}$. Thus, the constituent π is of the form $\bigotimes_v \pi_v$ such that $\pi_v \simeq \pi_v^0$ for $v \notin T$. ■

Remark 5.11 Let $\widetilde{\pi}$ be a cuspidal automorphic representation of $\widetilde{\mathbf{G}}(\mathbb{A})$. Since $\mathbf{G}(\mathbb{A})$ is a subgroup of $\widetilde{\mathbf{G}}(\mathbb{A})$ sharing the same derived group, the restriction $\mathcal{H}(\widetilde{\pi})|_{\mathbf{G}(\mathbb{A})}$ of the space $\mathcal{H}(\widetilde{\pi})$ from $\widetilde{\mathbf{G}}(\mathbb{A})$ to $\mathbf{G}(\mathbb{A})$ is still a non-zero subspace of cusp forms on $\mathbf{G}(\mathbb{A})$. Due to the result [11, Theorem 2.1, p. 113] of Gelfand and Piatetski–Shapiro, we have the decomposition

$$\widetilde{\pi}|_{\mathbf{G}(\mathbb{A})} = \bigoplus m_\pi \pi.$$

Here π runs through all irreducible constituents of the restriction $\widetilde{\pi}|_{\mathbf{G}(\mathbb{A})}$ which are cuspidal automorphic representations of $\mathbf{G}(\mathbb{A})$, and m_π is the multiplicity of π.

Remark 5.12 Let $\pi = \bigotimes \pi_v$ be a cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$. Theorem 4.13 in [25] verifies that there exists a cuspidal automorphic representation $\widetilde{\pi} = \bigotimes \widetilde{\pi}_v$ of $\widetilde{\mathbf{G}}(\mathbb{A})$ such that π is an irreducible constituent of $\widetilde{\pi}|_{\mathbf{G}(\mathbb{A})}$. From Proposition 5.10, we note that π_v is an irreducible constituent of $\widetilde{\pi}_v|_{\mathbf{G}(\mathbf{F}_v)}$ for all v.

6 Transfer for Unitary Supercuspidal Representations under the Local JL-Type Correspondence

Using a local to global argument in Section 5, we prove that Plancherel measures attached to unitary supercuspidal representations are preserved under the local JL-type correspondence, assuming a working hypothesis that Plancherel measures are invariant on a certain finite set. Throughout Section 6, F denotes a p-adic field of characteristic 0, and M denotes an F-Levi subgroup of a connected reductive F-split group G such that

$$\Pi_{i=1}^r \mathrm{SL}_{n_i} \subseteq M \subseteq \Pi_{i=1}^r \mathrm{GL}_{n_i} \tag{6.1}$$

for positive integers r and n_i. Let G' be an F-inner form of G, and let M' be an F-Levi subgroup of G' that is an F-inner form of M. Then M' satisfies the following property

$$\Pi_{i=1}^{r} \mathrm{SL}_{m_i}(D_{d_i}) \subseteq M'(F) \subseteq \Pi_{i=1}^{r} \mathrm{GL}_{m_i}(D_{d_i}).$$

Here D_{d_i} denotes a central division algebra of dimension d_i^2 over F where $n_i = m_i d_i$. Write $\widetilde{M}(F) = \Pi_{i=1}^{r} \mathrm{GL}_{n_i}(F)$ and $\widetilde{M}'(F) = \Pi_{i=1}^{r} \mathrm{GL}_{m_i}(D_{d_i})$.

We denote by θ and θ' the subsets of Δ and Δ' such that $M = M_\theta$ and $M = M_{\theta'}$, respectively. We fix representatives $w \in G(F)$ and $w' \in G'(F)$ of $\tilde{w} \in W_G$ and $\tilde{w} \in W_{G'}$ such that $\tilde{w} = \varphi(\tilde{w}')$, $\tilde{w}(\theta) \subseteq \Delta$ and $\tilde{w}'(\theta') \subseteq \Delta'$ as stated in Section 2.3.

6.1 Statement of Theorem

In this section, we state the main result and its contributions. Let $\sigma \in \mathcal{E}_u^\circ\big(M(F)\big)$ and $\sigma' \in \mathcal{E}_u^\circ(M'(F))$ be under the local JL-type correspondence. Since both σ and σ' are supercuspidal, by Remark 4.2 and Definition 4.5, we have $\widetilde{\sigma} \in \mathcal{E}_u^\circ\big(\widetilde{M}(F)\big)$ and $\widetilde{\sigma}' \in \mathcal{E}_u^\circ\big(\widetilde{M}'(F)\big)$ such that

(a) σ and σ' are isomorphic to irreducible constituents of the restrictions $\widetilde{\sigma}|_{M(F)}$ and $\widetilde{\sigma}'|_{M'(F)}$, respectively,

(b) $\mathbf{C}(\tilde{\sigma}) = \tilde{\sigma}'$.

We recall the sets $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$ and $\Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ of equivalence classes of all irreducible constituents of $\widetilde{\sigma}|_{M(F)}$ and $\widetilde{\sigma}'|_{M'(F)}$, respectively.

Working Hypothesis 6.1 *Let σ_1' and σ_2' be given in $\Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$. Then we have*

$$\mu_{M'}(\nu', \sigma_1', w') = \mu_{M'}(\nu', \sigma_2', w')$$

for any $\nu' \in \mathfrak{a}^*_{M',\mathbb{C}}$.

Remark 6.2 Since $\Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ can be considered as an L-packet on M' (*cf.* Remark 4.4), this hypothesis is related to the L-packet invariance of the Plancherel measure. When G' is an F-inner form of SL_n and M' is any F-Levi subgroup of G', Working Hypothesis 6.1 is a consequence of Proposition 2.4.

The following states our main result.

Theorem 6.3 *Let $\sigma \in \mathcal{E}_u^\circ\big(M(F)\big)$ and $\sigma' \in \mathcal{E}_u^\circ\big(M'(F)\big)$ be under the local JL-type correspondence. Assume that Working Hypothesis 6.1 is valid. Then we have*

$$\mu_M(\nu, \sigma, w) = \mu_{M'}(\nu, \sigma', w')$$

for any $\nu \in \mathfrak{a}^*_{M,\mathbb{C}} \simeq \mathfrak{a}^*_{M',\mathbb{C}}$.

