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Uniqueness for a Competing Species Model

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Abstract. We show that a martingale problem associated with a competing species model has a unique solution. The proof of uniqueness of the solution for the martingale problem is based on duality technique. It requires the construction of dual probability measures.

1 Introduction

Measure-valued branching Markov processes (or superprocesses) arise as limits of branching particle systems undergoing random migration and critical (or asymptotically critical) branching.

Recently, there has been interest in the study of populations with interactions. The uniqueness of a solution for the martingale problems for these models has always been an important and often difficult question. For some models, the Dawson-Girsanov theorem [6] in its various versions helped to solve the problem of uniqueness (see *e.g.* Theorem 3.10 [12]). Many cases of interactions were treated in [19] with the help of a historical calculus.

The competing species model is a model with the most natural kind of interaction—"point interaction", in which an interaction only occurs if particles collide. This process was introduced in [12] as a solution for the martingale problem M^λ which will be formulated in Section 2. The existence of the competing species model in dimensions $d = 1, 2, 3$ and non-existence in $d > 3$ was proved in [12]. The uniqueness for M^λ was derived via Dawson-Girsanov theorem for dimension $d = 1$. The question of uniqueness was open for dimensions $d = 2, 3$. Moreover, it was proved that for $d = 3$ solutions to M^λ are singular (in law) w.r.t. the pair of independent super-Brownian motions, which indicates that it is impossible to use the usual Dawson-Girsanov arguments. However, Evans and Perkins [13] recently proved the uniqueness for the historical martingale problem associated with the competing species model.

In this paper we prove uniqueness for the "non-historical" martingale problem M^λ in $d = 1, 2, 3$ using duality arguments already used in [17], [18]. If it is not stated otherwise, we will always assume that $d \leq 3$.

Notation and Organization of the Paper If E is a completely regular topological space, $\mathcal{B}(E)$ denotes its Borel σ-algebra together with Borel measurable functions on E. Let $M_1(E)$ (resp. $M_{1,c}(E)$) denote the space of probability measures on $\big(E, \mathcal{B}(E)\big)$ (resp. with compact

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support) equipped with the topology of weak convergence. Let $\Rightarrow$ stand for weak convergence.

Let $C_E[0,\infty)$ (respectively, $D_E[0,\infty)$) denote the space of continuous (respectively, cadlag) E-valued paths with the compact-open (respectively, Skhorohod) topology. (For the Skhorohod topology in $D_E[0,\infty)$ with E completely regular, see [15].) Let $B(E)$ (respectively, $C(E)$, $\overline{C}(E)$, $C_c(E)$) denote the set of bounded (respectively, continuous, bounded continuous, continuous with compact support) complex-valued functions on E. $\|\cdot\|_\infty = \|\cdot\|_{\infty,E}$ will be supremum norm on $B(E)$, $\overline{C}(E)$, $C_c(E)$. We will often suppress the subscript E if there is no ambiguity over what space functions are used. Set $B_{\mathrm{R}}(E)$ (respectively, $C_{\mathrm{R}}(E)$, $\overline{C}_{\mathrm{R}}(E)$, $C_{c,\mathrm{R}}(E)$) to be the subset of real-valued valued functions in $B(E)$ (respectively, $C(E)$, $\overline{C}(E)$, $C_c(E)$). In general, if F is a set of complex-valued (resp. real-valued) functions write F_+ for the subset of functions with non-negative real parts, that is, $\{f = f_1 + if_2 \in F : \inf_{x\in E} f_1(x) \geq 0\}$ (resp. $\{f \in F : \inf_{x\in E} f(x) \geq 0\}$).

We will abbreviate "boundedly pointwise" by bp.

The paper is organized as follows. The precise formulation of the competing species model martingale problem M^λ along with our main uniqueness result is given in Section 2. The important properties of solutions to M^λ are described in Section 3. Section 4 is devoted to a duality technique and provides a motivation for our construction of dual probability measures. In Section 5 we introduce solutions for some evolution equations and in Section 6 a certain measure and distribution valued process is defined; these are the two main components for our construction of an approximating sequence of dual measures described in Section 7. Here we also establish the existence of a system of dual probability measures as a limit point of the approximating dual measures in an appropriate space. We prove that these dual measures satisfy a certain equation; the latter plays a key role in the proof of our main uniqueness result in Section 8. The Appendix is devoted to the existence, uniqueness and the properties of the equations introduced in Section 5.

