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ABSTRACT

Using the descent spectral sequence for a Galois extension of ring spectra, we compute the Picard group of the higher real K -theory spectra of Hopkins and Miller at height $n = p - 1$, for p an odd prime. More generally, we determine the Picard groups of the homotopy fixed points spectra E_n^{hG} , where E_n is Lubin–Tate E -theory at the prime p and height $n = p - 1$, and G is *any* finite subgroup of the extended Morava stabilizer group. We find that these Picard groups are always cyclic, generated by the suspension.

1. Introduction

1.1 Picard groups in stable homotopy theory

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be any symmetric monoidal category. Recall that an object $X \in \mathcal{C}$ is called *invertible* if there exists an object $X^{-1} \in \mathcal{C}$ such that $X \otimes X^{-1} \simeq \mathbf{1}$. We let $\text{Pic}(\mathcal{C})$ denote the group of isomorphism classes of invertible objects. For example, if $\mathcal{C} = \text{Mod}_R^{\text{cl}}$ is the category of (classical) R -modules, for R a commutative ring, then this recovers the classical notion of the Picard group of a ring. As another example, it is not hard to show that the Picard group of the stable homotopy category is \mathbb{Z} , generated by S^1 . Our goal is to develop tools based on descent theory for determining the Picard groups of *module categories* in stable homotopy theory, i.e. module spectra over an \mathbf{E}_∞ -ring spectrum, and to apply them to a specific case of interest in chromatic homotopy theory.

Let R be an \mathbf{E}_∞ -ring spectrum, and let Mod_R denote its symmetric monoidal ∞ -category of modules (i.e. module spectra). We are interested in the Picard group of Mod_R (or, equivalently, of its homotopy category). When the homotopy groups of R are a *regular* ring, it is often possible to reduce this problem to pure algebra (i.e. the Picard group of the graded ring $\pi_*(R)$), cf. [BR05], [MS16, Theorem 2.4.6], and [HM15, Theorem 6.4] for the most general statement. However, in general, the determination of the Picard group is a subtle and complicated problem.

Our basic setup is that we will be working with ring spectra which themselves do not have regular homotopy groups, but are nonetheless closely related to ring spectra that do and whose Picard groups can be easily determined. We refer to [MS16] for a more detailed introduction to these types of questions.

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BASIC PROBLEM. Consider an \mathbf{E}_∞ -ring B such that $\pi_*(B)$ is regular, together with an action of a finite group G on B as an \mathbf{E}_∞ -ring. From this, we can build a new \mathbf{E}_∞ -ring $A = B^{hG}$ as the homotopy fixed points of the G -action. As $\pi_*(B)$ was assumed regular, the Picard group of B is the Picard group of the graded ring $\pi_*(B)$. However, in general $\pi_*(A)$ will be very far from regular, so one expects the determination of the Picard group of A to be an interesting *homotopical* problem.

We will primarily be interested in the case where $A \rightarrow B$ is a G -Galois extension of \mathbf{E}_∞ -ring spectra in the sense of Rognes [Rog08]. That is, we assume there exists a G -action on B as an \mathbf{E}_∞ - A -algebra such that $A \simeq B^{hG}$ and the natural map $B \otimes_A B \rightarrow F(G_+, B)$ is an equivalence. We additionally assume that the extension is *faithful*, i.e. that for each A -module N with $N \otimes_A B \simeq *$, we have $N \simeq *$. In this case, work of Gepner and Lawson [GL16] gives a powerful tool for computing the Picard group of A via a derived form of Galois descent. For another account of this, the reader is referred to [MS16]. In this paper, we use related techniques to show that in a certain class of examples, one does not obtain new ‘exotic’ classes in $\text{Pic}(A)$.

1.2 The descent spectral sequence

To describe the Gepner–Lawson method, we will need to use higher homotopy coherences which we now review (again, see [MS16] for a more detailed exposition). Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal ∞ -category. Keeping track of all higher isomorphisms, we obtain a *Picard ∞ -groupoid* (or *Picard space*), $\mathcal{P}ic(\mathcal{C})$ of invertible objects in \mathcal{C} , whose connected components give the Picard group of \mathcal{C} .

Thanks to the symmetric monoidal product \otimes , $\mathcal{P}ic(\mathcal{C})$ is a grouplike \mathbf{E}_∞ -space, and so is the delooping of a connective spectrum $\mathfrak{p}ic(\mathcal{C})$ called the *Picard spectrum*. We have identifications

$$\begin{aligned} \pi_0 \mathcal{P}ic(\mathcal{C}) &\simeq \text{Pic}(\mathcal{C}), \\ \pi_1 \mathcal{P}ic(\mathcal{C}) &\simeq (\pi_0 \text{End}(\mathbf{1}))^\times, \\ \pi_k \mathcal{P}ic(\mathcal{C}) &\simeq \pi_{k-1} \text{End}(\mathbf{1}) \quad \text{for } k > 1. \end{aligned}$$

Example 1.1. Suppose that $\mathcal{C} = \text{Mod}_R^{\text{cl}}$ is the symmetric monoidal category of (discrete) modules over a commutative ring R . Then $\mathcal{P}ic(\text{Mod}_R^{\text{cl}})$ is a space with

$$\begin{aligned} \pi_0 \mathcal{P}ic(\text{Mod}_R^{\text{cl}}) &\simeq \text{Pic}(R), \\ \pi_1 \mathcal{P}ic(\text{Mod}_R^{\text{cl}}) &\simeq R^\times, \\ \pi_k \mathcal{P}ic(\text{Mod}_R^{\text{cl}}) &= 0 \quad \text{for } k > 1. \end{aligned}$$

Let R be an \mathbf{E}_∞ -ring. We write $\mathcal{P}ic(R)$ and $\mathfrak{p}ic(R)$ for $\mathcal{P}ic(\text{Mod}_R)$ and $\mathfrak{p}ic(\text{Mod}_R)$, respectively. The space $\mathcal{P}ic(R)$ (or the spectrum $\mathfrak{p}ic(R)$) provides a means of capturing the higher homotopy coherences that appear in descent processes. In particular, we have the following basic descent result due to Gepner and Lawson [GL16].

THEOREM 1.2 (Cf. [GL16, Theorem 6.15] and [MS16, §3.3]). *Suppose that $A \rightarrow B$ is a faithful G -Galois extension of \mathbf{E}_∞ -rings. Then there is an equivalence of connective spectra*

$$\mathfrak{p}ic(A) \simeq \tau_{\geq 0} \mathfrak{p}ic(B)^{hG}.$$

In the associated homotopy fixed-point spectral sequence (HFPS)

$$H^s(G, \pi_t \mathfrak{p}ic(B)) \Rightarrow \pi_{t-s}(\mathfrak{p}ic(B)^{hG}), \tag{1.1}$$

the abutment for $t = s$ is the Picard group $\text{Pic}(A)$.

Gepner and Lawson [GL16, § 7.2] have completely calculated this spectral sequence in the fundamental example of the C_2 -Galois extension $KO \rightarrow KU$, and have shown that $\text{Pic}(KO) \simeq \mathbb{Z}/8$, generated by ΣKO . This calculation, via a slightly different method, is originally due to Hopkins (unpublished). We refer also to [MS16] for an account of the calculation of this spectral sequence. In practice, the hope in applying Theorem 1.2 above is that $\text{Pic}(B)$ is easy to determine because the homotopy groups $\pi_*(B)$ are somehow simpler, and so the problem revolves around determining the behavior of the spectral sequence.

In fact, the above spectral sequence (1.1) turns out to be very calculable in specific instances by comparing with the additive spectral sequence

$$E_2^{s,t} = H^s(G, \pi_t B) \implies \pi_{t-s} A, \tag{1.2}$$

arising from the equivalence $A \simeq B^{hG}$. In practice, one already knows the behavior of the additive spectral sequence in computing $\pi_*(A)$. Since for $k > 1$ there is a G -equivariant isomorphism $\pi_k(\mathbf{pic}(B)) \simeq \pi_{k-1} B$, one might hope that there is a relationship between differentials in the HFPSs for $\pi_*(B^{hG})$ and for $\pi_*(\mathbf{pic}(B)^{hG})$. To this effect, the following comparison result is proved in [MS16, § 5].

LEMMA 1.3 (Comparison tool 5.2.4 of [MS16]). *Suppose that $A \rightarrow B$ is a faithful G -Galois extension. Whenever $2 \leq r \leq t - 1$, there is an equality of differentials $d_r^{s,t}(\mathbf{pic} B) = d_r^{s,t-1}(B)$ in the Picard and additive descent spectral sequences. Here $d_r^{s,t}$ denotes the differential $E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}$ in the associated spectral sequences.*

The above result arises from the fact that there is a canonical identification of truncated spectra

$$(\Sigma^{-1} \mathbf{pic}(R))_{[n,2n-1]} \simeq R_{[n,2n-1]} \tag{1.3}$$

for any $n > 0$. The existence of the equivalence (1.3) is a consequence of the Freudenthal suspension theorem. Of course, the main case of interest for these questions is when $s = t$.

Our main technical advance in this paper is a strengthening of this comparison result when certain primes are inverted in $\pi_0 A$. Motivated by the ‘truncated’ logarithm in algebra, we obtain the following expanded range of comparison of spectral sequences.

THEOREM 1.4. *Suppose that $A \rightarrow B$ is a faithful G -Galois extension and $(p - 1)!$ is invertible in $\pi_0 A$. Whenever $2 \leq r \leq (p - 1)(t - 1)$, there is an equality of differentials $d_r^{s,t}(\mathbf{pic} B) = d_r^{s,t-1}(B)$ in the Picard and additive descent spectral sequences.*

We also obtain a formula (in Theorem 7.1) for the first differential outside the above range in terms of power operations.

1.3 Higher real K -theories

The primary application of these methods in the present paper will be to the higher real K -theory spectra of Hopkins–Miller. In particular, we are interested in the case where $B = E_n$, the n th Morava E -theory, the Landweber-exact ring spectrum with

$$(E_n)_* \simeq \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]\langle u^{\pm 1} \rangle,$$

where $\mathbb{W}(\mathbb{F}_{p^n})$ denotes the Witt vectors over \mathbb{F}_{p^n} , described in more detail in Example 3.2. Here the u_i have degree 0, while u has degree -2 . This is an \mathbf{E}_∞ -ring spectrum [Rez98, GH04] with an action through \mathbf{E}_∞ -ring maps by a profinite group, $\mathbb{G}_n = \mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, the (extended)

Morava stabilizer group. Here S_n is the automorphism group of the Honda formal group law over \mathbb{F}_{p^n} .

Following the work of Devinatz and Hopkins [DH04], for any closed subgroup $G \subset \mathbb{G}_n$ there is an associated \mathbf{E}_∞ -ring spectrum E_n^G , which acts like a homotopy fixed-point spectrum and, indeed, agrees with the usual construction when G is finite. For G a maximal finite subgroup of \mathbb{G}_n , these sometimes are called *higher real K -theory spectra*; the nomenclature is inspired by the fact that when $n = 1$ and $p = 2$, the maximal finite subgroup of $\mathbb{G}_1 \simeq \mathbb{Z}_2^\times$ is $C_2 \simeq \{\pm 1\}$ and $E_1^{hC_2}$ is nothing other than 2-complete real K -theory.

Consider now the case of $G \subset \mathbb{G}_n$ a *finite* subgroup. Then, the natural map $E_n^G = E_n^{hG} \rightarrow E_n$ is a $K(n)$ -local G -Galois extension by [Rog08, Theorem 5.4.4] (see also § 3.1 below), so the Galois descent methods and the spectral sequence of (1.1) are applicable. Using our general tools and the Gepner–Lawson spectral sequence, we prove the following main result.

THEOREM 1.5. *Let p be an odd prime and $G \subset \mathbb{G}_{p-1}$ any finite subgroup. Then, the Picard group $\text{Pic}(E_{p-1}^{hG})$ is cyclic, generated by ΣE_{p-1}^{hG} .*

Remark 1.6. The theorem is also true when $n = 1$, $p = 2$, as can be shown using the same argument as given for $\text{Pic}(KO)$ in [MS16].

When $G \subset \mathbb{G}_{p-1}$ is a *maximal* finite subgroup, the above result is due to Hopkins in unpublished work. In this case, the result can be proved without the new technical tools we develop in the paper. However, when (for example) $G = C_p \subset \mathbb{G}_{p-1}$, our formula for the first differential is actually necessary to give an upper bound to the order of the Picard group of E_n^{hG} . We note that Hopkins has informally observed that in most known examples, the Picard groups of these types of constructions tend to be cyclic. Thus, our main result can be viewed as a further example of Hopkins’s observation.

1.4 Connections with chromatic homotopy

Let $K(n)$ be the n th Morava K -theory at an implicit prime p ; it is a complex orientable spectrum with homotopy groups $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$ (where $|v_n| = 2(p^n - 1)$), and its associated formal group law is the Honda formal group law. It is an insight of Hopkins [HMS94] that the Picard group Pic_n of the $K(n)$ -local category itself is quite large and interesting to study.

Here the calculation of Pic_n essentially breaks into two parts (which arise via the $K(n)$ -local profinite Galois extension $E_n^{\mathbb{G}_n} \rightarrow E_n$); there is an algebraic part, which can be detected by cohomological methods, and an exotic part κ_n . The latter consists of the $K(n)$ -local spectra such that the completed E_n -homology of X , $(E_n)_*^\vee X = \pi_* L_{K(n)}(E_n \wedge X)$ is isomorphic to $(E_n)_*$ as Morava modules, i.e. as complete $(E_n)_*$ -modules with a compatible action of \mathbb{G}_n (see [GHMR05, § 2]). This latter part always vanishes when $p \gg n$, but can be highly non-trivial otherwise.

Let us write $\text{Pic}_n^{\text{alg}}$ for the algebraic part (i.e. the Picard group of \mathbb{G}_n -equivariant $(E_n)_*$ -modules) and κ_n for the exotic part. In all *known* cases the map $\text{Pic}_n \rightarrow \text{Pic}_n^{\text{alg}}$ is surjective (although it is not known whether this is true in general), and there is a short exact sequence

$$1 \rightarrow \kappa_n \rightarrow \text{Pic}_n \rightarrow \text{Pic}_n^{\text{alg}} \rightarrow 1.$$

Moreover, the generators of κ_n and $\text{Pic}_n^{\text{alg}}$ are understood well enough to determine the extension problem, so that the following results completely determine Pic_n . However, the list of known cases is not very long, and consists of the following. The height 1 case is due to Hopkins *et al.* [HMS94], and the height 2 case has many contributions from a number of people: Hopkins for $p > 3$

(unpublished, but see [Beh12, Lad13]), Goerss *et al.* [GHMR15] for κ_2 at $p = 3$, and Karamanov [Kar10] for the algebraic Picard group $\text{Pic}_2^{\text{alg}}$ at $p = 3$. The height 2 case for $p = 2$ is the subject of ongoing work of Beaudry–Bobkova–Goerss–Henn.

THEOREM 1.7. *The Picard group of the $K(n)$ -local category is given by*

$$\text{Pic}_1^{\text{alg}} = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 & p = 2, \\ \mathbb{Z}_p \times \mathbb{Z}/2(p-1) & p > 2, \end{cases} \quad \kappa_1 = \begin{cases} \mathbb{Z}/2 & p = 2, \\ 0 & p > 2, \end{cases}$$

$$\text{Pic}_2^{\text{alg}} = \begin{cases} ? & p = 2, \\ \mathbb{Z}_3^2 \times \mathbb{Z}/16 & p = 3, \\ \mathbb{Z}_p^2 \times \mathbb{Z}/2(p^2-1) & p > 3, \end{cases} \quad \kappa_2 = \begin{cases} ? & p = 2, \\ \mathbb{Z}/3 \times \mathbb{Z}/3 & p = 3, \\ 0 & p > 3. \end{cases}$$

It is known that \mathbb{Z}_p always injects into Pic_n . However, apart from the vanishing of κ_n for large primes, essentially nothing is known above height 2. However, the program of Henn and collaborators [Hen07, GHMR05, Bea15, Bob14] entails studying the $K(n)$ -local sphere $S_{K(n)} = E_n^{\mathbb{G}_n}$ via finite resolutions by simpler homotopy fixed point spectra, namely E_n^{hG} with G a finite subgroup of \mathbb{G}_n . Once such a resolution of the $K(n)$ -local sphere is understood, one can, in principle, build resolutions of any element of κ_n by modifying the attaching maps in the resolution of the sphere [GHMR15]. To proceed with this program, a first step is to understand the basic building blocks, i.e. $\text{Pic}(E_n^{hG})$.