Remark 6.4 If the central character of $\widetilde{\sigma}'$ is unramified and the set $\Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ is associated to a tempered and tame regular semi-simple elliptic L-parameter, the assumption is no longer needed due to [10].

We note that any two members $\tau \in \Pi_{\widetilde{\sigma}}\big(M(F)\big)$ and $\tau' \in \Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ are under the JL-type correspondence (see Remark 4.7). Fix $\sigma' \in \mathcal{E}_u^\circ\big(M'(F)\big)$ in Theorem 6.3. By varying σ over $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$, we have the following.

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Proposition 6.5 *Let τ_1 and τ_2 be given in $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$. Assume that Working Hypothesis 6.1 is valid. Then we have*

$$\mu_M(\nu, \tau_1, w) = \mu_M(\nu, \tau_2, w)$$

for all $\nu \in \mathfrak{a}^*_{M,\mathbb{C}}$.

Remark 6.6 Since $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$ can be considered as an L-packet on M (*cf.* Remark 4.4), Proposition 6.5 supports the conjecture that Plancherel measures are invariant on L-packets. We also refer the reader to [2, 10, 15–17] for other cases.

Remark 6.7 Proposition 6.5 reduces the study of the Plancherel measures for tempered representations to the generic cases. To be precise, since the unitary supercuspidal representation $\widetilde{\sigma}$ of $\widetilde{M}(F)$ is generic with respect to a generic character ψ [32, Section 2.3], there exists a unique generic representation $\tau_0 \in \Pi_{\widetilde{\sigma}}\big(M(F)\big)$ with respect to ψ (*cf.* [48, Proposition 2.8]). Then the result [42, Corollary 3.6] of Shahidi gives an explicit formula of $\mu_M(\nu, \tau_0, w)$ in terms of local factors via the Langlands–Shahidi method. Therefore, due to Proposition 6.5, the Plancherel measure $\mu_M(\nu, \tau, w)$ attached to any $\tau \in \Pi_{\widetilde{\sigma}}\big(M(F)\big)$ has the same formula with $\mu_M(\nu, \tau_0, w)$. Moreover, Theorem 6.3 admits the same formula for $\mu_{M'}(\nu, \tau', w')$ with any $\tau' \in \Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$.

6.2 Proof of Theorem 6.3

In this section, we present the proof of Theorem 6.3 by a local to global argument in Section 5. The outline is as follows.

Step 1 Construct global data from given local data as explained in Section 5.1.
Step 2 Find two cuspidal automorphic representations as described in Proposition 6.9.
Step 3 Use Langlands' functional equation on Eisenstein series.

This is a more general version of the method of Muić and Savin in [37]. Along with their Siegel Levi subgroups ($\simeq \mathrm{GL}_n$) of the groups SO_{2n} and Sp_{2n}, our construction in Step 1 treats any F-Levi subgroup of connected reductive groups over a p-adic field of characteristic 0. Further, Step 2 can be applied to any connected reductive group M satisfying condition (6.1).

The rest of this section is devoted to the proof of Theorem 6.3. Due to Remarks 2.2 and 2.5, it suffices to consider the case when G is semisimple (so is G') and M is maximal (so is M'). It then turns out that $\mathfrak{a}^*_{M,\mathbb{C}} \simeq \mathfrak{a}^*_{M',\mathbb{C}} \simeq \mathbb{C}$. Thus, we shall show that: for all $s \in \mathbb{C}$,

$$\mu_M(s, \sigma, w) = \mu_{M'}(s, \sigma', w').$$

We start with the following lemma which is immediately a consequence of Theorem 5.4.

Lemma 6.8 *Let F, G, G', M, M', $\widetilde{M}$ and $\widetilde{M}'$ be as above. Then there exist a number field $\mathbf{F}$, a non-empty finite set S of finite places of $\mathbf{F}$, a connected reductive $\mathbf{F}$-split group $\mathbf{G}$*

and its $\mathbf{F}$-inner form $\mathbf{G}'$, their $\mathbf{F}$-Levi subgroups $\mathbf{M}$ and $\mathbf{M}'$ ($\mathbf{F}$-inner form of $\mathbf{M}$), a connected reductive $\mathbf{F}$-split group $\widetilde{\mathbf{M}}$ and its $\mathbf{F}$-inner form $\widetilde{\mathbf{M}}'$ such that

(a) *for all $v \in S$, $\mathbf{F}_v \simeq F$, $\mathbf{G}_v \simeq G$, $\mathbf{G}'_v \simeq G'$, $\mathbf{M}_v \simeq M$, $\mathbf{M}'_v \simeq M'$, $\widetilde{\mathbf{M}}_v \simeq \widetilde{M}$ and $\widetilde{\mathbf{M}}'_v \simeq \widetilde{M}'$ over $\mathbf{F}_v$,*

(b) *for all $v \notin S$, $\mathbf{G}_v \simeq \mathbf{G}'_v$, $\mathbf{M}_v \simeq \mathbf{M}'_v$ and $\widetilde{\mathbf{M}}_v \simeq \widetilde{\mathbf{M}}'_v$ over $\mathbf{F}_v$,*

(c) $\mathbf{M}_{\mathrm{der}} = \widetilde{\mathbf{M}}_{\mathrm{der}} \subseteq \mathbf{M} \subseteq \widetilde{\mathbf{M}}$ *and* $\mathbf{M}'_{\mathrm{der}} = \widetilde{\mathbf{M}}'_{\mathrm{der}} \subseteq \mathbf{M}' \subseteq \widetilde{\mathbf{M}}'$.