2 Competing Species Model and the Main Result

Let $M_{F,w}$ be the space of finite measures on R^d with the weak topology. For $\mu \in M_{F,w}$ and $f \in B(\mathrm{R}^d)$ let $\mu(f) = \langle \mu, f\rangle = \langle f, \mu\rangle \equiv \int f\, d\mu$. Let $p_t(x)$ be the standard Brownian density and $\{S_t\}$ be the semigroup with generator $\frac{1}{2}\Delta$ and

$$\mathcal{D}\left(\frac{1}{2}\Delta\right) \equiv \left\{\phi \in \overline{C}(\mathrm{R}^d) : \frac{1}{2}\Delta\phi \in \overline{C}(\mathrm{R}^d)\right\}$$

be domain of $\frac{1}{2}\Delta$. Let $(\Omega, \mathcal{F}, P)$ be a probability space which is sufficiently rich to contain all the processes defined below, and for any process X defined on $(\Omega, \mathcal{F}, P)$ let $\mathcal{F}_t^X = \bigcap_{\epsilon>0} \sigma(X_s, s \leq t+\epsilon)$.

We will use the following definition of the collision local time and the collision measure for two continuous $M_{F,w}$-valued processes.

Definition 2.1 Let X^1 and X^2 be continuous $M_{F,w}$-valued stochastic processes defined on

$(\Omega, \mathcal{F}, P)$. If $\epsilon > 0$ and $\phi \in B(\mathrm{R}^d)$ let

$$L_t(S_\epsilon X^1, S_\epsilon X^2)(\phi) \equiv \int_0^t \int_{\mathrm{R}^d} (S_\epsilon X_u^1)(x)(S_\epsilon X_u^2)(x)\phi(x)\,dx\,du$$
$$= \int_0^t \int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_\epsilon(x_1 - x)p_\epsilon(x_2 - x)\phi(x)\,X_u^1(dx_1)\,X_u^2(dx_2)\,dx\,du.$$

The collision local time of (X^1, X^2) is a continuous non-decreasing $M_{F,w}$-valued process $L_t(X^1, X^2)$ such that $L_t(S_\epsilon X^1, S_\epsilon X^2)(\phi) \xrightarrow{\mathrm{P}} L_t(X^1, X^2)(\phi)$ as $\epsilon \downarrow 0$ for each $\phi \in \overline{C}(\mathrm{R}^d)$ and $t > 0$.

Remark 2.2 In [5] and [12] other approximating sequences are used. It can be checked directly that all the definitions are equivalent for the processes considered in this paper.

Definition 2.3 Let X^1 and X^2 be continuous $M_{F,w}$-valued stochastic processes with a collision local time $L_t(X^1, X^2)$. Suppose there exists a progressively measurable measure-valued process $K_t(X^1, X^2)$ such that

$$L_t(X^1, X^2)(\phi) = \int_0^t K_s(X^1, X^2)(\phi)\,ds \tag{2.1}$$

for all $\phi \in B(\mathrm{R}^d)$ and $t \geq 0$ a.s. Then K_t is called the *collision measure* of X^1 and X^2.

The martingale problem M^λ for the competing species model is stated as follows. Let $\lambda > 0$. We say that an $M_{F,w} \times M_{F,w}$-valued process $X = (X^1, X^2)$ solves M^λ if

$$M^\lambda \begin{cases} \text{For all } \phi_1, \phi_2 \in \mathcal{D}\left(\frac{1}{2}\Delta\right), \\ X_t^1(\phi_1) \equiv X_0^1(\phi_1) + \int_0^t X_s^1\left(\frac{1}{2}\Delta\phi_1\right)\,ds + M_t^1(\phi_1) - \lambda L_t(X^1, X^2)(\phi_1), \\ X_t^2(\phi_2) \equiv X_0^2(\phi_1) + \int_0^t X_s^2\left(\frac{1}{2}\Delta\phi_2\right)\,ds + M_t^2(\phi_2) - \lambda L_t(X^1, X^2)(\phi_2), \\ \text{where } M^j(\phi_j) \text{ are continuous martingales such that} \\ \langle M^1(\phi_1)\rangle_t = \int_0^t X_s^1(\phi_1^2)\,ds, \quad \langle M^2(\phi_2)\rangle_t = \int_0^t X_s^2(\phi_2^2)\,ds, \\ \langle M^2(\phi_2), M^1(\phi_1)\rangle_t = 0. \end{cases}$$

Remark 2.4 All the processes (in particular martingales) are supposed to be complex-valued if it is not stated otherwise. It is a simple exercise to check that the above martingale problem is equivalent to the martingale problem introduced in [12] with real-valued test functions.