Let $\kappa_{p-1}(G)$ denote the elements of $\text{Pic}(E_{p-1}^{hG})$ that are detected in filtration at least 2 in the Picard spectral sequence (1.1) (in fact, we will see that, by sparsity, such classes will only be in filtration at least $2p - 1$). Our calculations show that $\kappa_{p-1}(G) \cong \mathbb{Z}/p$, whenever G contains an element of order p . One can check that base-change gives a well-defined morphism $\phi : \kappa_{p-1} \rightarrow \kappa_{p-1}(G)$ which, in the cases of $n = 1, p = 2$ and $n = 2, p = 3$, and G the maximal finite subgroup, is known to be split surjective, cf. [MS16, § 8.5] and [GHMR15, Example 5.1 and Theorem 5.5]. This leads to the following question.

Question 1.8. Is the map $\phi : \kappa_{p-1} \rightarrow \kappa_{p-1}(G) \simeq \mathbb{Z}/p$ non-zero for any G containing an element of order p ?

1.5 Organization

The paper is arranged as follows. In the next section we review the basic properties of Morava E -theory, the classification of the maximal finite subgroups of \mathbb{G}_n , and recall some information about the action of these finite subgroups on $(E_{p-1})_*$. In § 3 we prove some reduction results at general height, in particular showing that one can reduce the proof of cyclicity of $\text{Pic}(E_n^{hG})$ to that of $\text{Pic}(E_n^{hG'})$, where G' is the maximal p -subgroup of G ; we then perform the main computation of this paper in § 4. This will rely on general techniques for comparing differentials in the additive and multiplicative algebraic and topological spectral sequences, which are described in the second part of the paper. More precisely, in § 5, we analyze truncated logarithms to give comparison results and a universal formula for the differentials in the Picard spectral sequence, and in § 6 we use the techniques of § 5 to determine the algebraic Picard group $H^1(C_p, (E_{p-1})_0^\times)$.

PART I. GENERALITIES AND COMPUTATIONS

2. Finite subgroups of \mathbb{G}_n and the additive spectral sequence

In this section, we recall the necessary information about the action of finite subgroups of \mathbb{G}_n on E_n , particularly in the case $n = p - 1$. We will, in particular, recall the computation of

the associated HFPSS for the homotopy of the higher real K -theories and their variants for non-maximal subgroups. In the following, only the result of the computation for the group C_p will be used.

2.1 The Morava stabilizer group

We start this section by reviewing more thoroughly the Morava stabilizer group \mathbb{S}_n , as well as the extended Morava stabilizer group \mathbb{G}_n . We use [Frö68, ch. III] as a general reference. Recall from the introduction that \mathbb{S}_n is the automorphism group of the height n Honda formal group law Γ_n over \mathbb{F}_{p^n} ; this is the formal group law $F_{\Gamma_n}(X, Y) \in \mathbb{F}_{p^n}[[X, Y]]$ with p -series $[p]_{\Gamma_n}(X) = X^{p^n}$.

DEFINITION 2.1. The group \mathbb{S}_n of *automorphisms* is given by those power series $f(X) \in \mathbb{F}_{p^n}[[X]]$ such that:

- (i) $f(0) = 0$ (i.e. f has no constant term);
- (ii) $f'(0) \neq 0 \in \mathbb{F}_{p^n}$;
- (iii) $F_{\Gamma_n}(f(X), f(Y)) = f(F_{\Gamma_n}(X, Y))$.

Observe that the group structure of \mathbb{S}_n is given by composition of power series. The group \mathbb{S}_n has a topology induced from the X -adic topology, and is a profinite group with respect to this topology.

We have a homomorphism

$$\mathbb{S}_n \rightarrow \mathbb{F}_{p^n}^\times, \quad f(X) \mapsto f'(0). \tag{2.1}$$

More invariantly, this homomorphism sends an automorphism f of the formal group F to its action on the *tangent space* of F (dual to the *Lie algebra*). This homomorphism is surjective, and its kernel (which consists of automorphisms that are congruent to the identity modulo X^2) is a pro- p -group.

The group \mathbb{S}_n can be described as the units in the maximal order \mathcal{O}_n of the central division algebra \mathbb{D}_n of Hasse invariant $1/n$ over \mathbb{Q}_p . More explicitly, we have that \mathcal{O}_n is the non-commutative ring

$$\mathcal{O}_n = \mathbb{W}(\mathbb{F}_{p^n})\langle S \rangle / (S^n = p, Sw = w^\phi S),$$

where $w \in \mathbb{W}(\mathbb{F}_{p^n})$ and ϕ is a lift of the Frobenius to $\mathbb{W}(\mathbb{F}_{p^n})$. Then \mathcal{O}_n is the endomorphism ring of the formal group Γ_n , and $\mathbb{S}_n = \mathcal{O}_n^\times$.

Finally, because Γ_n is already defined over \mathbb{F}_p , the Galois group $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on \mathbb{S}_n , and we define the extended Morava stabilizer group (which we will often just call the Morava stabilizer group) as the semidirect product $\mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, i.e. there is a split extension

$$1 \rightarrow \mathbb{S}_n \rightarrow \mathbb{G}_n \rightarrow \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \rightarrow 1. \tag{2.2}$$

2.2 The finite subgroups of \mathbb{G}_n

Before proceeding with the calculations, let us recall the classification of the maximal finite subgroups of \mathbb{G}_n , as well as their action on the coefficients of E_n . Bujard and Hewett [Hew95, Buj12] have determined the maximal finite subgroups of $\mathbb{S}_n \subset \mathbb{G}_n$. Specifically, let $p > 2$ and $n = (p - 1)p^{k-1}m$ for m prime to p , and denote by n_α the quotient $n/\phi(p^\alpha)$, where ϕ is Euler’s totient function. There are exactly $k + 1$ conjugacy classes of maximal finite subgroups of \mathbb{S}_n , represented by

$$G_0 = C_{p^{n-1}} \quad \text{and} \quad G_\alpha = C_{p^\alpha} \rtimes C_{(p^{n_\alpha-1})(p-1)} \quad \text{for } 1 \leq \alpha \leq k.$$

When $p - 1$ does not divide n , the only class of maximal finite subgroups is that of G_0 , so, in particular, there are no finite subgroups of order divisible by p .

Remark 2.2. Given the description of \mathbb{S}_n in § 2.1, we note that an element of order dividing $p^n - 1$ is always easy to describe: it is given by a Teichmüller lift η of a primitive $(p^n - 1)$ st root of unity, $\eta \in \mathbb{W}(\mathbb{F}_{p^n})^\times \subset \mathbb{S}_n$.

Consider now the special case when $n = p - 1$. For convenience, we drop the subscripts after \mathbb{G}, \mathbb{S} . In this situation, there are two conjugacy classes of maximal finite subgroups of \mathbb{S} .

- (i) One of order not divisible by p , namely the cyclic group C_{p^n-1} .
- (ii) One of order divisible by p , namely the semi-direct product $F = C_p \rtimes C_{(p-1)^2}$. In the latter case, the action of $C_{(p-1)^2}$ on C_p is given by a projection

$$C_{(p-1)^2} \rightarrow C_{p-1} \simeq \text{Aut}(C_p).$$

Remark 2.3. Using the description of \mathbb{S}_n in § 2.1, we can give an explicit equation of an order p element when $p = 3$. Let ω be a primitive eighth root of unity in $\mathbb{W}(\mathbb{F}_9)$, then we can take $-\frac{1}{2}(1 + \omega S)$. In general, however, it is difficult or impossible to give a similar expression with rational coefficients for an element of order p when $n > 2$, see [GMS98, Remark 5.3].

Finally, we recall from (2.2) that the extended Morava stabilizer group \mathbb{G} sits in a split extension

$$1 \rightarrow \mathbb{S} \rightarrow \mathbb{G} \rightarrow \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \rightarrow 1,$$

and, therefore, any finite subgroup G of \mathbb{G} sits in a (not necessarily split) extension

$$1 \rightarrow G_{\mathbb{S}} \rightarrow G \rightarrow G_{\text{Gal}} \rightarrow 1, \tag{2.3}$$

where $G_{\mathbb{S}} = G \cap \mathbb{S}$, and G_{Gal} is a subgroup of $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong C_n$.

In the case when $G_{\mathbb{S}}$ is a maximal abelian subgroup of \mathbb{S} , Bujard [Buj12, Theorem 4.13] has determined that G_{Gal} can be maximally large, i.e. it is always possible that $G_{\text{Gal}} \cong \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. This is also known to be true when $G_{\mathbb{S}}$ is a maximal finite subgroup of \mathbb{S} , cf. [Hen07, Proposition 20].

The proof of Theorem 1.5 for G will consist of a reduction first from G to its subgroup $G_{\mathbb{S}}$, and then further to the p -Sylow subgroup of $G_{\mathbb{S}}$.

2.3 The action on E_*

The first explicit description of the action of \mathbb{G} on E_* appears in unpublished work of Hopkins and Miller. The general situation for \mathbb{G} acting on E_* is determined by Devinatz and Hopkins [DH95].¹ We focus on the action of \mathbb{S} . For our purposes, a nice summary of what we need can be found in the work of Nave [Nav10, § 2].

Write an element $g \in \mathbb{S}$ as $g = \sum_{j=0}^{n-1} a_j S^j$, where $a_j \in \mathbb{W}(\mathbb{F}_{p^n})$ and a_0 is a unit. Recall that $u \in \pi_{-2}E_n$ is a generator. The next result follows from [DH95, Proposition 3.3 and Theorem 4.4].

THEOREM 2.4. *Let ϕ denote the lift of the Frobenius to $\mathbb{W}(\mathbb{F}_{p^n})$. Then if $g = \sum_{j=0}^{n-1} a_j S^j$ as above, we have*

$$g_*(u) \equiv a_0 u + a_{n-1}^\phi u u_1 + \cdots + a_1^{\phi^{n-1}} u u_{n-1} \pmod{(p, \mathfrak{m}^2)} \tag{2.4}$$

$$g_*(u u_i) \equiv a_0^{\phi^i} u u_i + \cdots + a_{i-1}^\phi u u_1 \pmod{(p, \mathfrak{m}^2)}. \tag{2.5}$$

¹ Both of these sources treat general p and n .

In the case of elements g of finite order, we can be much more explicit. By the classification of finite subgroups of \mathbb{S} , such a g either has order equal to p , or prime to p :

- (i) if g has order p , it is known [Nav10, Lemma 2.2] that $a_0 \equiv 1 \pmod{p}$ and a_1 is a unit;
- (ii) If g has order prime to p , then g is a Teichmüller lift of a , and it follows from [Hen07] that the induced map of rings $g_* : E_* \rightarrow E_*$ is given by

$$g_*(u) = au \quad \text{and} \quad g_*(u_i) = a^{p^i-1}u_i. \tag{2.6}$$

2.4 The HFPSS

Because the action of finite subgroups on E_* is well understood in our case of $n = p - 1$ (which we implicitly assume in this subsection), so is the computation of the associated group cohomology. In fact, the differentials in the HFPSS are also known. The calculations are due to Hopkins and Miller, although their work has not appeared in print. Published references include the thesis and subsequent paper of Nave [Nav99, Nav10], or for a detailed computation at $n = 2, p = 3$, the work of Goerss *et al.* [GHM04, GHMR05].

We start by considering the prime-to- p case.

Example 2.5. If p does not divide the order of G , then the higher cohomology of G with coefficients in E_* vanishes, giving that

$$(E^{hG})_* = (E_*)^G.$$

Using (2.6), we can (at least, in principle) determine these invariants. In particular, we see that if m is the order of $G_{\mathbb{S}}$, then u^m is $G_{\mathbb{S}}$ -invariant and is a minimal periodicity element of $(E_*)^{G_{\mathbb{S}}}$. However, this element is also fixed under the Galois quotient of G , so E^{hG} has periodicity $2m = 2|G_{\mathbb{S}}|$.

The most interesting case for us is when $G = C_p$. In this case, one has the following calculation of the E_2 -page modulo transfers.

PROPOSITION 2.6 (Hopkins–Miller). *There is an exact sequence*

$$E_* \xrightarrow{\text{tr}} H^*(C_p, E_*) \rightarrow \mathbb{F}_{p^n}[\alpha, \beta, \delta^{\pm 1}]/(\alpha^2) \rightarrow 0,$$

of graded groups, where the (s, t) -bidegrees are $|\alpha| = (1, 2n)$, $|\beta| = (2, 2pn)$, and $|\delta| = (0, 2p)$.

We note that because we have not taken Galois invariants anywhere, we see copies of \mathbb{F}_{p^n} (rather than \mathbb{F}_p) in the above spectral sequence. The passage to \mathbb{F}_p can be achieved by adding in the Galois group. For example, we also state the following result of Hopkins–Miller.

PROPOSITION 2.7 (Hopkins–Miller). *When $G \subset \mathbb{G}$ is a maximal finite subgroup containing p -torsion, then there is an exact sequence*

$$E_* \xrightarrow{\text{tr}} H^*(G, E_*) \rightarrow \mathbb{F}_p[\alpha, \beta, \Delta^{\pm 1}]/(\alpha^2),$$

with $|\Delta| = (0, 2pn^2)$ and α, β as above.

The above determines the E_2 -page of the HFPSS for $\pi_*(E^{hG})$ in two important cases. The differentials in the HFPSS are also well known. For any finite subgroup $G \subset \mathbb{G}$, there are maps

$$\text{Ext}_{BP_*BP}(BP_*, BP_*) \rightarrow H^*(\mathbb{G}, E_*) \rightarrow H^*(G, E_*).$$

If G contains an element of order p , then, by [Rav78], the composite sends α_1 to α and β_1 to β , at least up to a non-zero scalar.

One can also show this directly using the functoriality of the Greek letter construction, as in [GHM04, Proposition 7]. For this, recall that there is a class $v_1 \in \text{Ext}_{BP_*BP}^{0,2(p-1)}(BP_*, BP_*/(p))$ which defines a permanent cycle in the Adams–Novikov spectral sequence for the mod- p Moore spectrum $V(0)$. Via the Greek letter construction, v_1 gives rise to an element $\alpha_1 \in \pi_{2p-3}S^0_{(p)} \simeq \mathbb{Z}/p$ detected by a class, which we also denote by α_1 , in $\text{Ext}_{BP_*BP}^{1,2p-2}(BP_*, BP_*)$. Recalling that v_1 is sent to u_1u^{1-p} under the orientation map $BP \rightarrow E$, one can check that the image of v_1 defines a non-trivial element in $H^0(G, E_*/(p))$. Using the short exact sequence

$$0 \rightarrow E_* \rightarrow E_* \rightarrow E_*/(p) \rightarrow 0,$$

one can see that this class has non-trivial image in $\delta^0(v_1) \in H^1(G, E_{2p-2})$ (which, if G is maximal, is \mathbb{F}_p). The result follows since this latter group is generated by α . One can make a similar argument for β , and since α_1 and β_1 are permanent cycles, so are α and β . We also know via the calculations of Toda [Tod67, Tod68] that $\alpha_1\beta_1^p = 0$ and $\beta_1^{pn+1} = 0$ in π_*S^0 . Sparsity in the HFPSS, as well as the multiplicative structure, then allow the complete determination of the differentials.

In the following we use the notation \doteq to denote an equality that holds up to multiplication by a non-zero scalar.