Note that $\widetilde{\mathbf{M}}(\mathbf{F})$ and $\widetilde{\mathbf{M}}'(\mathbf{F})$ are of the forms $\prod_{i=1}^{r} \mathrm{GL}_{n_i}(\mathbf{F})$ and $\prod_{i=1}^{r} \mathrm{GL}_{m_i}(\mathbf{D}_{d_i})$, respectively. Here $\mathbf{D}_i$ denotes a central division algebra of dimension d_i^2 over $\mathbf{F}$ and $n_i = m_i d_i$. For all $v \in S$, it turns out that $\mathbf{G}'_v$, $\mathbf{M}'_v$ and $\widetilde{\mathbf{M}}'_v$ are *non quasi-split F_v*-inner forms of $\mathbf{G}_v$, $\mathbf{M}_v$ and $\widetilde{\mathbf{M}}_v$ respectively. Also, $\mathbf{G}_v$, $\mathbf{G}'_v$, $\mathbf{M}_v$ and $\mathbf{M}'_v$ are all *quasi-split* over $\mathbf{F}_v$ unless $v \in S$.

From now on, we fix a number field $\mathbf{F}$ and a finite set S of finite places of $\mathbf{F}$ as defined in Lemma 6.8, so that $\mathbf{F}_v \simeq F$ for all $v \in S$ and S consists of all non-split places. Next, we find two following cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of $\mathbf{M}(\mathbb{A})$ and $\pi' = \bigotimes_v \pi'_v$ of $\mathbf{M}'(\mathbb{A})$.

Proposition 6.9 *Let $\sigma \in \mathcal{E}_u^{\circ}\big(M(F)\big)$ and $\sigma' \in \mathcal{E}_u^{\circ}\big(M'(F)\big)$ be under the local JL-type correspondence. Then there exist a finite set V of places of $\mathbf{F}$ containing S and all archimedean places, and two cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of $\mathbf{M}(\mathbb{A})$ and $\pi' = \bigotimes_v \pi'_v$ of $\mathbf{M}'(\mathbb{A})$ such that*

(a) *for all $v \in S$, $\pi_v \simeq \sigma$ and $\pi'_v \in \Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$,*

(b) *for all $v \in V - S$, π_v and π'_v are irreducible constituents of the restriction of an irreducible representation of $\widetilde{\mathbf{M}}(\mathbf{F}_v)$ to $\mathbf{M}(\mathbf{F}_v)$,*

(c) *for all $v \notin V$, π_v and π'_v are isomorphic and unramified with respect to $\mathbf{M}(\mathcal{O}_v)$.*

Proof of Proposition 6.9 From Proposition 5.7, we construct a cuspidal automorphic representation $\pi = \bigotimes \pi_v$ of $\mathbf{M}(\mathbb{A})$ such that $\pi_v \simeq \sigma$ for all $v \in S$. By Remark 5.12, we also have a cuspidal automorphic representation $\widetilde{\pi} = \bigotimes \widetilde{\pi}_v$ of $\widetilde{\mathbf{M}}(\mathbb{A})$ such that π is an irreducible constituent of $\widetilde{\pi}|_{\mathbf{M}(\mathbb{A})}$ and π_v is an irreducible constituent of $\widetilde{\pi}_v|_{\mathbf{M}(\mathbf{F}_v)}$ for all v.

We claim that $\widetilde{\pi}$ is in the image of the map Φ defined in Theorem 3.16. To see this, we need to show that $\widetilde{\pi}$ is D-compatible. For all $v \in S$, since π_v is in $\mathcal{E}_u^{\circ}\big(\mathbf{M}(\mathbf{F}_v)\big)$, $\widetilde{\pi}_v$ is also in $\mathcal{E}_u^{\circ}\big(\widetilde{\mathbf{M}}(\mathbf{F}_v)\big)$ by Remark 4.2. It follows from Remark 3.9 that $\widetilde{\pi}_v$ is d_v-compatible for all $v \in S$. Hence, from Definition 3.15, $\widetilde{\pi}$ is D-compatible. We note from Proposition 3.17 that $\widetilde{\pi}'$ is cuspidal since $\widetilde{\pi}$ is cuspidal. Therefore, we have a unique cuspidal automorphic representation $\widetilde{\pi}'$ of $\widetilde{\mathbf{M}}'(\mathbb{A})$ such that $|\mathbf{LJ}|_v(\pi_v) = \pi'_v$ for all v. Since both $\widetilde{\pi}_v$ and $\widetilde{\pi'}_v$ are supercuspidal for $v \in S$ and S consists of non-split places, we note that $|\mathbf{LJ}|_v(\widetilde{\pi}_v) = \mathbf{C}(\widetilde{\pi}_v)$ for all $v \in S$ by Remark 3.12 and $\widetilde{\pi}'_v \simeq \widetilde{\pi}_v$ for all $v \notin S$.

Now we consider the restriction $\widetilde{\pi}'|_{\mathbf{M}'(\mathbb{A})}$ of $\widetilde{\pi}'$ to $\mathbf{M}'(\mathbb{A})$. From Remark 5.11, there exists a cuspidal automorphic representation $\pi' = \bigotimes_v \pi'_v$ of $\mathbf{M}'(\mathbb{A})$ such that π'_v is an irreducible constituent of $\widetilde{\pi}'_v|_{\mathbf{M}'(\mathbf{F}_v)}$ for all v.

The assertions (a), (b) and (c) are verified as follows. For all $v \in S$, since σ is an irreducible constituent of both $\widetilde{\sigma}|_{\mathbf{M}(\mathbf{F}_v)}$ and $\widetilde{\pi}_v|_{\mathbf{M}(\mathbf{F}_v)}$, it follows from Remark 4.3 that $\widetilde{\pi}_v \simeq \widetilde{\sigma} \otimes (\eta_v \circ \det)$ for some quasi-character η_v of $\mathbf{F}_v^{\times}$ (in fact, η_v is unitary).