Remark 2.5 Since $\mathcal{D}(\frac{1}{2}\Delta)$ is bp-dense in $B(\mathrm{R}^d)$, a standard construction allows us to extend M_t^j $(j = 1, 2)$ to an orthogonal martingale measure $\{M_t^j(\phi) : t \geq 0, \psi \in B(\mathrm{R}^d)\}$. That is, for each $\psi \in B(\mathrm{R}^d)$, $M_t^j(\phi)$ is a continuous square integrable martingale such that

$$\langle M^j(\phi)\rangle_t = \int_0^t \langle X_s^j, \phi^2\rangle\,ds. \tag{2.2}$$

Let us denote by $L^2_r(X^j, P)$ the following set of functions:

$$\left\{\phi\colon \Omega \times \mathrm{R}_+ \times \mathrm{R}^d \mapsto \mathrm{R} \text{ which is predictable (see p. 292 of [20]) and} \right.$$
$$\left. P\left[\int_r^t \int_{\mathrm{R}^d} |\phi(\cdot, s, y)|^2 X^j_s(y)\, dy\, ds\right] < \infty, \quad \forall t \geq r\right\}. \tag{2.3}$$

$L^2(X^j, P)$ stands for $L^2_0(X^j, P)$. Proceeding as in [20], for each $\phi \in L^2(X^j, P)$ one can define the stochastic integral

$$M^j_t(\phi) = \int_0^t \int_{\mathrm{R}^d} \phi(s, y)\, dM^j(s, y), \tag{2.4}$$

where $M^j_t(\phi)$ is a continuous square integrable martingale with quadratic variation

$$\int_0^t \int_{\mathrm{R}^d} \phi(s, y)^2\, X^j_s(dy)\, ds.$$

$L^2_r(X^j, P)$ $(j = 1, 2)$ is a complete space (see *e.g.* Exercise 2.5 in [20]).

Our concern is with the proof of the uniqueness of the solution for the martingale problem M^λ. Let us define the function $\tilde{f}\colon M_{F,w} \times M_{F,w} \mapsto [0, \infty]$ by

$$\tilde{f}(\mu_1, \mu_2) = \mu_1(1)^2 \mu_2(1)^2 + \mu_1(1)\mu_2(1)\big(\mu_1(1) + \mu_2(1)\big)$$
$$+ \int_{0+}^1 \int_{\mathrm{R}^d \times \mathrm{R}^d} p_s(x - y)\, \mu_1(dx)\, \mu_2(dy)\, ds, \tag{2.5}$$

and

$$M^*_1(M_{F,w} \times M_{F,w}) \equiv \left\{\nu \in M_1(M_{F,w} \times M_{F,w}) : \int \tilde{f}(\mu_1, \mu_2)\, \nu(d\mu_1, d\mu_2) < \infty\right\}.$$

Now we are ready to present our main result.

Theorem 2.6 *Let $d \leq 3$ and assume that $\nu \equiv P(X^1_0, X^2_0)^{-1} \in M^*_1(M_{F,w} \times M_{F,w})$. Then any two solutions for the martingale problem M^λ with $M_{F,w} \times M_{F,w}$-valued paths have the same finite dimensional distributions, which means that the law of any solution to M^λ is unique (on $C_{M_{F,w} \times M_{F,w}}\big([0, \infty)\big)$).*

3 Properties of Solutions to M^λ

In this section we assume that $X = (X^1, X^2)$ is any solution of the martingale problem for M^λ and $P(X^1_0, X^2_0)^{-1} \in M^*_1(M_{F,w} \times M_{F,w})$.