LEMMA 2.8. *In the spectral sequence for $\pi_*E^{hC_p}$, there are differentials*

$$d_{2p-1}(\delta) \doteq \alpha\beta^{p-1}\delta^{1-n^2} \quad \text{and} \quad d_{2n^2+1}(\delta^{n^3}\alpha) \doteq \beta^{n^2+1}.$$

*If G is a maximal finite subgroup, then in the spectral sequence for π_*E^{hG} , there are differentials*

$$d_{2p-1}(\Delta) \doteq \alpha\beta^{p-1} \quad \text{and} \quad d_{2n^2+1}(\Delta^n\alpha) \doteq \beta^{n^2+1}.$$

These differentials determine all others by multiplicativity. More precisely, in the C_p -HFPSS, the shorter differential d_{2p-1} is non-zero on all $\beta^b\delta^d$ with $d \not\equiv 0 \pmod p$, and classes of form $\alpha\beta^b\delta^d$ are hit whenever $b \geq p-1$ and $d \not\equiv -1 \pmod p$. Now consider a monomial $\alpha\beta^b\delta^d$ that survives to the $E_{2p} = E_{2n^2+1}$ -page; in particular, $d \equiv -1 \pmod p$ or $b < p-1$. If $d \not\equiv -1 \pmod p$ (so we would have that $b < p-1$), the potential elements for $d_{2n^2+1}(\alpha\beta^b\delta^d)$ (which are multiples of $\beta^{n^2+b+1}\delta^{d-n^3}$) have already been wiped out by d_{2p-1} . When $d \equiv -1 \pmod p$, the potential targets are still there, and Lemma 2.8 implies that indeed $d_{2n^2+1}(\alpha\beta^b\delta^d) \doteq \beta^{n^2+b+1}\delta^{d-n^3}$.

As a specific example, note that $\alpha\delta$ is a permanent cycle, but $\alpha\delta^{-1}$ is not.

Remark 2.9. This implies the following:

- the homotopy groups $\pi_*E_{p-1}^{hC_p}$ are $2p^2$ -periodic, with periodicity generator δ^p ; and
- the homotopy groups $\pi_*E_{p-1}^{hG}$ are $2p^2n^2$ -periodic, with periodicity generator Δ^p .

We demonstrate the computation (modulo some elements on the 0-line) of $\pi_*E^{hC_p}$ for $n = 2$, $p = 3$ in Figure 1.

3. General reduction steps

In this section, we prove some general results about the action of finite subgroups $G \subset \mathbb{G}_n$ on E_n that will be needed for the computation of the Picard group of E_n^{hG} (i.e. the proof that it is cyclic). First, we will give a detailed proof of the Devinatz–Hopkins–Rognes result (Proposition 3.6) that $E_n^{hG} \rightarrow E_n$ is a faithful G -Galois extension in the sense of Rognes [Rog08], based on the affineness machinery of [MM15] (where the proof is sketched). In particular, we obtain the descent spectral sequence for Picard groups that is our main tool.

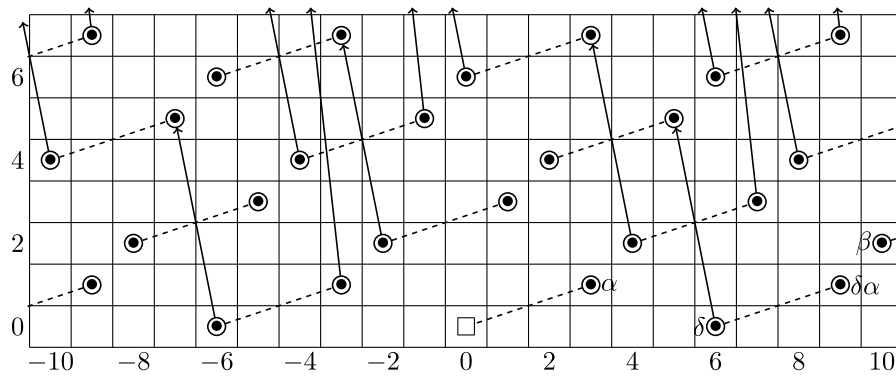


FIGURE 1. The HFPSS for $\pi_* E^{hC_p}$ in the case $n = 2, p = 3$. Circled dots refer to copies of \mathbb{F}_9 , and the square represents a copy of $\mathbb{W}(\mathbb{F}_9)$. The dotted lines represent multiplication by α .

We then make two basic reductions for proving the cyclicity of the Picard group, which show that it suffices to do the calculation in a single case: when G is a finite p -subgroup of \mathbb{S}_n . The analysis of the descent spectral sequence in this case (in our special case of interest, $n = p - 1$, G will then be C_p) will be carried out in the following section, using the technology developed in the second part of this paper.

We emphasize that n and p are arbitrary in this section.

3.1 The Galois property of the Hopkins–Miller spectra

In this subsection, we show that for any finite subgroup $G \subset \mathbb{G}_n$, the extension $E_n^{hG} \rightarrow E_n$ is a faithful G -Galois extension of ring spectra. This result is essentially due (at least in the $K(n)$ -local case) to Devinatz and Hopkins [DH04] and Rognes [Rog08]. We note also that, for $n = p - 1$ (the case in which we are primarily interested here), the first author in [Hea15] has given a computational proof of the vanishing of the Tate construction, which is a key component of the result.

Using the results of [MM15], it will suffice to show that the stack $(\text{Spec } \pi_0 E_n)/G$ is affine over the moduli stack M_{FG} of formal groups. This is easy to check at the closed point, and follows in general by a formal argument that we describe here, involving ‘free actions’.

We need to recall the following classical definition.

DEFINITION 3.1. Let A be a commutative ring with an action of a finite group G . Then G is said to *act freely* on A if for every non-zero commutative ring A' (it suffices to take A' to be a field), the G -action on the set $\text{Hom}(A, A')$ of ring-homomorphisms $A \rightarrow A'$ is free. Equivalently, for every prime ideal $\mathfrak{p} \subset A$, the subgroup $G_{\mathfrak{p}} \subset G$ stabilizing \mathfrak{p} acts faithfully on the residue field $k(\mathfrak{p})$. In this case, the inclusion of rings $A^G \rightarrow A$ induces a G -torsor $\text{Spec } A \rightarrow \text{Spec } A^G$, and $\text{Spec } A^G \simeq (\text{Spec } A)/G$ in the category of stacks.

We refer to [SGA1, Exp. V] for a detailed treatment and to [Lur11, § 4] for a discussion with a view towards ring spectra. The following is an important example.

Example 3.2. We let k be a perfect field of characteristic p . We recall (cf. [Ser79, ch. II, §§ 5–6]) some properties of the functorial (in k) construction of the p -complete ring $\mathbb{W}(k)$ of p -typical Witt vectors, which is uniquely determined by the condition that $\mathbb{W}(k)$ should be torsion-free and $\mathbb{W}(k)/p \simeq k$. The ring $\mathbb{W}(k)$ has the following universal property for p -complete rings: given

any p -complete ring A_0 , to give a map $\mathbb{W}(k) \rightarrow A_0$ is equivalent to giving a map of \mathbb{F}_p -algebras $k \rightarrow A_0/p$.

We will need the following basic fact about Witt vectors. Let k be a perfect field of characteristic p and let G be any finite subgroup of $\text{Aut}(k)$. Then, the G -action on $\mathbb{W}(k)$ is free. In particular, the extension $\mathbb{W}(k^G) \rightarrow \mathbb{W}(k)$ is a G -Galois extension of commutative rings (cf. [SGA1, Exp. I, Theorem 6.1]).

We now specialize to the case of interest in this subsection.

Construction 3.3. We consider the category consisting of pairs (B, Γ) where B is a commutative ring and Γ is a formal group over B . A morphism $(B, \Gamma) \rightarrow (B', \Gamma')$ consists of a morphism of commutative rings $f : B \rightarrow B'$ and an isomorphism of formal groups $f_*\Gamma \simeq \Gamma'$. This category is cofibered in groupoids over the category of commutative rings, and is encoded in the *moduli stack of formal groups* M_{FG} : that is, it is (by definition) the category of maps $\text{Spec } B \rightarrow M_{FG}$ from affine schemes to M_{FG} .

We will consider group actions in this category, or equivalently morphisms of algebraic stacks of the form $(\text{Spec } B)/G \rightarrow M_{FG}$. In particular, giving a G -action on a pair (B, Γ) yields for each $\sigma \in G$ the data of an automorphism $\sigma : B \simeq B$ and an isomorphism of formal groups $f_\sigma : \sigma_*\Gamma \simeq \Gamma$ which satisfy a homomorphism and cocycle condition, respectively. We will represent Γ by a formal group law over B given by a power series $F(X, Y) \in B[[X, Y]]$. In this case, $f_\sigma = f_\sigma(X)$ is a power series in $B[[X]]$ such that

$$F(f_\sigma(X), f_\sigma(Y)) = f_\sigma(\sigma_*F(X, Y)).$$

We will need a general criterion to show that a map of the form $(\text{Spec } B)/G \rightarrow M_{FG}$ is affine. For this, it will suffice to consider (for various n) the faithfully flat cover $M_{FG}^{(n)} \rightarrow M_{FG}$ where $M_{FG}^{(n)}$ is the moduli stack of formal groups together with a coordinate modulo degrees at least $n + 1$.

PROPOSITION 3.4. *Suppose G acts on the pair (B, Γ) where Γ is a formal group over the commutative ring B with the above notation. For any prime ideal $\mathfrak{p} \subset B$, we let $I_{\mathfrak{p}} \subset G$ be the subgroup of elements $\sigma \in G$ such that $\sigma(\mathfrak{p}) = \mathfrak{p}$ and σ induces the identity automorphism of $k(\mathfrak{p})$.*

Fix n , and suppose that if \mathfrak{p} is any prime ideal and $\sigma \in I_{\mathfrak{p}}$, then at least one of the first n coefficients of $f_\sigma(X) - X \in B[[X]]$ has non-zero image in $k(\mathfrak{p})$. Then the G -action on the scheme $\text{Spec } B \times_{M_{FG}} M_{FG}^{(n)}$ is free, so that $(\text{Spec } B)/G \times_{M_{FG}} M_{FG}^{(n)}$ is representable by an affine scheme.

Proof. Let B' be the B -algebra which classifies coordinates modulo degrees at least $n + 1$ on the formal group Γ . Abstractly, $B' \simeq B[x_1^{\pm 1}, x_2, \dots, x_n]$. The G -action on (B, Γ) induces a G -action on (B', Γ') , where Γ' is the extension of scalars of Γ to B' .

We need to show that G acts freely on $\text{Hom}(B', k)$ for any field k . Let $\sigma \in G \setminus \{1\}$ fix an element of $\text{Hom}(B', k)$ given by a map $\phi : B' \rightarrow k$. We get that σ fixes the restriction $\phi|_B : B \rightarrow k$, which means that the prime ideal $\mathfrak{p} = \ker(\phi|_B)$ is fixed by σ , and that σ acts trivially on the residue field $k(\mathfrak{p})$, so that $\sigma \in I_{\mathfrak{p}}$. Note that the fiber above $\phi|_B$ of the map (which is preserved by σ)

$$\text{Hom}(B', k) \rightarrow \text{Hom}(B, k) \tag{3.1}$$

is given by the set of coordinates to degree n on the formal group $(\phi|_B)_*(\Gamma)$, and the action of σ on this set of coordinates is given by composing with the image of f_σ . However, since the image

of $f_\sigma \in k[[X]]$ is not the identity, it acts freely on the above set of coordinates up to degree n . In particular, the action of σ on the fiber of (3.1) above $\phi|_B$ has no fixed points, which is a contradiction and which completes the proof. \square

LEMMA 3.5. *Suppose (R, \mathfrak{m}) is a complete local ring with perfect residue field k of characteristic $p > 0$, and let Γ be a formal group over R . Let a finite group G act on the pair (R, Γ) . Let $S \subset G$ be the subgroup that acts trivially on the residue field. Suppose that every element $\sigma \in S \setminus \{1\}$ induces a non-trivial automorphism of the reduction of Γ modulo \mathfrak{m} . Then the morphism $(\text{Spec } R)/G \rightarrow M_{FG}$ is affine.*

Proof. We will apply Proposition 3.4. Fix a prime ideal $\mathfrak{p} \subset R$ and $\sigma \in I_{\mathfrak{p}}$. The first claim is that $\sigma \in S$. In fact, this follows from the surjection $R/\mathfrak{p} \rightarrow R/\mathfrak{m} = k$. Since σ acts trivially on R/\mathfrak{p} , it must act trivially on k .

In applying Proposition 3.4, it suffices to consider the inertia subgroups $I_{\mathfrak{p}}$; since these are contained in S , we conclude that it suffices to consider the case when G itself acts trivially on the residue field. Choose n large enough such that each power series $f_\sigma(X) \in R[[X]]$ for $\sigma \neq 1$ has a coefficient in degrees at most n different modulo \mathfrak{m} from X . The claim is that $(\text{Spec } R)/G \times_{M_{FG}} M_{FG}^{(n)}$ is representable by an affine scheme. This now follows from Proposition 3.4 because each $f_\sigma(X) - X$ for $\sigma \neq 1$ has a coefficient which is a *unit* in some degree at most n . \square

PROPOSITION 3.6. *For any finite subgroup $G \subset \mathbb{G}_n$, the natural map $E_n^{hG} \rightarrow E_n$ exhibits the target as a faithful G -Galois extension of the source.*

Proof. By [MM15, Theorem 5.8], it suffices to show that $(\text{Spec } \pi_0 E_n)/G \rightarrow M_{FG}$ is an affine morphism of stacks. However, this follows from Lemma 3.5 (with $S = G_{\mathbb{S}} = G \cap \mathbb{S}$) because \mathbb{S}_n is defined as a subgroup of automorphisms of the Honda formal group over \mathbb{F}_{p^n} . The above reasoning shows that there exists m such that $(\text{Spec } \pi_0 E_n)/G \times_{M_{FG}} M_{FG}^{(m)}$ is affine. \square

It follows, in particular, that there is a descent spectral sequence for computing $\text{Pic}(E_n^{hG})$ (for any finite subgroup $G \subset \mathbb{G}_n$) based on the equivalence of spectra

$$\text{pic}(E_n^{hG}) \simeq \tau_{\geq 0} \text{pic}(E_n)^{hG}.$$

We know that $\text{Pic}(E_n)$ is cyclic (by regularity, cf. [BR05]), and we want to prove that $\text{Pic}(E_n^{hG})$ is also cyclic. In the rest of this section, we will reduce the analysis of this spectral sequence to the case where G is a p -subgroup of \mathbb{S}_n .

Remark 3.7. As a consequence, we can prove that the two possible notions of Picard group that one might be interested in here are equivalent. Recall that if R is a $K(n)$ -local \mathbf{E}_∞ -ring, one can consider two different Picard groups (cf. [HMS94]):

- (i) the (usual) Picard group $\text{Pic}(R)$ of all R -modules;
- (ii) the Picard group $\text{Pic}_{K(n)}(R)$ of the ∞ -category $L_{K(n)} \text{Mod}(R)$ of $K(n)$ -local R -modules (with the $K(n)$ -localized smash product).

The first group $\text{Pic}(R)$ is always a subgroup of $\text{Pic}_{K(n)}(R)$, because any invertible R -module is automatically perfect, hence $K(n)$ -local itself and is easily seen to be $K(n)$ -locally invertible too. Given an element of $\text{Pic}_{K(n)}(R)$, represented by a $K(n)$ -local R -module M , it belongs to the first group if and only if M is perfect as an R -module.

We claim that if $R = E_n^{hG}$ for any finite $G \subset \mathbb{G}_n$, the two notions coincide. Equivalently, if M is a $K(n)$ -local R -module which is $K(n)$ -locally invertible, then M is invertible as an R -module to begin with.

To see this, we consider the faithful G -Galois extension $R \rightarrow E_n$. Then E_n is a perfect R -module by [Rog08, Proposition 6.2.1]. Observe that the E_n -module $E_n \otimes_R M$ is still $K(n)$ -local as a result, and it is $K(n)$ -locally invertible. This implies that $E_n \otimes_R M$ belongs to $\text{Pic}_{K(n)}(E_n)$. However, $\text{Pic}_{K(n)}(E_n) = \text{Pic}(E_n) = \mathbb{Z}/2$ (by [BR05]); in particular, $E_n \otimes_R M$ is a perfect E_n -module, and by descent (cf. [Mat16, Proposition 3.27]) M is a perfect R -module.

3.2 Elimination of the Galois piece

In this subsection, we make a basic reduction that enables us to eliminate the Galois piece. In other words, we reduce to proving cyclicity of the Picard group of E_n^{hG} when $G \subset \mathbb{S}_n$.