Remark 3.2 yields that $\widetilde{\pi}'_v \simeq \widetilde{\sigma}' \otimes (\eta_v \circ \mathrm{Nrd})$. So, we have from Remark 4.3 that π'_v lies in $\Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ for all $v \in S$. Since $\widetilde{\pi}'_v \simeq \widetilde{\pi}_v$ for all $v \notin S$, Proposition 5.10 allows us to have a finite set V containing S such that π_v and π'_v are isomorphic and unramified for all $v \notin V$ with respect to $\mathbf{M}(\mathcal{O}_v)$. Also, for all $v \in V - S$, π_v and π'_v are irreducible constituents of the restriction of $\widetilde{\pi}'_v \simeq \widetilde{\pi}_v$ of $\widetilde{\mathbf{M}}(\mathbf{F}_v)$ to $\mathbf{M}(\mathbf{F}_v)$. This completes the proof of Proposition 6.9. ∎

We consider the standard global intertwining operator $M(s,\pi) := \bigotimes_v A(s,\pi_v,w)$ from the global induced representation $I(s,\pi)$ to $I\big(-s,w(\pi)\big)$. We refer to [11, Sections 4 and 7]. For $f \in I(s,\pi)$, we note that

$$f \in \bigotimes_{v\in V} I(s,\pi_v) \otimes \Big(\bigotimes_{v\notin V} f_v^0\Big),$$

where f_v^0 is a unique spherical vector such that $f_v^0|_{\mathbf{M}(\mathcal{O}_v)} = 1$. Let $M(s,\pi')$, $I(s,\pi')$ and $f_v'^0$ have the corresponding meaning for the cuspidal automorphic representation π' of $\mathbf{M}'(\mathbb{A})$. Due to Proposition 6.9, we identify f_v^0 and $f_v'^0$. Given $f = \bigotimes_v f_v \in I(s,\pi)$ and $f' = \bigotimes_v f'_v \in I(s,\pi')$, we get

$$M(s,\pi)f = \bigotimes_{v\in V} A(s,\pi_v,w)f_v \otimes \Big(\bigotimes_{v\notin V} A(s,\pi_v,w)f_v^0\Big),$$

$$M(s,\pi')f' = \bigotimes_{v\in V} A(s,\pi'_v,w')f'_v \otimes \Big(\bigotimes_{v\notin V} A(s,\pi'_v,w')f_v^0\Big).$$

So, we have the following functional equations by Eisenstein series

$$M(s,\pi)M\big(-s,w(\pi)\big) = \mathrm{id},$$

$$M(s,\pi')M\big(-s,w(\pi')\big) = \mathrm{id}.$$

It then follows that

$$\prod_{v\in V} \mu_{\mathbf{M}_v}(s,\pi_v,w)\gamma_{\tilde{w}}(\mathbf{G}_v|\mathbf{M}_v)^{-2} \prod_{v\notin V} c_v(s,\pi_v)c_v\big(-s,w(\pi_v)\big) = 1, \tag{6.2}$$

$$\prod_{v\in V} \mu_{\mathbf{M}'_v}(s,\pi'_v,w')\gamma_{\tilde{w}}(\mathbf{G}'_v|\mathbf{M}'_v)^{-2} \prod_{v\notin V} c_v(s,\pi'_v)c_v\big(-s,w(\pi'_v)\big) = 1. \tag{6.3}$$

Here $c_v(\cdot\,,\,\cdot)$ is a quotient of the local Langlands L-functions for unramified representations (see [41, (2.7) p. 554]). Since $\pi_v \simeq \pi'_v$ for all $v \notin V$, we have

$$c_v(s,\pi_v)c_v\big(-s,w(\pi_v)\big) = c_v(s,\pi'_v)c_v\big(-s,w(\pi'_v)\big).$$

For all $v \in V - S$, since π_v and π'_v are irreducible constituents of the restriction of an irreducible representation (denoted by $\widetilde{\tau}_v$) of $\widetilde{\mathbf{M}}(\mathbf{F}_v)$ to $\mathbf{M}(\mathbf{F}_v)$, both $\mu_{\mathbf{M}_v}(s,\pi_v,w)$ and $\mu_{\mathbf{M}'_v}(s,\pi'_v,w')$ can be expressed in terms of the Artin L-function and root number attached to the L-parameter of $\widetilde{\tau}_v$ (see [9, Proposition 2.6] and [27, Theorem 3.1] for non-archimedean places, [1, Section 3] for archimedean places). It then follows that

$\mu_{\mathbf{M}_v}(s, \pi_v, w) = \mu_{\mathbf{M}'_v}(s, \pi'_v, w')$. Further, we note that $\gamma_{\bar{w}}(\mathbf{G}_v|\mathbf{M}_v) = \gamma_{\bar{w}}(\mathbf{G}'_v|\mathbf{M}'_v)$ by [3, p. 89]. So, we have

$$\prod_{v \in S} \mu_{\mathbf{M}_v}(s, \pi_v, w) = \prod_{v \in S} \mu_{\mathbf{M}'_v}(s, \pi'_v, w').$$

For all $v \in S$, we note that: $\mathbf{M}_v \simeq M$ and $\mathbf{M}'_v \simeq M'$ over $\mathbf{F}_v$ by Lemma 6.8; $\pi_v \simeq \sigma$ by Proposition 6.9; $\mu_{\mathbf{M}'_v}(s, \pi'_v, w') = \mu_{\mathbf{M}'_v}(s, \sigma', w')$ by Working Hypothesis 6.1. Hence, we deduce from equations (6.2) and (6.3) that

$$\mu_M(s, \sigma, w)^m = \mu_{M'}(s, \sigma', w')^m. \tag{6.4}$$

Here m denotes the cardinality of S. Since Plancherel measures are holomorphic and non-negative along the unitary axis $\mathrm{Re}(s) = 0$, we therefore have that

$$\mu_M(s, \sigma, w) = \mu_{M'}(s, \sigma', w')$$

for all $s \in \mathbb{C}$. This completes the proof of Theorem 6.3.

7 Applications

In this section, we present some applications of Theorem 6.3. We continue with the notation in Section 6. Throughout Section 7, we assume that M and M' are maximal. For $\sigma \in \mathrm{Irr}\big(M(F)\big)$ and $\sigma' \in \mathrm{Irr}\big(M'(F)\big)$, we define $W(\sigma) := \{w \in W_M : {}^w\sigma \simeq \sigma\}$ and $W(\sigma') := \{w' \in W_{M'} : {}^{w'}\sigma' \simeq \sigma'\}$.