Let us derive some simple properties of (X^1, X^2). As it follows from Theorem 5.1 of [5], we may assume the existence of dominating independent super-Brownian motions (Y^1, Y^2) starting at (X^1_0, X^2_0) $(X^1 \leq Y^1, X^2 \leq Y^2)$ enlarging the probability space if necessary. This assumption and the notation Y^1, Y^2 for the dominating pair will be valid throughout the whole paper.

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Lemma 3.1 *Let X be any solution of the martingale problem for M^λ, $P(X_0^1, X_0^2)^{-1} \in M_1^*(M_{F,w} \times M_{F,w})$, and $\tilde{f}$ is as in (2.5). Then $P(X_t^1, X_t^2)^{-1} \in M_1^*(M_{F,w} \times M_{F,w})$ for each $t \geq 0$.*

Proof Immediately follows from the definition of $M_1^*(M_{F,w} \times M_{F,w})$ and the fact that $P[\tilde{f}(X_t^1, X_t^2)] < \infty$ for all $t > 0$. (The domination of (X^1, X^2) by the pair of independent superprocesses (Y^1, Y^2) reduces the proof to the calculation of the moments of superprocesses; these calculations are standard (see *e.g.* [8]).) ■

Remark 3.2 In fact, one can prove stronger result:

$$P[\sup_{t \leq T} \tilde{f}(X_t^1, X_t^2)] < \infty, \quad \forall T > 0,$$

which in turn will allow to prove the strong Markov property of the (unique) solution. However strong Markov property follows easily from uniqueness of the solution for M^λ given by Theorem 2.6 and Theorem 8.2 in [13]. Therefore we decided to not to include the self-contained proof of the strong Markov property into the paper, while mentioning that it is possible.

Lemma 3.3 *There exists a version $\tilde{K}_s(X^1, X^2)$ of $K_s(X^1, X^2)$ such that $\tilde{K}_s(X^1, X^2) \leq K_s(Y^1, Y^2)$ for each $s > 0$.*

Proof For all $0 < s < t$, $\phi \in \overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+$,

$$\begin{aligned} L_{s,t}(X^1, X^2)(\phi) &\equiv \lim_{\epsilon \downarrow 0} \int_s^t \int_{\mathrm{R}^d} (S_\epsilon X_u^1)(x)(S_\epsilon X_u^2)(x)\phi(x)\,dx\,du \\ &\leq \lim_{\epsilon \downarrow 0} \int_s^t \int_{\mathrm{R}^d} (S_\epsilon Y_u^1)(x)(S_\epsilon Y_u^2)(x)\phi(x)\,dx\,du \\ &\equiv L_{s,t}(Y^1, Y^2)(\phi). \end{aligned} \tag{3.1}$$

On the other hand (see [5, Remark 5.12(4)]) it is known that

$$L_{s,t}(Y^1, Y^2)(\phi) = \int_s^t K_u(Y^1, Y^2)(\phi)\,du, \tag{3.2}$$

$$L_{s,t}(X^1, X^2)(\phi) = \int_s^t K_u(X^1, X^2)(\phi)\,du. \tag{3.3}$$

The relations (3.1), (3.2), (3.3) hold for all $s < t$, $\phi \in \overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+$, therefore there exists a version $\tilde{K}_s(X^1, X^2)$ of measure valued process $K_s(X^1, X^2)$ such that $\tilde{K}_s(X^1, X^2) \leq K_s(Y^1, Y^2)$ for each s. ■

In the remainder of this work, we denote by $K_s(X^1, X^2)$ any version whose existence was proved in the previous lemma.

Lemma 3.4 *For all* $\psi, \phi, \phi_1, \phi_2 \in \overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+$,

$$P[K_s(X^1, X^2)(\psi)] \le P\Big[\int_{\mathrm{R}^d} S_s(X_0^1)(x) S_s(X_0^2)(x)\psi(x)\,dx\Big], \tag{3.4}$$

(3.5)

$$\begin{aligned} &P[X_s^j(\phi)K_s(X^1, X^2)(\psi)] \\ &\quad\le P\Big[\int_{\mathrm{R}^d} X_0^j(S_s\phi) S_s(X_0^1)(x) S_s(X_0^2)(x)\psi(x)\,dx \\ &\qquad + \int_{\mathrm{R}^d}\int_0^s\int_{\mathrm{R}^d} S_v(X_0^j)(y) S_{s-v}(\phi)(y) p_{s-v}(x-y)\,dy\,dv\, S_s(X_0^k)(x)\psi(x)\,dx\Big], \\ &\qquad j = 1, 2, \end{aligned}$$