We will need a few basic properties of $\mathbb{W}(k)$. Most of them can be encapsulated appropriately in the notion of ‘formal étaleness’, but we will spell out some details for the convenience of the reader.

Let $f : A_0 \rightarrow A'_0$ be a morphism of p -local commutative rings. We say that f is a *uniform \mathcal{F}_p -isomorphism* if there exists $N > 0$ such that:

- (i) $\ker(f)^N = 0$;
- (ii) given $x \in A'_0$, $x^{p^N} \in \text{im}(f)$.

LEMMA 3.8. *Let $f : A_0 \rightarrow A'_0$ be a uniform \mathcal{F}_p -isomorphism of p -complete rings. Then given any perfect field k of characteristic p , any map of commutative rings $\mathbb{W}(k) \rightarrow A'_0$ lifts uniquely to a map $\mathbb{W}(k) \rightarrow A_0$.*

Proof. It is a straightforward calculation (cf. [MNN15, Proposition 3.26]) that $f \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is also a uniform \mathcal{F}_p -isomorphism, so that we may assume that A_0 and A'_0 are \mathbb{F}_p -algebras by the universal property of the Witt vectors. It suffices to show that $\text{Hom}(k, A_0) \simeq \text{Hom}(k, A'_0)$.

Recall that an \mathbb{F}_p -algebra C_0 is said to be *perfect* if the Frobenius $F : C_0 \rightarrow C_0$ is an isomorphism. The inclusion of perfect \mathbb{F}_p -algebras into all \mathbb{F}_p -algebras has a right adjoint, called the *inverse limit perfection*, which sends an \mathbb{F}_p -algebra C_0 to the inverse limit of the Frobenius system $\cdots \rightarrow C_0 \xrightarrow{F} C_0$. Since k is perfect, and since the uniform \mathcal{F}_p -isomorphism condition guarantees that $A_0 \rightarrow A'_0$ induces an isomorphism on inverse limit perfections, an adjunction argument now gives the desired claim. □

LEMMA 3.9. *Let \tilde{G} be a finite group acting on a p -complete \mathbf{E}_∞ -ring R in such a way that $R^{h\tilde{G}} \rightarrow R$ is a faithful \tilde{G} -Galois extension. Let k be a perfect field of characteristic p . Then any map of commutative rings $\mathbb{W}(k) \rightarrow \pi_0(R)^{\tilde{G}}$ lifts uniquely to a map $\mathbb{W}(k) \rightarrow \pi_0(R^{h\tilde{G}})$.*

Proof. By the previous lemma, it suffices to show that the reduction map $\pi_0(R^{h\tilde{G}}) \rightarrow \pi_0(R)^{\tilde{G}}$ is a uniform \mathcal{F}_p -isomorphism. This follows as in [Mat15, § 4] from the fact that the HFPSS for the \tilde{G} -action on R has a horizontal vanishing line at a finite stage by the theory of ‘descendability’ (cf. [Mat16, §§ 3–4]) and the fact that everything in positive filtration at E_2 is annihilated by the p -part of $|\tilde{G}|$. □

PROPOSITION 3.10. *Let G be a finite group acting on an even-periodic \mathbf{E}_∞ -ring R . Suppose that:*

- (i) $\pi_0 R$ is a complete local ring with perfect residue field k of characteristic $p > 0$; let $\tilde{G} \subseteq G$ be the subgroup that acts trivially on k ;

- (ii) the morphism of \mathbf{E}_∞ -rings $R^{hG} \rightarrow R$ is a faithful G -Galois extension;
 - (iii) the Picard group of the \mathbf{E}_∞ -ring $R^{h\tilde{G}}$ is cyclic and generated by the suspension $\Sigma R^{h\tilde{G}}$.
- Then the Picard group of R^{hG} is cyclic and generated by the suspension ΣR^{hG} .

Proof. Let $\Gamma = G/\tilde{G}$ be the quotient group, so that by the Galois correspondence, the morphism of \mathbf{E}_∞ -rings $A = R^{hG} \rightarrow B = R^{h\tilde{G}}$ is a faithful Γ -Galois extension. We claim first that it is actually a Galois extension on homotopy groups, i.e. $A \rightarrow B$ is étale, in the sense that $\pi_0(A) \rightarrow \pi_0(B)$ is étale and the natural map $\pi_*A \otimes_{\pi_0A} \pi_0B \rightarrow \pi_*B$ is an isomorphism; this will enable us to make the desired conclusion. We refer to [BR07] for the relationship between Galois extensions on homotopy and Galois extensions in the sense of Rognes [Rog08], and to [Lur11, § 4] as a convenient reference for some of the facts we need.

In particular, by [Lur11, Corollary 4.15], we need to show that Γ acts freely on $\pi_0(B)$. We will do this by functoriality. By the universal property of the (p -typical) Witt vectors, we obtain a G -equivariant map

$$\mathbb{W}(k) \rightarrow \pi_0(R),$$

where G acts on $\mathbb{W}(k)$ via the induced action on k . Of course, the G -action on $\mathbb{W}(k)$ factors through $\Gamma = G/\tilde{G}$, and we claim that this lifts to a Γ -equivariant map

$$\mathbb{W}(k) \rightarrow \pi_0(B) = \pi_0(R^{h\tilde{G}}).$$

The equivariance alone gives the Γ -equivariant lift $\mathbb{W}(k) \rightarrow \pi_0(R)^{\tilde{G}}$, which of course receives a map from $\pi_0(R^{h\tilde{G}})$. Using Lemma 3.9, we can upgrade it to the claimed Γ -equivariant map $\mathbb{W}(k) \rightarrow \pi_0(B)$.

The Γ -action on $W(k)$ is free (Example 3.2). It follows that the Γ -action on $\pi_0(B)$ is free. Applying [Lur11, Corollary 4.15], we conclude that $A \rightarrow B$ is an algebraic Γ -Galois extension in that $\pi_0(A) \rightarrow \pi_0(B)$ is one, and $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_*(B)$ is an isomorphism.

Finally, we prove the result about the Picard group. Let M be an invertible A -module. By assumption, $M \otimes_A B$ is a suspension of a free B -module of rank one; by desuspending M appropriately, we may assume that $M \otimes_A B \simeq B$. We would like to see that $M \simeq A$. Since B is faithfully flat over A , we can conclude that $\pi_0(M)$ is an invertible $\pi_0(A)$ -module and that the map $\pi_0(M) \otimes_{\pi_0(A)} \pi_*(A) \rightarrow \pi_*(M)$ is an isomorphism (because this happens after base-change to B). In particular, M simply arises from an invertible module over $\pi_0(A)$.

Therefore, it suffices to show that the Picard group of the (ordinary) commutative ring $\pi_0(A)$ is trivial. This will follow if we can show that $\pi_0(A)$ is a (possibly non-noetherian) local ring. Since $\pi_0(A) \rightarrow \pi_0(B)$ is finite étale, it suffices to show that $\pi_0(B)$ is a local ring, i.e. that its spectrum has a unique closed point. However, $\pi_0(B) \rightarrow \pi_0(R)^{\tilde{G}}$ is an \mathcal{F}_p -isomorphism as above, so that it induces an isomorphism on Zariski spectra. Therefore, the fact that $\pi_0(R)^{\tilde{G}}$ is a local ring forces $\pi_0(B)$ to be one, and completes the proof. \square

In our example of interest, R will be E_n , G will be a finite subgroup of the extended Morava stabilizer group, and $\tilde{G} = G_{\mathbb{S}}$, as in (2.3).

3.3 Reduction to the p -Sylow

Here, we will use basic facts about the structure of \mathbb{S}_n , for which we refer the reader to § 2.1.

PROPOSITION 3.11. *Let G be a finite subgroup of \mathbb{S}_n at some height n and at the implicit prime p . Let $G_p \subseteq G$ be a p -Sylow subgroup. Suppose the Picard group of the \mathbf{E}_∞ -ring $E_n^{hG_p}$ is cyclic, generated by the suspension. Then the Picard group of E_n^{hG} is also cyclic, generated by the suspension.*

Proof. Observe first that $\text{Pic}(E_n^{hG}) = \pi_0(\mathbf{pic}(E_n)^{hG})$ is a torsion group (cf. [MS16, § 5.3]), so it suffices to show that the suspension generates the q -part $\text{Pic}(E_n^{hG})_q$ of $\text{Pic}(E_n^{hG})$ for any prime number q . We consider two cases.

(i) $q = p$. In this case, the map

$$\phi_p : \text{Pic}(E_n^{hG})_p \rightarrow \text{Pic}(E_n^{hG_p})_p$$

(induced by base-change along $E_n^{hG} \rightarrow E_n^{hG_p}$) is injective thanks to the transfer map

$$\text{Pic}(E_n^{hG_p})_p \simeq \pi_0 \mathbf{pic}(E_n)^{hG_p} \rightarrow \text{Pic}(E_n^{hG})_p \simeq \pi_0 \mathbf{pic}(E_n)^{hG}.$$

Namely, recall that if X is a p -local spectrum with a G -action, the composite map $X^{hG} \rightarrow X^{hG_p} \xrightarrow{\text{tr}} X^{hG}$ is an equivalence. Since the map ϕ_p carries the suspension of the unit to itself, the fact that this class generates the target implies that it must generate the source.

(ii) $q \neq p$. Let $G_q \subseteq G$ be a q -Sylow subgroup. As above, we observe that the map ϕ_q defined via

$$\phi_q : \text{Pic}(E_n^{hG})_q \rightarrow \text{Pic}(E_n^{hG_q})_q$$

is injective. So, it suffices to consider the case where G is itself a q -group.

We consider the HFPSS converging to $\text{Pic}(E_n^{hG}) = \pi_0 \mathbf{pic}(E_n)^{hG}$. In this case, since G is a q -group, we observe that the only contributions come from $H^0(G, \mathbb{Z}/2)$ (which is $\mathbb{Z}/2$, generated by the suspension) and $H^1(G, (E_n)_0^\times) \simeq H^1(G, (\mathbb{F}_{p^n})^\times)$ via the Teichmüller lift. In other words, we need to show (to prove cyclicity of the Picard group in this case) that the group $H^1(G, (\mathbb{F}_{p^n})^\times) = \text{Hom}(G, \mathbb{F}_{p^n}^\times)$ is generated by the class ψ corresponding to the G -action on $\pi_{-2}(E_n) \otimes_{(E_n)_0} \mathbb{F}_{p^n}$. (This will give the order of the Picard group as well as solve the extension problem.) However, we know (by general facts about even-periodic ring spectra) that this is simply the action on the Lie algebra, i.e. the homomorphism of (2.1). However, as we remarked before, the kernel of (2.1) is pro- p , so its restriction to G must be injective. Hence, G injects into $\mathbb{F}_{p^n}^\times$, which forces $H^1(G, \mathbb{F}_{p^n}^\times)$ to be generated by the class of the injection. \square

4. Cyclicity of the Picard group

We now fix an odd prime p , and work at height $n = p - 1$; we will omit subscripts from our notation. In this section, we will use the HFPSS to compute $\text{Pic}(E^{hG})$ for all finite subgroups $G \subset \mathbb{G}$, referring to future sections of the document for some technical results required. We will use the results of the previous section to reduce to the case where $G = C_p$ is the p -Sylow subgroup of a maximal finite subgroup of \mathbb{S} . We also sketch a direct proof when G is a maximal finite subgroup of \mathbb{G} with p -torsion, to illustrate to the reader that taking Galois invariants simplifies the calculation somewhat.

4.1 The general case

Our goal in this subsection is to prove the main result of this paper, namely the following theorem.

THEOREM 4.1. *Let $G \subset \mathbb{G}$ be a finite subgroup. Then $\text{Pic}(E^{hG})$ is cyclic, generated by ΣE^{hG} .*

By combining Propositions 3.10, and 3.11, it is enough to prove the result when $G = C_p$. To handle this case, we start by recalling from Proposition 2.6 that, modulo the image of the transfer,

$$H^*(C_p, E_*)/\text{transfers} \simeq \mathbb{F}_{p^n}[\alpha, \beta, \delta^{\pm 1}]/(\alpha^2), \tag{4.1}$$

where $|\alpha| = (1, 2n)$, $|\beta| = (2, 2pn)$, and $|\delta| = (0, 2p)$. Recall also that the homotopy groups $\pi_*(E^{hC_p})$ are $2p^2$ -periodic; this sets $\mathbb{Z}/(2p^2)$ as a lower bound on the order of the Picard group. We will use the Picard spectral sequence to determine an upper bound. Since we will be comparing the Picard spectral sequence with the additive one (for computing $\pi_*E^{hC_p}$), we will write $d_{k,\times}$ for the Picard differentials and d_k for the additive ones.

Our first task is to work out the E_2 -term $H^s(C_p, \pi_t \mathbf{pic}(E))$, and we are interested in the abutment for $t - s = 0$. In the range where $t \geq 2$, this follows immediately from (4.1). Namely, we have

$$H^s(C_p, \pi_t \mathbf{pic}(E)) = (\mathbb{F}_{p^n}[\alpha, \beta, \delta^{\pm 1}]/(\alpha^2))^{s,t-1} \quad \text{when } t \geq 2, s \geq 1$$

since in this case $\pi_t \mathbf{pic}(E) = \pi_{t-1}E$, equivariantly. (The superscript notation on the right-hand side designates the bidegree.)

Let us single out the classes in the $t - s = 0$ column, i.e. those with potential contribution to the Picard group of E^{hC_p} , that come from the additive spectral sequence (so, $s > 1$). They are

$$e_c = \alpha\beta^{pc-1}\delta^{n-1-c(pn-1)},$$

indexed by integers $c \geq 1$, and have cohomological degree $s(e_c) = 2pc - 1$.

LEMMA 4.2. *The classes e_c for $c \neq 1$ do not survive the Picard spectral sequence.*

Proof. In the additive spectral sequence, by Lemma 2.8, the classes e_c are never the source of a d_{2p-1} differential, but can be the target. In particular, there are additive differentials

$$d_{2p-1}(\beta^a\delta^b) \doteq b\alpha\beta^{a+p-1}\delta^{b-n^2}.$$

For this image to be (a multiple of) e_c , we must have $b = n^2 + n - 1 - c(pn - 1) \equiv c - 1 \pmod p$.

- If $c \not\equiv 1 \pmod p$, then e_c is the target of d_{2p-1} on an element with $t = 2p(c - 1)$. Since such $c \geq 2$, we have that $t \geq 2p > 2p - 1$, so that Theorem 1.4 implies that this differential can be imported in the Picard spectral sequence. Hence, e_c itself is killed.
- If $c \equiv 1 \pmod p$, this differential is zero (as then $b \equiv 0 \pmod p$), so e_c survives the d_{2p-1} .

The next and only possible additive differential is a d_{2n^2+1} . We have

$$d_{2n^2+1}(\alpha\delta^{n^3}) \doteq \beta^{n^2+1},$$

which determines the rest. Namely, we have, for all $a, b \in \mathbb{Z}$, $a \geq 0$,

$$d_{2n^2+1}(\alpha\beta^a\delta^{bp-1}) \doteq \beta^{a+n^2+1}\delta^{pb-n^3-1}.$$

When $c \equiv 1 \pmod p$, we have that $n - 1 - c(pn - 1) \equiv -1 \pmod p$, hence

$$d_{2n^2+1}(e_c) \doteq \beta^{pc+n^2}\delta^{c(1-pn)+n-n^3-1}.$$

To import this differential to the Picard spectral sequence, we need to have that

$$2n^2 + 1 \leq t(e_c) = 2pc - 2.$$

This holds whenever

$$p - 2 + \frac{5}{2p} \leq c \iff c \geq p - 1.$$

Since $c \equiv 1 \pmod{p}$, we see we can import all differentials except that involving e_1 , and the result follows. \square

Remark 4.3. This lemma says nothing about the fate of e_1 ; it does not survive in the additive spectral sequence, but we shall see later that it must survive in the multiplicative case.

We can now prove our main result.