Proposition 7.1 *Let $\sigma \in \mathcal{E}_u^\circ\big(M(F)\big)$ and $\sigma' \in \mathcal{E}_u^\circ\big(M'(F)\big)$ be under the local JL-type correspondence. Suppose that $|W(\sigma)| = |W(\sigma')| = 2$. Let $\nu_0 \in \mathbb{R}$ be given. Then under Working Hypothesis 6.1, $i_{G,M}(\nu, \sigma)$ is reducible at $\nu = \nu_0$ if and only if $i_{G',M'}(\nu, \sigma')$ is reducible at $\nu = \nu_0$. Moreover, ν_0 is either 0 or $\pm x_0$ for some positive real number x_0.*

Proof If $\mu_M(0, \sigma, w) \neq 0$, we have $\mu_{M'}(0, \sigma', w') \neq 0$ from Theorem 6.3. Since $|W(\sigma)| = |W(\sigma')| = 2$, we note from [45, Corollary 5.4.2.3] that both $i_{G,M}(0, \sigma)$ and $i_{G',M'}(0, \sigma')$ are reducible. Also, it follows from [46, Lemma 1.3] that both $\mu_M(\nu, \sigma, w)$ and $\mu_{M'}(\nu, \sigma', w')$ are holomorphic for all $\nu \in \mathbb{R} - \{0\}$. Hence, by [45, Lemma 5.4.2.4], both $i_{G,M}(\nu, \sigma)$ and $i_{G',M'}(\nu, \sigma')$ are irreducible for all $\nu \in \mathbb{R} - \{0\}$.

If $\mu_M(0, \sigma, w) = 0$, Theorem 6.3 implies that $\mu_{M'}(0, \sigma', w') = 0$. So, by [45, Corollary 5.4.2.3], both $i_{G,M}(0, \sigma)$ and $i_{G',M'}(0, \sigma')$ are irreducible. It follows from [46, Lemma 1.2] that, for $\nu \in \mathbb{R}$, there exists a unique $x_0 > 0$ such that $\mu_M(\nu, \sigma, w)$ has a (simple) pole at $\nu = \pm x_0$. We note from [45, Lemma 5.4.2.4] that $i_{G,M}(\nu, \sigma)$ with $\nu \in \mathbb{R}$ is reducible only at $\nu = \pm x_0$. By Theorem 6.3, the same is true for $i_{G',M'}(\nu, \sigma')$. Therefore, both are irreducible for all $\nu \in \mathbb{R} - \{\pm x_0\}$. ■

Remark 7.2 If $|W(\sigma)| = |W(\sigma')| = 1$, both $i_{G,M}(\nu, \sigma)$ and $i_{G',M'}(\nu, \sigma')$ are irreducible for any $\nu \in \mathbb{R}$ due to [46, Lemma 1.3] and [45, Lemma 5.4.2.4].

For $\sigma \in \mathrm{Irr}_u\big(M(F)\big)$ and $\nu \in \mathbb{R}$, we say that $i_{G,M}(\nu, \sigma)$ is *in the complementary series* if it is unitarizable. The following is immediately a consequence of Proposition 7.1.

Corollary 7.3 *Assume the hypotheses of Proposition 7.1. If $i_{G,M}(0, \sigma)$ is irreducible, then $i_{G,M}(\nu, \sigma)$ is in the complementary series if and only if $i_{G',M'}(\nu, \sigma')$ is in the complementary series if and only if $|\nu| < x_0$. If $i_{G,M}(0, \sigma)$ is reducible, then neither $i_{G,M}(\nu, \sigma)$ nor $i_{G',M'}(\nu, \sigma')$ is in the complementary series for $\nu > 0$.*

Remark 7.4 Proposition 7.1 and Corollary 7.3 imply that the reducibility of $i_{G,M}(\nu, \sigma)$ and the edges of the complementary series are transferred to those of $i_{G',M'}(\nu, \sigma')$ for $\nu \in \mathbb{R}$.

More explicit values of x_0 are made in the following corollary which is a consequence of Remark 6.7, Proposition 7.1, Corollary 7.3 and [42, Theorem 8.1].

Corollary 7.5 *Assume the hypotheses of Proposition 7.1. Suppose $i_{G,M}(0, \sigma)$ is irreducible. Choose a unique i, $i = 1$ or 2, such that $P_{\sigma,i}(1) = 0$ (see [42, Corollary 7.6] for details). Then*

(a) *the real number x_0 in Proposition 7.1 is either $\frac{1}{2}$ or 1,*
(b) *for $0 < |\nu| < 1/i$, both $i_{G,M}(\nu, \sigma)$ and $i_{G',M'}(\nu, \sigma')$ are in the complementary series,*
(c) *for $|\nu| > 1/i$, neither $i_{G,M}(\nu, \sigma)$ nor $i_{G',M'}(\nu, \sigma')$ is in the complementary series.*

In what follows, we prove that both the reducibility of $i_{G,M}(\nu, \sigma)$ and the edges of complementary series are invariant on the set $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$ when M is maximal.

Proposition 7.6 *Let $\sigma_1, \sigma_2 \in \Pi_{\widetilde{\sigma}}\big(M(F)\big)$ be given. Suppose that $|W(\sigma_1)| = |W(\sigma_2)| = 2$. Let $\nu_0 \in \mathbb{R}$ be given. Then, under Working Hypothesis 6.1, $i_{G,M}(\nu, \sigma_1)$ is reducible at $\nu = \nu_0$ if and only if $i_{G,M}(\nu, \sigma_2)$ is reducible at $\nu = \nu_0$. Moreover, ν_0 is either 0 or $\pm x_0$ for some positive real number x_0.*

Proof Fix $\sigma' \in \Pi_{\widetilde{\sigma}'}\big(M'(F)\big)$ in Proposition 7.1. By varying σ over $\Pi_{\widetilde{\sigma}}\big(M(F)\big)$, we have the proposition. ∎

In a similar way to Corollary 7.3, we have the following.