(3.6)

$$\begin{aligned} &P[X_s^1(\phi_1)X_s^2(\phi_2)K_s(X^1, X^2)(\psi)] \\ &\quad\le P\Big[\int_{\mathrm{R}^d}\Big(X_0^1(S_s\phi_1)S_sX_0^1(x) \\ &\qquad\qquad + \int_0^s\int_{\mathrm{R}^d} S_{v_1}(X_0^1)(y_1) S_{s-v_1}(\phi_1)(y_1) p_{s-v_1}(x-y_1)\,dy_1\,dv_1\Big) \\ &\qquad \times\Big(X_0^2(S_s\phi_2)S_sX_0^2(x) \\ &\qquad\qquad + \int_0^s\int_{\mathrm{R}^d} S_{v_2}(X_0^2)(y_2) S_{s-v_2}(\phi_2)(y_2) p_{s-v_2}(x-y_2)\,dy_2\,dv_2\Big)\psi(x)\,dx\Big]. \end{aligned}$$

Proof We outline a proof of (3.6); the inequalities (3.4), (3.5) are established similarly.
Lemma 3.3, the representation

$$K_s(Y^1, Y^2)(\psi) = \lim_{\epsilon\downarrow 0}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_\epsilon(x-y)\psi\big((x+y)/2\big)\,Y_s^1(dx)\,Y_s^2(dy)$$

and the Fatou lemma imply that

$$\begin{aligned} &P[X_s^1(\phi_1)X_s^2(\phi_2)K_s(X^1, X^2)(\psi)] \\ &\quad\le \liminf_{\epsilon\downarrow 0} P\Big[Y_s^1(\phi_1)Y_s^2(\phi_2)\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_\epsilon(x-y)\psi\big((x+y)/2\big)\,Y_s^1(dx)\,Y_s^2(dy)\Big]. \end{aligned}$$

The remainder of the proof rests on a standard calculation of the moments of the pair of superprocesses (Y^1, Y^2). ■

Lemma 3.5 *For each* $0 < \beta < T$,

$$\limsup_{\epsilon\downarrow 0}\ \sup_{\beta\le s\le T} E[K_s(S_\epsilon Y^1, S_\epsilon Y^2)(1)^2] < \infty.$$

Proof The proof involves a calculation of the second moments of superprocesses. ■

The previous lemma yields the following corollary.

Corollary 3.6

$$\limsup_{\epsilon\downarrow 0}\sup_{\beta\le s\le T} E[K_s(S_\epsilon X^1, S_\epsilon X^2)(1)^2] < \infty, \tag{3.7}$$

$$\sup_{\beta\le s\le T} E[K_s(Y^1, Y^2)(1)^2] < \infty, \tag{3.8}$$

$$\sup_{\beta\le s\le T} E[K_s(X^1, X^2)(1)^2] < \infty. \tag{3.9}$$

Proof The estimate (3.7) is a consequence of Lemma 3.5 and the domination property: $X^1 \le Y^1, X^2 \le Y^2$.

Further, in Section 4 of [11] it is proved that for every $s > 0$

$$K_s(Y^1, Y^2)(1) = \lim_{\epsilon\downarrow 0}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_{2\epsilon}(x-y)\, Y^1_s(dx)\, Y^2_s(dy) = \lim_{\epsilon\downarrow 0} K_s(S_\epsilon Y^1, S_\epsilon Y^2)(1), \text{ a.s.}$$

Now (3.8) is a consequence of Lemma 3.5 and the Fatou lemma.