Proof of Theorem 4.1. We have seen that in the range $t \geq 2$ of the HFPSS for $\pi_{t-s} \mathbf{pic}(E^{hC_p})$, we only have a single possible contribution to the 0-stem, namely a group of order at most p^n , arising from the class e_1 . This class has bidegree $(2p - 1, 2p - 1)$, and it follows from Theorem 7.1 that there is a differential

$$d_{2n^2+1, \times}(e_1) \doteq d_{2n^2+1}(e_1) + \zeta \beta \mathcal{P}^{n/2}(e_1), \tag{4.2}$$

for $\zeta \in \mathbb{F}_p^\times$. Let $f_t = \beta^{n^2+n+1} \delta^{-n^2(n+1)}$ so that we have (in the additive spectral sequence)

$$d_{2n^2+1}(e_1) \doteq f_t.$$

While it is possible to work out exactly what $\beta \mathcal{P}^{n/2}(e_1)$ is, we will not need to be so precise. Instead we claim that the group $\mathbb{F}_{p^n} e_1$ on the E_2 -page can contribute at most an element of order p to Picard group.

Choose $\xi' \in \mathbb{F}_{p^n}$ such that $\beta \mathcal{P}^{n/2}(e_1) = \xi' f_t$. Then we have

$$d_{2n^2+1}(e_1) = \xi f_t \quad \text{and} \quad \beta \mathcal{P}^{n/2}(e_1) = \xi' f_t,$$

for $\xi, \xi' \in \mathbb{F}_{p^n}$ and $\xi \neq 0$. It follows from (4.2) that, for any $a \in \mathbb{F}_{p^n}$, we have

$$\begin{aligned} d_{2n^2+1, \times}(ae_1) &= d_{2n^2+1}(ae_1) + \zeta \beta \mathcal{P}^{n/2}(ae_1) \\ &= a \xi f_t + a^p \zeta \beta \mathcal{P}^{n/2}(e_1) \\ &= a \xi f_t + a^p \zeta \xi' f_t \\ &= (a \xi + a^p \zeta \xi') f_t. \end{aligned}$$

Since $\xi \neq 0$, there are at most p choices of a that make the above vanish, and so we do indeed have an upper bound of \mathbb{Z}/p for the contribution of e_1 in $\mathbf{Pic}(E^{hC_p})$.

Finally, we know from Proposition 6.5 that $H^1(C_p, \pi_1 \mathbf{pic}(E)) \simeq \mathbb{Z}/p$, while as usual it follows from [BR05] that the invariants $H^0(C_p, \pi_0 \mathbf{pic}(E))$ are $\mathbb{Z}/2$. Thus, we have the following in the 0-stem of the Picard spectral sequence:

- a group of order at most 2 in $t = s = 0$;
- a group of order at most p in $t = s = 1$; and
- a group of order at most p in $t = s = 2p - 1$.

This gives an upper bound of $2p^2$ for the order of the Picard group, but as noted previously this is also a lower bound, and the result follows. A picture of the Picard spectral sequence for $n = 2$, $p = 3$ and $G = C_3$ is shown in Figure 2. \square

4.2 The maximal finite subgroup

In this short section we sketch a direct proof in the case that G is a maximal finite subgroup of \mathbb{G}_n with p -torsion. We include this to show that after taking Galois invariants the calculation is

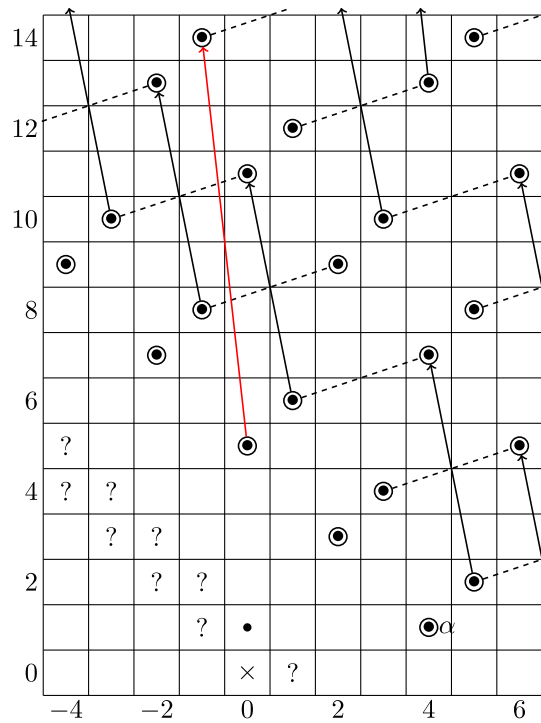


FIGURE 2. The HFPSS for $\pi_* \text{pic } E_p^{hC_p}$ in the case $n = 2, p = 3$. The differential out of $(0, 5)$ has kernel $\mathbb{Z}/3$. (Here \bullet denotes $\mathbb{Z}/3$, \odot denotes \mathbb{F}_9 , and \times denotes $\mathbb{Z}/2$.)

slightly easier; in particular, we do not need to use Theorem 7.1. The calculation of $\text{Pic}(E_n^{hG})$ in this case was previously known to Hopkins.

We start by recalling that, modulo the image of the transfer, we have

$$H^*(G, E_*)/\text{transfers} \simeq \mathbb{F}_p[\alpha, \beta, \Delta^{\pm 1}]/(\alpha^2), \tag{4.3}$$

where $|\Delta| = (0, 2pn^2)$, and that the homotopy groups $\pi_*(E^{hG})$ are $2p^2n^2$ periodic; this sets $\mathbb{Z}/(2p^2n^2)$ as a lower bound on the order of the Picard group. As usual, we will use the Picard spectral sequence to determine an upper bound.

In the range $t \geq 2$ we have classes

$$e_c = \alpha\beta^{pc-1}\Delta^{(c(1-pn)+n-1)/n^2},$$

in the 0-stem, indexed by integers $c \geq 1$, which have cohomological degree $s(e_c) = 2pc - 1$. Once again, one can check that the classes e_c for $c \neq 1$ do not survive the Picard spectral sequence. One key difference to the case of C_p is that we see immediately that e_1 can contribute a group of at most order p to $\text{Pic}(E^{hG})$, thus avoiding the use of Theorem 7.1.

We are then left with calculating the contributions from $t = 0$ and $t = 1$. It follows from [BR05] that $H^0(G, \pi_0 \text{pic}(E)) \simeq H^0(G, \mathbb{Z}/2) \simeq \mathbb{Z}/2$, and so we are left to determine $H^1(G, \pi_1 \text{pic}(E)) \simeq H^1(G, E_0^\times)$. This can be done in at least a couple of different ways; for example, as in Proposition 6.5, or using an exponential map and several Bockstein spectral sequences as in [Kar10]. In any case, the result is a group of order pn^2 .

Putting all of this together, we obtain an upper bound $2 \cdot pn^2 \cdot p$ for the abutment, i.e. for the Picard group of E^{hG} . However, by Remark 2.9 we know the Picard group contains a cyclic group

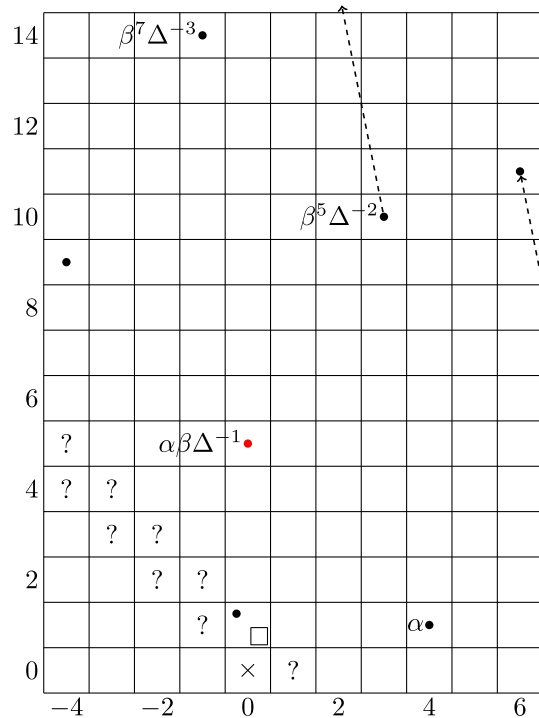


FIGURE 3. The HFPSS for $\pi_* \text{pic } E^{hG}$ in the case $n = 2, p = 3$ for $G \subset \mathbb{G}$ a maximal finite subgroup whose order is divisible by p . (Here \square denotes $\mathbb{Z}/4$, \bullet denotes $\mathbb{Z}/3$, and \times denotes $\mathbb{Z}/2$.)

of order $2p^2n^2$, and so it follows that $\text{Pic}(E^{hG})$ is indeed cyclic of order $2p^2n^2$. We illustrate this for $n = 2, p = 3$ in Figure 3.

PART II. COMPUTATIONAL TOOLS

5. Truncated logarithms and differentials in the algebraic setting

5.1 Overview

In this section, we describe a tool for working with the following general situation. Let R be a complete noetherian local ring with maximal ideal $\mathfrak{m} \subset R$ such that the residue field R/\mathfrak{m} has characteristic p . Suppose G is a finite group acting on R . Our goal is to calculate the cohomology groups $H^i(G, R^\times)$. We assume that the additive analog, the cohomology groups $H^i(G, R)$, are understood by other means.

One approach towards $H^i(G, R^\times)$ is to consider the \mathfrak{m} -adic filtration on R . It induces a G -stable filtration on R^\times ,

$$R^\times \supset 1 + \mathfrak{m} \supset 1 + \mathfrak{m}^2 \supset \dots, \tag{5.1}$$

which leads to a spectral sequence

$$E_1^{i,j} = H^i(G, (1 + \mathfrak{m}^j)/(1 + \mathfrak{m}^{j+1})) \implies H^i(G, R^\times). \tag{5.2}$$

Here the differentials run $d_k : E_k^{i,j} \rightarrow E_k^{i+1, j+k}$. For $j > 0$, there is a G -equivariant isomorphism (sending $1 + x \mapsto x$)

$$(1 + \mathfrak{m}^j)/(1 + \mathfrak{m}^{j+1}) \simeq \mathfrak{m}^j/\mathfrak{m}^{j+1}$$

between the associated graded module of the filtration (5.1) and the \mathfrak{m} -adic filtration on R itself. The latter, of course, leads to a spectral sequence (indexed similarly)

$$E_1^{i,j} = H^i(G, \mathfrak{m}^j / \mathfrak{m}^{j+1}) \implies H^i(G, R). \tag{5.3}$$

In practice, one may (and is likely to) understand the additive spectral sequence (5.3) concretely in specific instances, for instance, because of the additional tools provided by multiplicativity. The additive and the multiplicative spectral sequences have almost the same E_1 pages, and in this section we will give a tool for comparing differentials in a range of dimensions.

5.2 The p -truncated exponential

Let R be a commutative ring and let $I \subset R$ be an ideal such that $I^2 = 0$. Then there is an isomorphism of groups

$$I \simeq 1 + I \subset R^\times, \quad x \mapsto 1 + x.$$

We now review a crucial and elementary piece of algebra that will enable generalizations of this fact when certain primes are inverted in R .

DEFINITION 5.1. Let R be a $\mathbb{Z}[1/(p - 1)!]$ -algebra. We define a function

$$\exp_p : R \rightarrow R, \quad \exp_p(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{p-1}}{(p - 1)!}, \tag{5.4}$$

called the p -truncated exponential, and a function

$$\log_p : R \rightarrow R, \quad \log_p(x) = \sum_{n=1}^{p-1} (-1)^{n-1} \frac{(x - 1)^n}{n}, \tag{5.5}$$

called the p -truncated logarithm.

The significance arises from the following well-known application.

LEMMA 5.2. If $I \subset R$ is an ideal with $I^p = 0$ and R is a $\mathbb{Z}[1/(p - 1)!]$ -algebra, then the map

$$\exp_p : I \rightarrow 1 + I \subset R^\times$$

is a group isomorphism (whose inverse is given by the truncated logarithm).

We note that the inverse isomorphisms \exp_p, \log_p between I and $1 + I$ are compatible with the I -adic filtration, and with the isomorphisms on associated graded modules $I^k / I^{k+1} \simeq (1 + I^k) / (1 + I^{k+1})$ given by $1 + x \mapsto x$.

Example 5.3. Suppose (A, \mathfrak{m}) is a local ring whose residue field is of characteristic p , a prime larger than p , or zero. Then for each $k > 0$, we have a functorial isomorphism of groups

$$(1 + \mathfrak{m}^k) / (1 + \mathfrak{m}^{pk}) \simeq \mathfrak{m}^k / \mathfrak{m}^{pk},$$

given by the truncated logarithm. This leads to an identification of the spectral sequences (5.3) and (5.2) in a ‘stable’ range of dimensions (which we will make more precise below).

We will actually need a slight generalization of this fact (which measures the first-order failure of \exp_p to be a homomorphism in general).

DEFINITION 5.4. Given $x_1, \dots, x_k \in R$, we define

$$\sigma_p(x_1, \dots, x_k) \stackrel{\text{def}}{=} \frac{(\sum_i x_i)^p - \sum_i x_i^p}{p!} = \sum_{\substack{j_1, \dots, j_k \in [0, p-1] \\ j_1 + \dots + j_k = p}} \frac{x_1^{j_1} x_2^{j_2} \dots x_k^{j_k}}{j_1! \dots j_k!}, \tag{5.6}$$

so that we obtain a function $\sigma_p(\cdot, \dots, \cdot) : R^k \rightarrow R$. If M is a $\mathbb{Z}[1/(p-1)!]$ -module, we will also use σ_p to denote the induced function

$$\sigma_p : M^k \rightarrow \text{Sym}^p M,$$

defined by the same formula.

PROPOSITION 5.5. *Let R be a $\mathbb{Z}[1/(p-1)!]$ -algebra. Suppose that $x_1, \dots, x_k \in R$ and let $I = (x_1, \dots, x_k) \subset R$. Then*

$$\prod \exp_p(x_i) - \exp_p\left(\sum x_i\right) \equiv \sigma_p(x_1, \dots, x_k) \pmod{I^{p+1}}. \tag{5.7}$$

Proof. This follows by a straightforward computation. □

With notation as above, observe that $\sigma_p(x_1, \dots, x_k) \in I^p$ and that $\exp_p(x) \in 1 + I$ for $x \in I$. As a result, we also find that, if I is contained in the Jacobson radical of R , then we have the formula

$$\frac{\exp_p(\sum x_i)}{\prod \exp_p(x_i)} \equiv 1 - \sigma_p(x_1, \dots, x_k) \pmod{I^{p+1}}. \tag{5.8}$$

For example, taking $k = 2$ and taking $x_1 = y, x_2 = -y$, we find

$$\exp_p(y) \exp_p(-y) \equiv 1 + \sigma_p(y, -y) \pmod{I^{p+1}}. \tag{5.9}$$

Note, in addition, that $\sigma_p(y, -y) = y^p \sigma_p(1, -1)$.

COROLLARY 5.6. *If $y_0, \dots, y_n \in I$ and I is contained in the radical of R , then*

$$\frac{\exp_p(\sum_{i=0}^n (-1)^i y_i)}{\prod_{i=0}^n \exp_p(y_i)^{(-1)^i}} \equiv 1 - \sigma_p(y_0, -y_1, \dots, (-1)^n y_n) + \sum_{i \text{ odd}} \sigma_p(y_i, -y_i) \pmod{I^{p+1}}. \tag{5.10}$$

Proof. This follows from (5.8) taking $x_i = (-1)^i y_i$. We multiply it by $\prod_{i \text{ odd}} (\exp_p(-y_i) / \exp_p(y_i)^{-1})$ and invoke (5.9). □

We give a name to the discrepancy term; given y_0, \dots, y_n , we set

$$\mu_p(y_0, \dots, y_n) \stackrel{\text{def}}{=} \sigma_p(y_0, -y_1, \dots, (-1)^n y_n) - \sum_{i \text{ odd}} \sigma_p(y_i, -y_i). \tag{5.11}$$

5.3 The stable range and first differential

We begin by describing a version of the general setup of comparison of spectral sequences from the introduction to this section.