Corollary 7.7 *Assume the hypotheses of Proposition 7.6. If $i_{G,M}(0, \sigma_1)$ is irreducible, then $i_{G,M}(\nu, \sigma_1)$ is in the complementary series if and only if $i_{G,M}(\nu, \sigma_2)$ is in the complementary series if and only if $|\nu| < \nu_0$. If $i_{G,M}(0, \sigma_1)$ is reducible, then neither $i_{G,M}(\nu, \sigma_1)$ nor $i_{G,M}(\nu, \sigma_2)$ is in the complementary series for $\nu > 0$.*

8 A Generalization

Let F be a p-adic field of characteristic 0, and let M be an F-Levi subgroup of a connected reductive F-group G. Let G' be an F-inner form of G, and let M' be an F-Levi subgroup of G' that is an F-inner form of M. In this section, we extend Theorem 6.3 to the case that unitary supercuspidal representations of $M(F)$ and $M'(F)$ have the same L-parameter under the following assumption.

Assumption 8.1 *There exist a finite set V of places of $\mathbf{F}$ containing S and all archimedean places, and two cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of $\mathbf{M}(\mathbb{A})$ and $\pi' = \bigotimes_v \pi'_v$ of $\mathbf{M}'(\mathbb{A})$ such that*

(a) *for all $v \in S$, $\pi_v \simeq \sigma$ and π'_v is in the L-packet of σ',*
(b) *for all $v \in V - S$, π_v and π'_v are irreducible constituents of the restriction of an irreducible representation of $\widetilde{\mathbf{M}}(\mathbf{F}_v)$ to $\mathbf{M}(\mathbf{F}_v)$,*
(c) *for all $v \notin V$, π_v and π'_v are isomorphic and unramified.*

Remark 8.2 This assumption has been fulfilled in Proposition 6.9 when a given Levi subgroup M satisfies condition (6.1), and generalizes the global Jacquet–Langlands correspondence for GL_n to any connected reductive group.

Now we state the following proposition which is a generalization of Theorem 6.3.

Proposition 8.3 *Let σ and σ' be irreducible unitary supercuspidal representations of $M(F)$ and $M'(F)$ having the same L-parameter. Suppose that Assumption 8.1 is valid and that Plancherel measures are invariant on the L-packet of σ'. Then we have*

$$\mu_M(\nu, \sigma, w) = \mu_{M'}(\nu, \sigma', w')$$

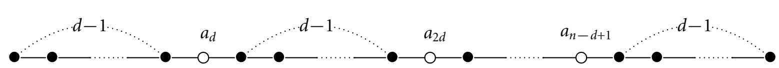
for $\nu \in \mathfrak{a}^*_{M,\mathbb{C}} \simeq \mathfrak{a}^*_{M',\mathbb{C}}$.

Proof This is proved by replacing Proposition 6.9 with Hypothesis 8.1 from the proof of Theorem 6.3. ∎

A Examples

We continue with the notation in Sections 2 and 6. We give a few examples of an F-Levi subgroup M and its F-inner form M' satisfying condition (6.1) based on the Satake classification [40, Section 3]. In the following diagrams (Satake diagrams) a black vertex indicates a root in the set of simple roots of the fixed minimal F-Levi subgroup M'_0 of G'. So, we remove only white vertices to obtain an F-Levi subgroup M' (see [40, Section 2.2] and [7, Section I.3]). We focus on the case that M is maximal (*cf.* Remark 2.5).

(1) A_n **cases**



Set $\theta = \Delta - \{\alpha_j\}$, where $\alpha_j = e_j - e_{j+1}$ for $j = d, 2d, \cdots, md$. Note that $md = n - d + 1$.

(a) Let $G = \mathrm{GL}_{n+1}$. Note that $G'(F) = \mathrm{GL}_{m+1}(D_d)$, where $n + 1 = d(m + 1)$. Then we have
$$M = M_\theta = \mathrm{GL}_{m_1 d} \times \mathrm{GL}_{m_2 d} = \widetilde{M},$$
where $m_1 d + m_2 d = n + 1$. So, $M'(F) = \mathrm{GL}_{m_1}(D_d) \times \mathrm{GL}_{m_2}(D_d)$.

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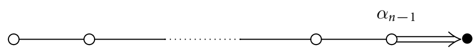
(b) Let $G = \mathrm{SL}_{n+1}$. Note that $G'(F) = \mathrm{SL}_{m+1}(D_d)$, where $n+1 = d(m+1)$. Then we have

$$M = M_\theta = G \cap (\mathrm{GL}_{m_1 d} \times \mathrm{GL}_{m_2 d}) \hookrightarrow \mathrm{GL}_{m_1 d} \times \mathrm{GL}_{m_2 d} = \widetilde{M},$$

where $m_1 d + m_2 d = n+1$. So, $M'(F) = G'(F) \cap \left(\mathrm{GL}_{m_1}(D_d) \times \mathrm{GL}_{m_2}(D_d)\right)$.

(2) B_n **cases**



Set $\theta = \Delta - \{\alpha_{n-1}\}$, where $\alpha_{n-1} = e_{n-1} - e_n$.

(a) Let $G = \mathrm{Spin}_{2n+1}$. Then we have

$$M = M_\theta \simeq \mathrm{GL}_n \times \mathrm{SL}_2 \hookrightarrow \widetilde{M} = \mathrm{GL}_n \times \mathrm{GL}_2 .$$

So, $M'(F) \simeq \mathrm{GL}_n(F) \times \mathrm{SL}_1(D_2)$.

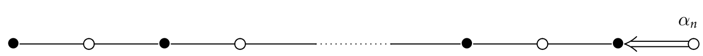
(b) Let $G = \mathrm{GSpin}_{2n+1}$. Then we have

$$M = M_\theta \simeq \mathrm{GL}_n \times \mathrm{GL}_2 = \widetilde{M}.$$

So, $M'(F) \simeq \mathrm{GL}_n(F) \times \mathrm{GL}_1(D_2)$.