Further, (3.9) follows from (3.8) and Lemma 3.3. ■

4 Duality Tools

Our goal is to prove that any two solutions to M^λ have the same finite-dimensional distributions. It is known from Theorem 4.4.2 of [10] that it suffices to verify uniqueness of the one-dimensional distributions. But attempts to use Theorem 4.4.2 of [10] directly meet some technical difficulties (see discussion after Lemma 2.1 in [18]). The following lemma is just a reformulation of Theorem 4.4.2 for our case:

Lemma 4.1 *Suppose that for each initial distribution $\nu = P(X^1_0, X^2_0)^{-1} \in M^*_1(M_{F,w} \times M_{F,w})$ any two solutions (X^1, X^2), $(\hat{X}^1, \hat{X}^2)$ of the martingale problem M^λ have the same one-dimensional distributions, that is, for each $t > 0$,*

$$P\{(X^1_t, X^2_t) \in \Gamma\} = P\{(\hat{X}^1_t, \hat{X}^2_t) \in \Gamma\}, \quad \Gamma \in \mathcal{B}\big((M_{F,w} \times M_{F,w})\big). \tag{4.1}$$

Then any two solutions of the martingale problem M^λ have the same finite-dimensional distributions. (That is, uniqueness holds.)

Proof The proof is completely analogous to the proof of Theorem 4.4.2 of [10]. However at some stage of the proof we need the following fact: if $P(X^1_0, X^2_0)^{-1} \in M^*_1(M_{F,w} \times M_{F,w})$ then $P(X^1_t, X^2_t)^{-1}$ is also in $M^*_1(M_{F,w} \times M_{F,w})$ for each $t > 0$. But Lemma 3.1 assures us that this is indeed the case. ■

Let us introduce the following notation. Denote by $S(R^d)$ the (Schwartz) space of rapidly decreasing real-valued functions on R^d and by S' the topological dual of $S(R^d)$, the space of tempered distributions. We endow S' with the strong topology. Let M_F be the space of finite measures on R^d. We consider M_F as a subspace of S' with the relative topology. Note that M_F differs from $M_{F,w}$ by the topology induced on it. Let $\tilde{S}(\mathrm{R}^d)$ (resp. $\tilde{S'}$) be the space of complex valued rapidly decreasing functions on R^d with positive real part (resp. the space of complex valued tempered distributions with measure-valued real part), that is

$$\tilde{S}(\mathrm{R}^d) \equiv \{f : f = f_1 + if_2,\, f_1 \in S(R^d)_+,\, f_2 \in S(R^d)\},$$
$$\tilde{S'} \equiv \{\mu : \mu = \mu_1 + i\mu_2, \mu_1 \in M_F, \mu_2 \in S'\}.$$

Now we can define a class of functions on $M_{F,w} \times M_{F,w}$ that separates the measures in $M_1^*(M_{F,w} \times M_{F,w})$. Let

$$L \equiv \text{ linear span } \{F \in \overline{C}(M_{F,w} \times M_{F,w}) : F_\phi(\mu_1, \mu_2) \equiv \exp\{-\langle \mu_1, \phi\rangle + i\langle \mu_2, \overline{\phi}\rangle\}, \phi \in \tilde{S}(\mathrm{R}^d)\}.$$

Lemma 4.2 *The set of functions L is separating on $M_1(M_{F,w} \times M_{F,w})$.*

Proof Let $\tilde{\mu}_1 = \mu_1 + i\mu_2$, $\tilde{\mu}_2 = \mu_1 - i\mu_2$. Since the transformation $(\mu_1, \mu_2) \mapsto (\tilde{\mu}_1, \tilde{\mu}_2)$ is one-to-one, it suffices to show that the set of functions

$$\tilde{L} \equiv \text{ linear span } \{\tilde{F}_{\phi_1,\phi_2}(\tilde{\mu}_1, \tilde{\mu}_2) \equiv \exp\{-\langle \tilde{\mu}_1, \phi_1\rangle + i\langle \tilde{\mu}_2, \phi_2\rangle\}, \phi_1 \in S(R^d)_+,\ \phi_2 \in S(R^d)\}$$

is separating on $M_1(M_{F,w} \times S')$. But this follows from Corollary 1.9 of [2]. ■

If $\{\tilde{P}_t, t \geq 0\}$ is a set of probability measures in $M_1(M_F \times S')$, then for any function $f \in \mathcal{B}(M_F \times S')$ we denote $\tilde{P}_t[f(\tilde{X}_t, \tilde{Y}_t)] \equiv \int_{M_F \times S'} f(\mu_1, \mu_2)\tilde{P}_t\,(d\mu_1, d\mu_2)$.