PROPOSITION 5.7. *Let G be a finite group acting on a local noetherian ring (R, \mathfrak{n}) with $(p-1)!$ invertible. Consider the two truncated spectral sequences*

$$\begin{aligned} E_1^{s,t} &= H^s(G, \mathfrak{n}^t / \mathfrak{n}^{t+1}) \implies H^i(G, \mathfrak{n}^k / \mathfrak{n}^{pk}), \\ E_1^{s,t} &= H^i(G, \mathfrak{n}^t / \mathfrak{n}^{t+1}) \implies H^i(G, (1 + \mathfrak{n}^k) / (1 + \mathfrak{n}^{pk})), \end{aligned}$$

where $t \in [k, pk - 1]$. Then the natural identification of E_1 -terms extends to an isomorphism of spectral sequences given by the p -truncated exponential.

We now describe the setup that will enable us to ultimately give a formula for the first unequal differential. For convenience, we will work with *nonunital* commutative rings here. If A is a nonunital commutative ring, we define an abelian monoid $1 + A$ consisting of formal elements $1 + x, x \in A$ with the obvious multiplication rule.

Construction 5.8. Suppose \mathfrak{m}^\bullet is a *nonunital* cosimplicial commutative $\mathbb{Z}[1/(p-1)!]$ -algebra such that $\mathfrak{m}^{\bullet,p+1} = 0$.² We have a descending multiplicative filtration by powers

$$\mathfrak{m}^\bullet \supset \mathfrak{m}^{\bullet,2} \supset \dots$$

Consider also the cosimplicial abelian group $1 + \mathfrak{m}^\bullet$.³ We also have a filtration of cosimplicial abelian groups

$$1 + \mathfrak{m}^\bullet \supset 1 + \mathfrak{m}^{\bullet,2} \supset \dots,$$

and the successive quotients are isomorphic to those of the additive one.

Both lead to spectral sequences starting from the associated graded modules $\mathfrak{m}^{\bullet,j}/\mathfrak{m}^{\bullet,j+1}$ (for $1 \leq j \leq p$) converging to the cohomology of \mathfrak{m}^\bullet (respectively $1 + \mathfrak{m}^\bullet$), i.e. the E_1 -pages are identified. We have spectral sequences

$$E_1^{i,j} = H^i(\mathfrak{m}^{\bullet,j}/\mathfrak{m}^{\bullet,j+1}) \implies H^i(\mathfrak{m}^\bullet) \tag{5.12}$$

$$E_1^{i,j,\times} = H^i(\mathfrak{m}^{\bullet,j}/\mathfrak{m}^{\bullet,j+1}) \implies H^i(1 + \mathfrak{m}^\bullet). \tag{5.13}$$

The E_1 -terms are identified. We will denote the differentials at the k th page by d_k and d_k^\times , respectively.

Using the truncated exponential, we obtain isomorphisms of filtered cosimplicial abelian groups

$$\mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,p} \simeq (1 + \mathfrak{m}^\bullet)/(1 + \mathfrak{m}^{\bullet,p}), \quad \mathfrak{m}^{\bullet,2} \simeq 1 + \mathfrak{m}^{\bullet,2}$$

inducing the above isomorphism on the associated graded modules.

The above construction yields the following result.

PROPOSITION 5.9 (Spectral sequence comparison result, stable range). *In the spectral sequences $E_k^{*,*}$ and $E_k^{*,*,\times}$ above, we have isomorphisms $E_k^{*,*} \simeq E_k^{*,*,\times}$ for $k \leq p - 1$. This identification is compatible with the above identification at $k = 1$ and is compatible with the differentials d_k for $k < p - 1$.*

The most important goal is to obtain a description of the first ‘unstable’ differential d_{p-1} (in the present setup of nonunital algebras with $(p + 1)$ -fold products zero, it is also the last differential in the spectral sequence).

Let P^\bullet be a cosimplicial $\mathbb{Z}[1/(p-1)!]$ -module. Then we can consider the cosimplicial $\mathbb{Z}[1/(p-1)!]$ -module $\text{Sym}^p P^\bullet$ obtained by applying the p th symmetric power levelwise. We construct an operation

$$H^i(P^\bullet) \rightarrow H^{i+1}(\text{Sym}^p P^\bullet).$$

Construction 5.10. Let $x \in P^i$. We define

$$\phi(x) \stackrel{\text{def}}{=} \mu_p(\partial^0 x, \partial^1 x, \partial^2 x, \dots, \partial^i x) \in \text{Sym}^p P^{i+1}.$$

² The notation $\mathfrak{m}^{\bullet,k}$ means the k th power of the cosimplicial ring \mathfrak{m}^\bullet .

³ Since \mathfrak{m}^\bullet is nilpotent, this is a cosimplicial abelian group and not only a cosimplicial abelian monoid.

Here the ∂^k are the cosimplicial operators. We claim that:

- (i) if x is a cocycle in P^i , then $\phi(x)$ is a cocycle in $\text{Sym}^p P^{i+1}$;
- (ii) if $x, y \in P^i$ are cohomologous cocycles, then $\phi(x)$ and $\phi(y)$ are cohomologous in $\text{Sym}^p P^{i+1}$.

The proofs of both these claims will be contained in Proposition 5.11 below. As a result, we will also abuse notation and use ϕ to denote the induced map in cohomology.

Return to the previous spectral sequences for computing the cohomology of the cosimplicial abelian groups \mathfrak{m}^\bullet and $1 + \mathfrak{m}^\bullet$. The $(p - 1)$ th differential in each spectral sequence, namely d_{p-1} or $d_{p-1, \times}$, runs from a sub-object of $H^i(\mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,2})$ to a quotient of $H^{i+1}(\mathfrak{m}^{\bullet,p})$.

PROPOSITION 5.11 (First unstable differential). *The operator ϕ satisfies the claimed properties (i) and (ii) above, and we have $d_{p-1, \times} = d_{p-1} + \phi$.*

Let us explain this formula. We have a natural (multiplication) morphism of cosimplicial abelian groups $\text{Sym}^p(\mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,2}) \rightarrow \mathfrak{m}^{\bullet,p}$. The operator ϕ gives us a natural map from $H^i(\mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,2})$ to $H^{i+1}(\text{Sym}^p(\mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,2}))$ and we compose that with the multiplication map.

Proof. We unwind the definitions. A class $x \in H^i(\mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,2})$ that has survived to the E_{p-1} stage of the spectral sequence, by definition, lifts to a class in $H^i(\mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,p})$, which we represent by a degree i cycle \tilde{x} . The short exact sequence of cosimplicial abelian groups

$$0 \rightarrow \mathfrak{m}^{\bullet,p} \rightarrow \mathfrak{m}^\bullet \rightarrow \mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,p} \rightarrow 0,$$

leads to a connecting homomorphism

$$\delta : H^i(\mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,p}) \rightarrow H^{i+1}(\mathfrak{m}^{\bullet,p}).$$

The differential $d_{p-1}(x)$ is given by the image of $\delta(\tilde{x})$.

In the multiplicative setting, we know that the cycle $\exp_p(\tilde{x}) \in (1 + \mathfrak{m}^\bullet)/(1 + \mathfrak{m}^{\bullet,p})$ lifts the image of x (or rather, $1 + x$) in the E_1 -page of the multiplicative spectral sequence. Now $d_{p-1, \times}(x)$ is given as follows: one considers the short exact sequence of cosimplicial abelian groups

$$0 \rightarrow (1 + \mathfrak{m}^{\bullet,p}) \rightarrow (1 + \mathfrak{m}^\bullet) \rightarrow (1 + \mathfrak{m}^\bullet)/(1 + \mathfrak{m}^{\bullet,p}) \rightarrow 0,$$

and the induced connecting homomorphism

$$\delta_\times : H^i((1 + \mathfrak{m}^\bullet)/(1 + \mathfrak{m}^{\bullet,p})) \rightarrow H^{i+1}(1 + \mathfrak{m}^{\bullet,p}).$$

Unwinding the definitions, the multiplicative differential $d_{p-1, \times}(x)$ is given by

$$d_{p-1, \times}(x) = \delta_\times(\exp_p(\tilde{x})) \in H^{i+1}(1 + \mathfrak{m}^{\bullet,p}) \simeq H^{i+1}(\mathfrak{m}^{\bullet,p}),$$

or rather the image in the quotient of this at the E_{p-1} -page.

It follows now that we need to compare the additive and the multiplicative boundary maps δ, δ^\times . Given a cycle $\tilde{x} \in \mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,p}$ in degree i , projecting to a cycle and induced cohomology class $x \in \mathfrak{m}^\bullet/\mathfrak{m}^{\bullet,2}$, we claim that

$$\delta_\times(\exp_p(\tilde{x})) = 1 + \delta(\tilde{x}) + \phi(\tilde{x}) \in H^{i+1}(1 + \mathfrak{m}^{\bullet,p}) \simeq H^{i+1}(\mathfrak{m}^{\bullet,p}). \tag{5.14}$$

To prove this, we unwind the definitions again. Lift \tilde{x} to a chain $x' \in \mathfrak{m}^\bullet$ (in degree i). The additive connecting homomorphism $\delta(\tilde{x})$ is obtained by considering the image in cohomology of

the alternating sum $y' = \sum_k (-1)^k \partial^k x'$, which is an $(i + 1)$ -cocycle in $\mathbf{m}^{\bullet,p}$. For the multiplicative version, we have to consider the cocycle

$$\delta_{\times}(\exp_p(\tilde{x})) = \prod_k (\partial^k \exp_p(x'))^{(-1)^k} = \prod_k \exp_p(\partial^k x')^{(-1)^k} \in 1 + \mathbf{m}^{\bullet,p+1}.$$

Now we use (5.10), which gives us

$$\begin{aligned} \delta_{\times}(\exp_p(\tilde{x})) &= \exp_p\left(\sum (-1)^k \partial^k x'\right) + \mu_p(\partial^0 x', \partial^1 x', \dots, \partial^i x') \\ &= \exp_p(y') + \mu_p(\partial^0 x', \partial^1 x', \dots, \partial^i x') \\ &= 1 + y' + \mu_p(\partial^0 x', \partial^1 x', \dots, \partial^i x'). \end{aligned}$$

In cohomology, this is the desired claim, provided we can justify the claims about ϕ made in Construction 5.10.

The justification of Construction 5.10 now follows from reversing the above argument. Keep the notation from there. We consider the commutative nonunital cosimplicial algebra

$$A^{\bullet} = \bigoplus_{0 < k \leq p} \text{Sym}^k P^{\bullet}$$

and its quotient $\bar{A}^{\bullet} = \bigoplus_{0 < k < p} \text{Sym}^k P^{\bullet}$. Choose a cycle $x \in P^{\bullet} \subset \bar{A}^{\bullet}$, which clearly extends to a cycle in A^{\bullet} . Then $\exp_p(x)$ is a cycle in $1 + \bar{A}^{\bullet}$. The coboundary of this class in $\text{Sym}^p P^{\bullet} \simeq 1 + \text{Sym}^p P^{\bullet} \subset 1 + A^{\bullet}$ is given by $\phi(x)$ by the above analysis. This shows that $\phi(x)$ is a cycle and that its cohomology class depends only on that of x . \square

5.4 Identification with $\beta\mathcal{P}^0$

Let P^{\bullet} be a cosimplicial $\mathbb{Z}[1/(p-1)!]$ -module. In the previous subsection, we described a natural operation (based on a formula)

$$\phi : H^i(P^{\bullet}) \rightarrow H^{i+1}(\text{Sym}^p P^{\bullet}). \tag{5.15}$$

Suppose $i > 0$.⁴ In the setting of cosimplicial \mathbb{F}_p -vector spaces, the work of Priddy [Pri73] has classified all such operations, which amount to determining the cohomology of the symmetric powers of a cosimplicial vector space (one reduces to an Eilenberg–MacLane complex). In particular, for cosimplicial $\mathbb{Z}[1/(p-1)!]$ -modules, the only such natural operations are the scalar multiples of a single operation called $\beta\mathcal{P}^0$, as one sees from Priddy’s work (and the Bockstein spectral sequence). In other words, in the ‘universal’ case when $H^*(P^{\bullet})$ is a $\mathbb{Z}[1/(p-1)!]$ (generated by ι) concentrated in degree i , we have $H^{i+1}(\text{Sym}^p P^{\bullet}) \simeq \mathbb{Z}/p$, generated by the class $\beta\mathcal{P}^0\iota$.

Given a cosimplicial \mathbb{F}_p -vector space P^{\bullet} , the cosimplicial commutative ring $\text{Sym}^* P^{\bullet}$ yields (upon taking the totalization) an \mathbf{E}_{∞} -algebra over \mathbb{F}_p . Then $\beta\mathcal{P}^0$ is identified with the power operation on \mathbf{E}_{∞} -algebras over \mathbb{F}_p with the same name.

In this section, we will prove the following proposition.

PROPOSITION 5.12. *For cosimplicial $\mathbb{Z}[1/(p-1)!]$ -modules, we have $\phi = -\beta\mathcal{P}^0$. As a result, in the situation of Proposition 5.11, we have*

$$d_{p-1,\times} = d_{p-1} - \beta\mathcal{P}^0. \tag{5.16}$$

⁴ When $i = 0$, the power operation $\beta\mathcal{P}^0 = 0$ and the operation ϕ is identically zero, so we shall not discuss this case.

While one can compare explicit formulas, we will approach the determination of the operator ϕ by considering examples. We note that this formula will not be used in the following. Rather, it will suffice to know that ϕ is Frobenius-semilinear.

First of all, we give an example to show that ϕ is not identically zero, i.e. that the additive and multiplicative versions of d_{p-1} can, in fact, differ.

Example 5.13. Consider the commutative algebra $\mathbb{F}_p[x]/x^{p+1}$ and the ideal $\mathfrak{m} = (x)$, so that $\mathfrak{m}^{p+1} = 0$. We then have an isomorphism

$$(1 + \mathfrak{m})/(1 + \mathfrak{m}^p) \simeq (\mathbb{Z}/p)^{p-1},$$

as one sees via the truncated exponential and logarithm. However, we also have

$$1 + \mathfrak{m} \simeq \mathbb{Z}/p^2 \oplus (\mathbb{Z}/p)^{p-2}.$$

In fact, the unit $1 + x \in 1 + \mathfrak{m}$ has order exactly p^2 . Meanwhile, $\exp_p(x^2), \dots, \exp_p(x^{p-1})$ each have order p (one could also take $1 + x^2, \dots, 1 + x^{p-1}$). One sees that the map

$$1 + \mathfrak{m} \rightarrow (1 + \mathfrak{m})/(1 + \mathfrak{m}^p)$$

can be identified with the direct sum of the reduction map $\mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p$ and the identity $(\mathbb{Z}/p)^{p-2} \rightarrow (\mathbb{Z}/p)^{p-2}$.

Now let X_\bullet be a simplicial set and suppose that there exists a class in $H^n(X_\bullet, \mathbb{Z}/p)$ which does not lift to $H^n(X_\bullet, \mathbb{Z}/p^2)$. Form the nonunital cosimplicial commutative algebra $F(X_\bullet, \mathfrak{m}) = \mathfrak{m}^{X_\bullet}$, whose group of units is $F(X_\bullet, 1 + \mathfrak{m})$. We can apply the machinery of the previous subsection to this example, where the filtration in question is the \mathfrak{m} -adic filtration.

The additive spectral sequence converges to the cohomology of X_\bullet with coefficients in $(\mathbb{Z}/p)^p$ and the multiplicative spectral sequence converges to the cohomology with coefficients in $\mathbb{Z}/p^2 \oplus (\mathbb{Z}/p)^{p-2}$. We see in particular that, while there are no differentials additively (the filtration splits), there is necessarily a $d_{p-1, \times}$ since there are cohomology classes in $H^n(X_\bullet, \mathbb{Z}/p)$ which do not lift to $H^n(X_\bullet, \mathbb{Z}/p^2)$.

We now describe an example where the additive spectral sequence supports a differential but the multiplicative spectral sequence does not.