(3) C_n **cases**

(**n: even**)



(every other dot black)

Set $\theta = \Delta - \{\alpha_n\}$, where $\alpha_n = 2e_n$.

(a) Let $\mathbf{G} = \mathrm{Sp}_{2n}$. Then we have

$$M = M_\theta \simeq \mathrm{GL}_n = \widetilde{M},$$

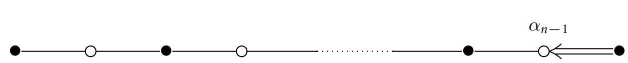
which is the Siegel Levi subgroup. So, $M'(F) \simeq \mathrm{GL}_{n/2}(D_2)$.

(b) Let $G = \mathrm{GSp}_{2n}$. Then we have

$$M = M_\theta \simeq \mathrm{GL}_n \times \mathrm{GL}_1 = \widetilde{M}.$$

So, $M'(F) \simeq \mathrm{GL}_{n/2}(D_2) \times \mathrm{GL}_1(F)$.

(**n: odd**)



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(every other dot black)

Set $\theta = \Delta - \{\alpha_{n-1}\}$, where $\alpha_{n-1} = e_{n-1} - e_n$.

(c) Let $G = \mathrm{Sp}_{2n}$. Then we have

$$M = M_\theta \simeq \mathrm{GL}_{n-1} \times \mathrm{SL}_2 \hookrightarrow \mathrm{GL}_{n-1} \times \mathrm{GL}_2 = \widetilde{M}.$$

So, $M'(F) \simeq \mathrm{GL}_{(n-1)/2}(D_2) \times \mathrm{SL}_1(D_2)$.

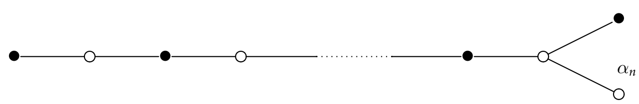
(d) Let $G = \mathrm{GSp}_{2n}$. Then we have

$$M = M_\theta \simeq \mathrm{GL}_n \times \mathrm{GL}_2 = \widetilde{M}.$$

So, $M'(F) \simeq \mathrm{GL}_{(n-1)/2}(D_2) \times \mathrm{GL}_1(D_2)$.

(4) D_n **cases**

($\mathbf{D_n - 1}$)



(n : even)

Set $\theta = \Delta - \{\alpha_n\}$, where $\alpha_n = e_{n-1} + e_n$.

(a) Let $G = \mathrm{Spin}_{2n}$. From [28, 41] we have

$$M_{\mathrm{der}} = \mathrm{SL}_n \hookrightarrow M = M_\theta \hookrightarrow \mathrm{GL}_1 \times \mathrm{GL}_n = \widetilde{M}.$$

So, $M'(F) \hookrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_{n/2}(D_2) = \widetilde{M}'(F)$.

(b) Let $G = \mathrm{GSpin}_{2n}$. Then we have

$$M = M_\theta \simeq \mathrm{GL}_1 \times \mathrm{GL}_n = \widetilde{M}.$$

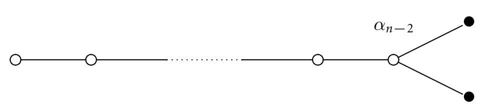
So, $M'(F) \simeq \mathrm{GL}_1 \times \mathrm{GL}_{n/2}(D_2)$.

(c) Let $G = \mathrm{SO}_{2n}$. Then we have

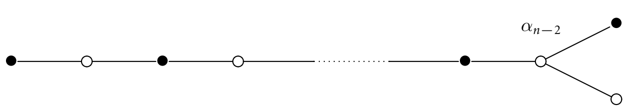
$$M = M_\theta \simeq \mathrm{GL}_n = \widetilde{M},$$

which is the Siegel Levi subgroup. So, $M'(F) \simeq \mathrm{GL}_{n/2}(D_2)$.

($\mathbf{D_n - 2}$)



(any n)



(n : even)

Set $\theta = \Delta - \{\alpha_{n-2}\}$, where $\alpha_{n-2} = e_{n-2} - e_{n-1}$. Note that, for each M of type $D_n - 2$, there are two inequivalent F-inner forms M'.

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(d) Let $G = \mathrm{Spin}_{2n}$. From [28, 41] we have

$$M_{\mathrm{der}} \simeq \mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2 \hookrightarrow M = M_\theta \hookrightarrow \mathrm{GL}_1 \times \mathrm{GL}_{n-2} \times \mathrm{GL}_2 \times \mathrm{GL}_2 = \widetilde{M}.$$

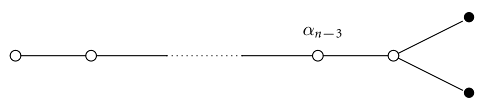
So, $M'(F) \hookrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_{n-2}(F) \times \mathrm{GL}_1(D_2) \times \mathrm{GL}_1(D_2) = \widetilde{M}'(F)$ for the upper diagram (any n); $M'(F) \hookrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_{n-2}(F) \times \mathrm{GL}_1(D_2) \times \mathrm{GL}_2(F) = \widetilde{M}'(F)$ for the lower diagram (n: even).

(e) Let $G = \mathrm{GSpin}_{2n}$. Then we have

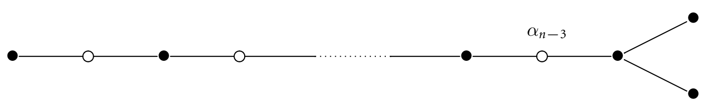
$$M = M_\theta \simeq \mathrm{GL}_{n-2} \times \mathrm{GL}_2 \times \mathrm{GL}_2 = \widetilde{M}.$$

So, $M'(F) \simeq \mathrm{GL}_1(F) \times \mathrm{GL}_{n-2}(F) \times \mathrm{GL}_1(D_2) \times \mathrm{GL}_1(D_2)$ for the upper diagram (any n); $M'(F) \simeq \mathrm{GL}_1(F) \times \mathrm{GL}_{n-2}(F) \times \mathrm{GL}_1(D_2) \times \mathrm{GL}_2(F)$ for the lower diagram (n: even).

$(\mathbf{D_n - 3})$



(any n)



(n : odd)

Set $\theta = \Delta - \{\alpha_{n-3}\}$, where $\alpha_{n-3} = e_{n-3} - e_{n-2}$.

(f) Let $G = \mathrm{Spin}_{2n}$. From [28, 41] we have

$$M_{\mathrm{der}} \simeq \mathrm{SL}_{n-3} \times \mathrm{SL}_4 \hookrightarrow M = M_\theta \hookrightarrow \mathrm{GL}_1 \times \mathrm{GL}_{n-3} \times \mathrm{GL}_4 = \widetilde{M}.$$

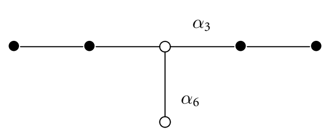
So, $M'(F) \hookrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_{n-3}(F) \times \mathrm{GL}_2(D_2) = \widetilde{M}'(F)$ for the upper diagram (any n); $M'(F) \hookrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_{(n-3)/2}(D_2) \times \mathrm{GL}_1(D_4) = \widetilde{M}'(F)$ for the lower diagram (n: odd).

(g) Let $G = \mathrm{GSpin}_{2n}$. Then we have

$$M = M_\theta \simeq \mathrm{GL}_{n-2} \times \mathrm{GL}_4 = \widetilde{M}.$$

So, $M'(F) \simeq \mathrm{GL}_{n-2}(F) \times \mathrm{GL}_2(D_2)$ for the upper diagram (any n); $M'(F) \simeq \mathrm{GL}_{(n-2)/2}(D_2) \times \mathrm{GL}_1(D_4)$ for the lower diagram (n: odd).

(5) E_6 **cases**



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(a) $(\mathbf{E_6-1})$ Let G be a simply connected group of type E_6. Set $\theta = \Delta - \{\alpha_3\}$, where $\alpha_3 = e_3 - e_4$. From [28, 41] we have

$$M_{\mathrm{der}} \simeq \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2 \hookrightarrow M = M_\theta \hookrightarrow \mathrm{GL}_1 \times \mathrm{GL}_3 \times \mathrm{GL}_3 \times \mathrm{GL}_2 = \widetilde{M}.$$

So, $M'(F) \hookrightarrow \mathrm{GL}_1 \times \mathrm{GL}_1(D_3) \times \mathrm{GL}_1(D_3) \times \mathrm{GL}_2(F) = \widetilde{M}'(F)$.

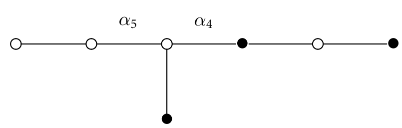
(b) ((**x**) in [34]) Let G be a simply connected group of type E_6. Set $\theta = \Delta - \{\alpha_6\}$, where $\alpha_6 = e_4 + e_5 + e_6 + \epsilon$. From [28, 41] we have

$$M_{\mathrm{der}} \simeq \mathrm{SL}_6 \hookrightarrow M = M_\theta \hookrightarrow \mathrm{GL}_1 \times \mathrm{GL}_6 = \widetilde{M}.$$

So, $M'(F) \hookrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_2(D_2) = \widetilde{M}'(F)$.

(c) Let G be a simply connected group of type E_6. Set $\theta = \Delta - \{\alpha_3, \alpha_6\}$. Then we have $M_{\mathrm{der}} \simeq \mathrm{SL}_3 \times \mathrm{SL}_3 \hookrightarrow M = M_\theta \hookrightarrow \mathrm{GL}_1 \times \mathrm{GL}_3 \times \mathrm{GL}_3 = \widetilde{M}$ and $M'(F) \hookrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_1(D_3) \times \mathrm{GL}_1(D_3) = \widetilde{M}'(F)$.
Note that (a), (b) and (c) above are all possible types of F-Levi subgroups of M'.

(6) E_7 **cases**



(a) $(\mathbf{E_7-1})$ Let G be a simply connected group of type E_7. Set $\theta = \Delta - \{\alpha_4\}$, where $\alpha_4 = e_4 - e_5$. From [28, 41] we have

$$M_{\mathrm{der}} \simeq \mathrm{SL}_2 \times \mathrm{SL}_3 \times \mathrm{SL}_4 \hookrightarrow M = M_\theta \hookrightarrow \mathrm{GL}_1 \times \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_4 = \widetilde{M}.$$

So, $M(F) \hookrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_1(D_2) \times \mathrm{GL}_3(F) \times \mathrm{GL}_2(D_2) = \widetilde{M}'(F)$.

(b) $(\mathbf{E_7-4})$ Let G be a simply connected group of type E_7. Set $\theta = \Delta - \{\alpha_5\}$, where $\alpha_5 = e_5 - e_6$. From [28, 41] we have

$$M_{\mathrm{der}} \simeq \mathrm{SL}_6 \times \mathrm{SL}_2 \hookrightarrow M = M_\theta \hookrightarrow \mathrm{GL}_1 \times \mathrm{GL}_6 \times \mathrm{GL}_2 = \widetilde{M}.$$

So, $M(F) \hookrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_3(D_2) \times \mathrm{GL}_2(F) = \widetilde{M}'(F)$.

(7) E_8**,** F_4 **and** G_2 **cases** Any connected reductive algebraic F-group G of type E_8, F_4, or G_2 does not have non quasi-split F-inner forms of G (see Proposition 2.6).

Acknowledgements We are deeply indebted to Professor Freydoon Shahidi for his constant support and profound insights. We are also grateful to Professor Gordan Savin for his valuable advice and corrections to the manuscript. Thanks are due to Professor Dipendra Prasad for his helpful correspondences and comments on a local to global argument. This work is a part of the author's Ph.D. dissertation at Purdue University.

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Department of Mathematics, Purdue University, West Lafayette, IN 47907, U.S.A.

and

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078-1058, USA
e-mail: kchoiy@math.purdue.edu kwangho.choiy@okstate.edu