Example 5.14. Consider the complete local ring $R = \mathbb{Z}_p[\zeta_p]$ and the maximal ideal $\mathfrak{m} = (1 - \zeta_p)$. We consider the quotient $\overline{R} = R/\mathfrak{m}^{p+1}$ and the image $\overline{\mathfrak{m}}$ of \mathfrak{m} . We have:

- (i) $(1 + \overline{\mathfrak{m}})/(1 + \overline{\mathfrak{m}}^p) \simeq \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^p \simeq (\mathbb{Z}/p)^{p-1}$ (note here that $p \in \mathfrak{m}^{p-1} \setminus \mathfrak{m}^p$);
- (ii) $\overline{\mathfrak{m}} \simeq \mathbb{Z}/p^2 \oplus (\mathbb{Z}/p)^{p-2}$; the generator of the \mathbb{Z}/p^2 is $(1 - \zeta_p)$ and the generators of the remaining terms are $(1 - \zeta_p)^i, i = 2, \dots, p - 1$; the map $\overline{\mathfrak{m}} \rightarrow \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^p$ is a direct sum of the reduction $\mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p$ and the identity $(\mathbb{Z}/p)^{p-2} \rightarrow (\mathbb{Z}/p)^{p-2}$;
- (iii) $1 + \overline{\mathfrak{m}} \simeq (\mathbb{Z}/p)^p$. In fact, one sees that every element has order p here (since ζ_p does); the map $1 + \overline{\mathfrak{m}} \rightarrow (1 + \overline{\mathfrak{m}})/(1 + \overline{\mathfrak{m}}^p)$ is a surjection $(\mathbb{Z}/p)^p \rightarrow (\mathbb{Z}/p)^{p-1}$.

Let X_\bullet be a simplicial set as in the previous example. In this case, reasoning analogously, we find that the *additive* spectral sequence for $F(X_\bullet, \overline{\mathfrak{m}})$ must support a d_{p-1} but the multiplicative spectral sequence does not.

Proof of Proposition 5.12. We will analyze the last example more carefully to determine the precise multiple.⁵ Note that $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 \simeq \mathbb{F}_p$ (generated by the image of $1 - \zeta_p$) and $\bar{\mathfrak{m}}^p \simeq \mathbb{F}_p$ (generated by the image of $(1 - \zeta_p)^p$). The (additive) differential d_{p-1} comes from the short exact sequence

$$0 \rightarrow \bar{\mathfrak{m}}^p \rightarrow \bar{\mathfrak{m}} \rightarrow \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^p \rightarrow 0, \tag{5.17}$$

and we have the following commutative diagram of short exact sequences.

$$\begin{CD} 0 @>>> \mathbb{Z}/p @>{1 \mapsto p}>> \mathbb{Z}/p^2 @>>> \mathbb{Z}/p @>>> 0 \\ @. @VVV @VVV @VVV @. \\ 0 @>>> \bar{\mathfrak{m}}^p @>>> \bar{\mathfrak{m}} @>>> \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^p @>>> 0 \end{CD} \tag{5.18}$$

Here the map $\mathbb{Z}/p^2 \rightarrow \bar{\mathfrak{m}}$ carries $1 \mapsto 1 - \zeta_p$. Note that $p(1 - \zeta_p) \equiv (1 - \zeta_p)^p$ in $\bar{\mathfrak{m}}$ as one sees from the expression $\prod_{i=1}^{p-1} (T - \zeta_p^i) = T^{p-1} + T^{p-2} + \dots + 1$ with $T = 1$.

Take $X_\bullet = K(\mathbb{Z}/p, n)$, so that $H^n(X_\bullet, \mathbb{F}_p) = H^{n+1}(X_\bullet, \mathbb{F}_p) = \mathbb{F}_p$. We let ι be a generator of the former, so that $\beta\iota$ is a generator of the latter for β the Bockstein. Again, this analysis is based on the cosimplicial commutative ring $\bar{\mathfrak{m}}^{X_\bullet}$.

- (i) Note that $H^n(X_\bullet, \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^p) \simeq H^n(X_\bullet, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^p \simeq \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^p \iota$ and similarly for $H^{n+1}(X_\bullet, \bar{\mathfrak{m}}^p)$. The differential in question comes from the connecting homomorphism in the short exact sequence (5.17), so that by naturality of (5.18), it carries $(1 - \zeta_p)\iota$ to $(1 - \zeta_p)^p \beta\iota \in H^{n+1}(X_\bullet, \bar{\mathfrak{m}}^p)$.
- (ii) The associated graded module of $\bar{\mathfrak{m}}$ comes from the reduction of a \mathbb{Z} -algebra, so $\beta\mathcal{P}^0$ is the composite of the Bockstein and \mathcal{P}^0 . Now $\mathcal{P}^0((1 - \zeta_p)\iota) = (1 - \zeta_p)^p \iota \in H^n(X, \bar{\mathfrak{m}}^p)$ since \mathcal{P}^0 is the Frobenius on underlying rings. Applying β to that, we obtain $(1 - \zeta_p)^p \beta\iota$.

Putting these together, it follows that if the class corresponding to $(1 - \zeta_p)\iota$ is to survive in the multiplicative spectral sequence (as it must), the formula must be as desired. \square

We refer to [Mum66, Lecture 27] for an instance of the appearance of these Bockstein operators in the deformation theory of the Picard scheme in characteristic p . In that case, there are no algebraic obstructions, but the multiplicative operations are controlled by the Bocksteins.

6. The algebraic Picard group for the C_p -action on E_n

We keep the previous notation and fix a prime number p , setting $n = p - 1$. Let $C_p \subset \mathbb{G}$ be a maximal finite p -subgroup. Recall that $(E_n)_0 \simeq \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$ is a complete local ring; we denote the maximal ideal by \mathfrak{m} . It acquires a C_p -action by restriction.

In this section, we apply the tools of the preceding section and calculate the algebraic Picard group $H^1(C_p, (E_n)_0^\times)$, showing that it is cyclic of order p . This will be a key input into the topological descent spectral sequence for Picard groups. The approach will be to use the \mathfrak{m} -adic filtration on $(E_n)_0^\times$.

6.1 The additive spectral sequence

The associated graded module of the \mathfrak{m} -adic filtration of $(E_n)_0$ has been determined in unpublished work of Hopkins and Miller and studied additionally in work of Hill [Hil06].

Since $(E_n)_0$ is a regular local ring, one has canonical and C_p -equivariant isomorphisms

$$\mathfrak{m}^i/\mathfrak{m}^{i+1} \simeq \text{Sym}^i(\mathfrak{m}/\mathfrak{m}^2), \quad i \geq 0,$$

⁵ We note that the precise multiple will not be used in the following, only that it is not zero.

where the symmetric powers are taken over the residue field \mathbb{F}_{p^n} (upon which the group C_p acts trivially). The first main result is the following, which is a consequence of the coordinates given on [Nav10, p. 497].

PROPOSITION 6.1 (Hopkins–Miller). *As a C_p -representation, $\mathfrak{m}/\mathfrak{m}^2$ is the reduced regular representation $\bar{\rho}$ over \mathbb{F}_{p^n} .*

Recall that the *reduced regular representation* $\bar{\rho}$ of C_p over a field k is the quotient of the regular representation $k[C_p]$ by the trivial subrepresentation. As a corollary, we find that the associated graded module of $(E_n)_0$, as a C_p -representation and algebra, is the symmetric algebra

$$\text{gr}(E_n)_0 \simeq \text{Sym}^* \bar{\rho}$$

of the reduced regular representation $\bar{\rho}$.

Over a field k of characteristic p , the decomposition of the symmetric powers of $\bar{\rho}$ as C_p -representations up to free summands is calculated in work of Almkvist and Fossum (and was done by Hopkins and Miller). We will let $\mathbf{1}$ denote the trivial representation.

PROPOSITION 6.2 (Almkvist–Fossum [AF78, ch. 3, Propositions 3.4–3.6]). *The C_p -representation $\text{Sym}^i \bar{\rho}$ decomposes as follows:*

- (i) *if $i \equiv 0 \pmod p$, then $\text{Sym}^i \bar{\rho} \simeq \mathbf{1} \oplus \text{free}$;*
- (ii) *if $i \equiv 1 \pmod p$, then $\text{Sym}^i \bar{\rho} \simeq \bar{\rho} \oplus \text{free}$;*
- (iii) *if $i \not\equiv 0, 1 \pmod p$, then $\text{Sym}^i \bar{\rho}$ is a free $k[C_p]$ -module.*

The invariant element of $\text{Sym}^p \bar{\rho}$ that generates the $\mathbf{1}$ summand is given as follows. The representation $\bar{\rho}$ is generated by elements $\epsilon_1, \dots, \epsilon_p$ with $\epsilon_1 + \dots + \epsilon_p = 0$ and with C_p acting on the ϵ_i via the natural permutation action. Then the (norm) product $\epsilon_1 \dots \epsilon_p \in \text{Sym}^p \bar{\rho}$ is C_p -invariant and generates the desired summand. The analogous invariant elements in $\text{Sym}^{pk} \bar{\rho}$ are the powers of this norm product.

We now describe the cohomology of the associated graded module of $(E_n)_0$. Recall that

$$H^*(C_p, \mathbb{F}_{p^n}) \simeq \mathbb{F}_{p^n}[a, b]/a^2, \quad |a| = 1, |b| = 2,$$

is the tensor product on an exterior algebra on a class in degree one and a polynomial algebra on a class in degree two, both over \mathbb{F}_{p^n} . Recall also that the cohomology of $\bar{\rho}$ as a graded module is given by

$$H^*(C_p, \bar{\rho}) \simeq H^*(C_p, \mathbb{F}_{p^n})\{u_0, u_1\}/(au_0 = 0, bu_0 = au_1), \quad |u_0| = 0, |u_1| = 1.$$

As a result, we can completely determine the cohomology modulo transfers (which only appear in H^0).

COROLLARY 6.3. *The cohomology $H^*(C_p, \text{gr}(E_n)_0)$ modulo transfers is given by*

$$H^*(C_p, \mathbb{F}_{p^n})[u, v, v']/(v^2 = v'^2 = 0, av = 0, bv = av').$$

The internal grading of u is p and that of v, v' is 1. Moreover, u, v are in cohomological degree zero and v' is in H^1 .

We display in Figure 4 the E_1 -page of the spectral sequence

$$H^i(C_p, \text{gr}^j((E_n)_0)) \implies H^i(C_p, (E_n)_0) \tag{6.1}$$

when $p = 3$.

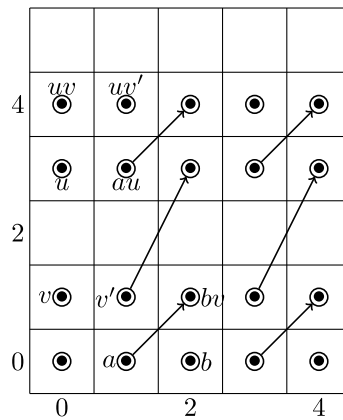


FIGURE 4. The spectral sequence for $H^*(C_p, (E_n)_0)$ for $p = 3$.

We note that none of the transfers support differentials in the spectral sequence thanks to the norm map. Moreover, this holds in the topological spectral sequence for the same reason.

It remains to determine the differentials. We do this by reverse-engineering the output of the Hopkins–Miller calculation.

PROPOSITION 6.4. *The spectral sequence (6.1) collapses at the p th page. The differentials are (up to non-zero scalars) determined by $d_1(a) \doteq bv$ and $d_{p-1}(v') \doteq bu$. The classes u, b, v are permanent cycles.*

Proof. We start by describing the permanent cycles. First, the class u in degree p is constructed as a multiplicative norm. This construction can be carried out in $(E_n)_0$, so u is necessarily a permanent cycle. The class b comes from $H^*(C_p, \mathbb{Z})$ and is also therefore a permanent cycle. The invariant class v in $H^0(C_p, \bar{\rho})$ is represented by the image of p in $\mathfrak{m}/\mathfrak{m}^2$. Since this is invariant in $(E_n)_0$, it is also a permanent cycle.

Next, we describe the first differential d_1 . We claim that, up to unit scalars, $d_1(a) = bv = av'$. One sees this by naturality with the filtration (by powers of the maximal ideal) on $\mathbb{W}(\mathbb{F}_{p^n})$. We have a C_p -equivariant, filtration-preserving map $\mathbb{W}(\mathbb{F}_{p^n}) \rightarrow (E_n)_0$, where the domain has trivial action. One easily computes the d_1 in the spectral sequence for $\mathbb{W}(\mathbb{F}_{p^n})$ (as a Bockstein), and uses the fact that the unique non-trivial map of C_p -representations $\mathbf{1} \rightarrow \bar{\rho}$ induces an isomorphism on H^2 . By naturality, this determines the d_1 in our spectral sequence.

Finally, we claim that $d_{p-1}(v') \doteq bu$ as desired. We know that this differential must happen, because otherwise $H^2(C_p, (E_n)_0)$ would have cardinality at least p^{2n} , whereas it has cardinality p^n by the Hopkins–Miller calculations (Proposition 2.6). \square

6.2 The multiplicative spectral sequence

We now describe the modifications to the above spectral sequence that occur for the multiplicative case, and determine the algebraic Picard group for the group C_p .

The following is our main result. As before, we let $\mathfrak{m} \subset (E_n)_0$ be the maximal ideal, with the induced C_p -action, and let ω denote $\pi_2 E_n$ as an (invertible) equivariant $(E_n)_0$ -module.

PROPOSITION 6.5. *We have $H^1(C_p, 1 + \mathfrak{m}) \simeq H^1(C_p, (E_n)_0^\times) \simeq \mathbb{Z}/p$, generated by the class of ω .*

Proof. We use the \mathfrak{m} -adic filtration on $1 + \mathfrak{m}$, which leads to a spectral sequence whose E_1 -page is identified with (6.1) with the bottom row removed, converging to $H^*(C_p, 1 + \mathfrak{m})$. We will denote the differentials by $d_{k,\times}$.

In the *algebraic* spectral sequence, we saw in Proposition 6.4 that everything that can contribute to the cohomological degree 1 abutment supports either a d_1 or a d_{p-1} . In filtration at least 2, we can import these differentials thanks to the truncated exponential and logarithm (Proposition 5.7). However, we *cannot* import the formula

$$d_{p-1}(v') \stackrel{\cdot}{=} bu,$$

into the multiplicative spectral sequence. Instead, we use the formula of Proposition 5.12, which gives the differential in the multiplicative spectral sequence. Suppose $d_{p-1}(v') = \xi bu, \beta P^0(v') = \xi'bu$ for some $\xi, \xi' \in \mathbb{F}_{p^n}$. For any $e \in \mathbb{F}_{p^n}$, one obtains:

$$\begin{aligned} d_{p-1,\times}(ev') &= d_{p-1}(ev') - \beta P^0(ev') \\ &= e\xi bu - e^p \beta P^0(v') \\ &= e\xi bu - e^p \xi' bu = (e\xi - e^p \xi')bu. \end{aligned}$$

Now we know that $\xi \neq 0$, so regardless of the value of ξ' , there will be *at most* p choices of e that make the above vanish. This gives an upper bound of \mathbb{Z}/p for the H^1 .

However, the Hopkins–Miller calculations (Proposition 2.6) also give a lower bound of \mathbb{Z}/p for the Picard group: in fact, it shows that $\omega = \pi_2 E_n$ has order p in the algebraic Picard group as $H^*(C_p, \omega^{\otimes j}) \neq H^*(C_p, (E_n)_0)$ for $j < p$. This completes the proof. \square

We emphasize that, in the above proof, the additive differentials were \mathbb{F}_{p^n} -linear, but the multiplicative differentials were *not*. This enables us to reduce a group that was *a priori* \mathbb{F}_{p^n} to a \mathbb{Z}/p . Also, one can directly show that $\xi' \neq 0$, although we do not need to do so here. The fact that the difference between the additive and multiplicative differentials is Frobenius-semilinear is all that was used, not any of the explicit formulas.

7. The topological comparison of differentials

7.1 Context

We begin by reviewing the context from [MS16, §§5 and 6]. Let R^\bullet be a cosimplicial \mathbf{E}_∞ -ring. In this case, there is a Bousfield–Kan spectral sequence (BKSS)

$$\pi^s \pi_t R^\bullet \implies \pi_{t-s} \text{Tot } R^\bullet. \tag{7.1}$$

Applying the functor \mathfrak{gl}_1 , one also obtains a BKSS

$$\pi^s \pi_t \mathfrak{gl}_1(R^\bullet) \implies \pi_{t-s} \text{Tot}(\mathfrak{gl}_1(R^\bullet)). \tag{7.2}$$

The latter spectral sequence is relevant to Picard group computations via descent theory, but the former spectral sequence (7.1) is usually better understood (e.g. because of the multiplicative structure).

For $t - s > 0$, these homotopy groups are determined by the BKSSs for the *spaces* $\Omega^\infty R^\bullet$ and $\Omega^\infty \mathfrak{gl}_1(R^\bullet)$, which have isomorphic connected components. As a result, the spectral sequences (7.1) and (7.2) can be compared in this range. However, it is the $t - s = -1$ component of (7.2) that is relevant to the Picard group calculations. Although at E_2 both spectral sequences (except for $s = 0, 1$) are the same in this column, the differentials vary. Nonetheless, in [MS16], the second

and third authors showed that there is a *stable range* in which differentials can be compared, even for $t - s = -1$, and the first differential beyond the stable range can be computed. In particular, one observes that in any range $[n, 2n - 1]$, there is a functorial equivalence of *spectra*

$$\tau_{[n, 2n-1]}R \simeq \tau_{[n, 2n-1]}\mathfrak{gl}_1(R), \tag{7.3}$$

for an \mathbf{E}_∞ -ring R . This gives the stable range, and the first differential beyond this range can be computed via reduction to a universal example.

In this section, we will prove that after arithmetic localization there is a larger range in which one can compare differentials in the two spectral sequences, analogous to the range that one obtains in algebra using the truncated logarithm (cf. § 5). It is possible to prove an analog of (7.3) in this context (and construct a ‘spectral’ truncated logarithm); this will be explored in forthcoming work by Clausen, Heuts, Mathew and Stojanoska.

Let R^\bullet be a cosimplicial \mathbf{E}_∞ -ring whose members are all $\mathbb{Z}[1/(p - 1)!]$ -local. Our main goal is to demonstrate the following.

THEOREM 7.1. *Suppose R^\bullet is a cosimplicial $\mathbb{Z}[1/(p - 1)!]$ -local \mathbf{E}_∞ -ring. Denote by $E_*^{s,t}$ and $E_{*,\times}^{s,t}$ the associated spectral sequences (7.1) and (7.2) above. Suppose given $t \geq 2$ and an element $x \in E_2^{t+1,t}$ such that x survives to the E_k -page. Then:*

- (i) *if $k \leq (p - 1)t$, then x survives to $E_{k,\times}$ in the \mathfrak{gl}_1 spectral sequence and $d_{k,\times}(x) = d_k(x)$;*
- (ii) *if t is even and $k = (p - 1)t + 1$, then $d_{k,\times}(x) = d_k(x) + \zeta\beta\mathcal{P}^{t/2}(x)$ for some $\zeta \in \mathbb{F}_p^\times$ (independent of R^\bullet).*

We will discuss the universal operation $\beta\mathcal{P}^{t/2}$ as well: it arises from the right derived functors of the symmetric algebra functor in the sense of Priddy [Pri73], or as a power operation for \mathbf{E}_∞ -ring spectra over $\mathbb{Z}[1/(p - 1)!]$. Again, the only fact we will need about it is its Frobenius-semilinearity property, and we will not try to determine the constant ζ . When $p = 2$, this recovers the comparison results of [MS16, §§ 5 and 6]. In particular, we shall assume throughout that $p > 2$. Unlike in [MS16], we will use the same notation for elements of the multiplicative and additive spectral sequences.

7.2 Proofs

Theorem 7.1 is proved using the same strategy as in [MS16, § 6], although we will not try to determine ζ . Namely, we consider the ‘universal’ example of a cosimplicial \mathbf{E}_∞ -ring R^\bullet (with $(p - 1)!$ inverted) together with a class in $E_k^{t+1,t}$ and prove the desired claims here, where it will follow by a sparsity argument. We need to look for the cosimplicial \mathbf{E}_∞ -ring (which is $\mathbb{Z}[1/(p - 1)!]$ -local) which corepresents the functor

$$X^\bullet \mapsto \Omega^\infty(\Sigma^{-1} \text{fib}(\text{Tot}_{t+k}(X^\bullet) \rightarrow \text{Tot}_t(X^\bullet))),$$

since a class in $E_k^{t+1,t}$ represents a class in $\pi_{-1} \text{Tot}_{t+k}(X^\bullet)$ trivialized in $\text{Tot}_t(X^\bullet)$.

In this subsection, everything will be $\mathbb{Z}[1/(p - 1)!]$ -localized.

Construction 7.2. To construct this universal cosimplicial \mathbf{E}_∞ -ring R^\bullet , we let Lan denote the functor of left Kan extension. We define the pointed cosimplicial space \mathcal{F}^\bullet via the homotopy pushout

$$\begin{array}{ccc} \text{Lan}_{\Delta^{\leq t} \rightarrow \Delta}(*)_+ & \longrightarrow & * \\ \downarrow & & \downarrow \\ \text{Lan}_{\Delta^{\leq t+k} \rightarrow \Delta}(*)_+ & \longrightarrow & \mathcal{F}^\bullet \end{array}$$

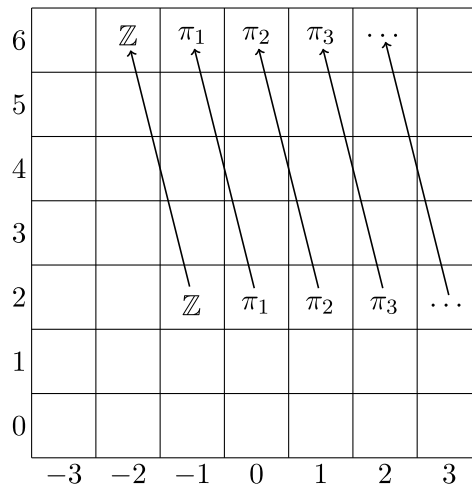


FIGURE 5. The BKSS for \mathcal{G}^\bullet for $t = 1, k = 4$.

The cosimplicial \mathbf{E}_∞ -ring R^\bullet is defined via

$$R^\bullet = \text{Free}(\Sigma^{-1}\Sigma^\infty\mathcal{F}^\bullet)[1/(p-1)!].$$

We write \mathcal{G}^\bullet for the cosimplicial spectrum $\Sigma^{-1}\Sigma^\infty\mathcal{F}^\bullet[1/(p-1)!]$, so that

$$R^\bullet = \text{Free}(\mathcal{G}^\bullet)[1/(p-1)!]. \tag{7.4}$$

Pointwise, the left Kan extensions have the homotopy types of wedges of spheres. As in [MS16, § 6.2], we find the following properties of the pointed cosimplicial space $\mathcal{F}^\bullet : \Delta \rightarrow \mathcal{S}_*$:

- (i) $\mathcal{F}(T)$ (for $T \in \Delta$) is a wedge of S^{t+1} and S^{t+k} ;
- (ii) $\mathcal{F}^\bullet|_{\Delta_{\leq t}}$ is contractible; for any $T \in \Delta^{\leq t+k}$, $\mathcal{F}(T)$ is a wedge of copies of S^{t+1} .

By desuspending, this of course determines the pointwise homotopy type of $\mathcal{G}(T)$ for any $T \in \Delta$.

The BKSS for the cosimplicial spectrum $\Sigma^\infty\mathcal{F}^\bullet$ is determined as in [MS16, § 6.4] (cf. [MS16, proof of Proposition 6.3.1]), and in particular follows from the calculation there of the BKSS for the cosimplicial spectrum $\Sigma_+^\infty\text{Lan}_{\Delta_{\leq t} \rightarrow \Delta}(*).$ The BKSS for \mathcal{G}^\bullet is a shift by one of that of $\Sigma^\infty\mathcal{F}^\bullet[1/(p-1)!]$, and we describe it next. For the reader’s convenience, we also include an example in Figure 5 below.

PROPOSITION 7.3. (i) As graded modules over $\pi_*(S^0)[1/(p-1)!]$, we have

$$H^s(\pi_*(\mathcal{G}^\bullet)) \simeq \begin{cases} 0 & s \neq t+1, t+k+1, \\ \pi_*(S^t)[1/(p-1)!] & s = t+1, \\ \pi_*(S^{t+k-1})[1/(p-1)!] & s = t+k+1. \end{cases}$$

- (ii) We let $\iota \in E_2^{t+1,t}$ be the fundamental generating class. There is a d_k in the BKSS, which carries ι to a generator in $E_2^{t+k+1,t+k-1} \simeq \pi_{t+k-1}(S^{t+k-1})[1/(p-1)!]$.

Now we consider the BKSS for R^\bullet in the desired range, which depends only on the Postnikov section $\tau_{\leq pt}R^\bullet$. We observe that there is an equivalence of cosimplicial spectra

$$\tau_{\leq pt}R^\bullet \simeq S^0[1/(p-1)!] \vee \mathcal{G}^\bullet \vee \mathcal{G}_{h\Sigma_2}^{\bullet \wedge 2} \vee \dots \vee \mathcal{G}_{h\Sigma_p}^{\bullet \wedge p}.$$

We claim that in the range of concern, none of the terms besides \mathcal{G}^\bullet and $\mathcal{G}_{h\Sigma_p}^{\bullet\wedge p}$ can contribute; this follows from the next lemma.

LEMMA 7.4. *For any $2 \leq u \leq p - 1$, the BKSS for $(\mathcal{G}^{\bullet\wedge u})_{h\Sigma_u}$ is zero at E_2 in cohomological degrees below $(u - 1)(t + 1) + (t + k + 1)$.*

Proof. Consider the cosimplicial $\pi_*(S^0)[1/(p - 1)!]$ -module $M^\bullet = \pi_*\mathcal{G}^\bullet$, which is levelwise free. The cosimplicial module $\pi_*(\mathcal{G}_{h\Sigma_u}^{\bullet\wedge u})$ is obtained as $(M^{\otimes u})_{\Sigma_u}$ where the tensor product is taken over $\pi_*(S^0)$, and where Σ_u acts by permuting the factors with appropriate signs. Since the order of Σ_u is invertible, the operation of taking coinvariants is well-behaved (and comes from the induced operation on cohomology); we obtain that

$$H^*(\pi_*(\mathcal{G}_{h\Sigma_u}^{\bullet\wedge u})) \simeq (H^*(M^\bullet)^{\otimes u})_{\Sigma_u}.$$

For $s \leq (u - 1)(t + 1) + (t + k + 1)$, the only contribution to $H^*(\pi_*(\mathcal{G}^{\bullet\wedge u}))$ is in cohomological degree $u(t + 1)$ and is a free module of rank one over $\pi_*(S^0)[1/(p - 1)!]$. However, the Σ_u action is by the sign representation which annihilates this upon passage to the coinvariants. \square

Finally, we need to determine the contribution of $\mathcal{G}_{h\Sigma_p}^{\bullet\wedge p}$ in case $k = (p - 1)t + 1$. Here we assume that t is even. In the range in question, this only contributes via its π_{pt} , given by

$$\pi_{pt}(\mathcal{G}_{h\Sigma_p}^{\bullet\wedge pt}) \simeq \text{Sym}^p \pi_t \mathcal{G}^\bullet.$$

Proof of Theorem 7.1. Following the outline of [MS16], it suffices to determine the behavior of the tautological class in the BKSS for $\mathfrak{gl}_1(R^\bullet)$. In the BKSS for R^\bullet , the tautological class supports a d_k as in Proposition 7.3 (by naturality for the map $\mathcal{G}^\bullet \rightarrow R^\bullet$). We observe that the tautological class in the *multiplicative* spectral sequence cannot support a differential $d_{i,\times}$ for $i < k$ as there is no suitable target in the spectral sequence. Indeed, for connectivity reasons, there are only the terms $\mathcal{G}_{h\Sigma_i}^{\bullet\wedge i}$ for $i < p$ that can contribute here, and these do nothing thanks to Lemma 7.4.

If $k \leq (p - 1)t$, we find that the target for a $d_{k,\times}$ in the \mathfrak{gl}_1 spectral sequence is simply a $\mathbb{Z}[1/(p - 1)!]$; there are no additional contributions. As in [MS16], we conclude (by comparing with square-zero extensions) that we can identify the differentials d_k and $d_{k,\times}$.

Consider, finally, the case $k = (p - 1)t + 1$ and t even. In this case, the target of the differential $d_{k,\times}$ still contains $\mathbb{Z}[1/(p - 1)!]d_{k,t}$, but also has another factor as $H^{pt+2}(\pi_{pt}R^\bullet)$, which we need to determine.

Here we will need to use some of the theory of *right derived functors* of symmetric powers [Pri73]. They determine the cohomology of the symmetric powers of a cosimplicial object. The next proposition follows from [Pri73, §§ 3 and 4], and in particular [Pri73, Theorem 4.0.1], by running a Bockstein spectral sequence. We note also that the work of Kraines [Kra66] relates these operations to p -fold Massey powers.

PROPOSITION 7.5. *Let X^\bullet be a cosimplicial $\mathbb{Z}[1/(p - 1)!]$ -module which is levelwise free. Suppose $H^*(X^\bullet)$ is concentrated in degree k , k odd, and $H^k(X^\bullet) \simeq \mathbb{Z}[1/(p - 1)!]$ generated by a tautological class ι . Then for each $i \leq (k - 1)/2$, we have $H^{k+2i(p-1)+1}(\text{Sym}^p X^\bullet) \simeq \mathbb{Z}/p$ generated by a class $\beta\mathcal{P}^i\iota$.*

Now we take $k = t + 1$ in the above. It follows that $H^{pt+2}(\pi_{pt}(\mathcal{G}_{h\Sigma_p}^{\bullet\wedge pt})) \simeq \mathbb{Z}/p$, generated by $\beta\mathcal{P}^{t/2}\iota$. It follows that we must have a ‘universal’ formula

$$d_{k,\times}(\iota) = d_k(\iota) + \zeta\beta\mathcal{P}^{t/2}\iota,$$

where $\zeta \in \mathbb{F}_p$. We note that the coefficient of $d_k(\iota)$ is argued to be one similarly as in [MS16].

To complete the argument, we need to argue that $\zeta \neq 0$. To do this, we consider (for any even t) an \mathbf{E}_∞ -ring A under \mathbb{F}_p such that $\pi_*(A) \simeq \mathbb{F}_p[u]/(u^{p+1})$ for $|u| = t$. We can construct A by taking the free \mathbf{E}_∞ -ring over \mathbb{F}_p on a class in degree t and truncating at pt , i.e.

$$A = \tau_{\leq pt} \text{Free}_{\mathbb{F}_p}(\Sigma^t \mathbb{F}_p),$$

where $\text{Free}_{\mathbb{F}_p}$ denotes the free \mathbf{E}_∞ -algebra functor over \mathbb{F}_p . In Lemma 7.6 below, we will show that the natural map

$$K(\mathbb{F}_p, t) \rightarrow \text{GL}_1(A), \tag{7.5}$$

inducing an isomorphism on π_t , does not deloop. Of course, the analogous map $K(\mathbb{F}_p, t) \rightarrow \Omega^\infty A$ does deloop. Equivalently, the associated (Atiyah–Hirzebruch) spectral sequence (AHSS) for $\mathfrak{gl}_1(A)^{K(\mathbb{F}_p, t+1)}$ admits a differential on the tautological class in $H^{t+1}(K(\mathbb{F}_p, t+1), \pi_t A)$. By interpreting the AHSS as a BKSS, we find that there must be a non-zero $d_{k,\times}$ and therefore $\zeta \neq 0$. \square

LEMMA 7.6. *The map (7.5) does not deloop to a map $K(\mathbb{F}_p, t+1) \rightarrow \text{BGL}_1(A)$.*

Proof. To see that (7.5) is not a loop map, we use the universal properties of GL_1 (cf. [ABGHR14, §3]). If (7.5) were a loop map, we would have a morphism of associative ring spectra over \mathbb{F}_p

$$\mathbb{F}_p \wedge \Sigma_+^\infty K(\mathbb{F}_p, t) \rightarrow A,$$

inducing an isomorphism on π_t . However, the class in $\pi_t A$ has non-zero p th power, while the p th power of the tautological class in $\mathbb{F}_p \wedge \Sigma_+^\infty K(\mathbb{F}_p, t)$ vanishes by [CLM76, Lemma 6.1].⁶ \square

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