Lemma 4.3 *Let $f_\epsilon \in \mathcal{B}(M_F \times S')$ for each $\epsilon > 0$. Suppose that for each initial distribution $\nu = P(X_0^1, X_0^2)^{-1} \in M_1^*(M_{F,w} \times M_{F,w})$ and each $\phi = \phi_1 + i\phi_2 \in \tilde{S}(\mathrm{R}^d)$, there exists a set of probability measures $\{\tilde{P}_t, t \geq 0\}$ in $M_1(M_F \times S')$ such that $\tilde{P}_0 = \delta_{(\phi_1,\phi_2)}$ and*

$$P[e^{-\langle X_t^1,\phi\rangle - \langle X_t^2,\overline{\phi}\rangle}] = \lim_{\epsilon \downarrow 0} P \times \tilde{P}_t[e^{-\langle X_0^1, f_\epsilon(\tilde{X}_t + i\tilde{Y}_t)\rangle - \langle X_0^2, f_\epsilon(\tilde{X}_t - i\tilde{Y}_t)\rangle}] \tag{4.2}$$

for each $t \geq 0$ and each solution (X^1, X^2) to M^λ. Then for each initial distribution $\nu = P(X_0^1, X_0^2)^{-1} \in M_1^*(M_{F,w} \times M_{F,w})$ uniqueness holds for the martingale problem M^λ.

Proof With Lemma 4.1 and Lemma 4.2 in mind the proof is completely analogous to that of Theorem 4.4.7 of [10] (see also Theorem 1.3 in [17]). ■

To prove our main theorem we construct such a set of probability measures $\{\tilde{P}_t, t \geq 0\}$; henceforth this set will be called the set of *dual probability measures*. We use tightness arguments. We are motivated by the following considerations. Let us rewrite the martingale problem M^λ in an exponential form. Itô's formula implies that, for all $\phi = \phi_1 + i\phi_2 \in \tilde{S}(\mathrm{R}^d)$,

$$
\begin{aligned}
&e^{-X^1_t(\phi)-X^2_t(\overline{\phi})} - \int_0^t e^{-X^1_s(\phi)-X^2_s(\overline{\phi})}\Big(-X^1_s\left(\frac{1}{2}\Delta\phi\right) - X^2_s\left(\frac{1}{2}\Delta\overline{\phi}\right)\\
&\qquad\qquad + \frac{1}{2}\langle X^1_s + X^2_s, \phi_1^2 - \phi_2^2\rangle + i\langle X^1_s - X^2_s, \phi_1\phi_2\rangle\\
&\qquad\qquad + 2\lambda K_s(X^1, X^2)(\phi_1)\Big)\, ds
\end{aligned}
\tag{4.3}
$$

is a martingale.

Suppose that there exists a pair of $M_F \times S'$-valued processes $(\tilde{X}, \tilde{Y})$ which is defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and satisfies the following martingale problem:

$$
\begin{cases}
\text{For all } \psi_1, \psi_2 \in \mathcal{D}\left(\frac{1}{2}\Delta\right),\\
\tilde{X}_t(\psi_1) \equiv \tilde{X}_0(\psi_1) + \int_0^t \tilde{X}_s\left(\frac{1}{2}\Delta\psi_1\right) ds + M^1_t(\psi_1)\\
\qquad\qquad -\frac{1}{2}\int_0^t K_s\big((\tilde{X}-\tilde{Y}), (\tilde{X}+\tilde{Y})\big)(\psi_1)\, ds,\\
\tilde{Y}_t(\psi_2) \equiv \tilde{Y}_0(\psi_2) + \int_0^t \tilde{Y}_s\left(\frac{1}{2}\Delta\psi_2\right) ds + M^2_t(\psi_2) - \int_0^t K_s(\tilde{X}, \tilde{Y})(\psi_2)\, ds,\\
\text{where } M^j(\psi_j) \text{ are martingales such that}\\
\langle M^k(\psi_k), M^j(\psi_j)\rangle_t = \delta_{kj}\lambda\int_0^t (\tilde{X}_s)(\psi_k\psi_j)\, ds, \quad \forall\, k, j = 1, 2.
\end{cases}
\tag{4.4}
$$

Here $K_t(\tilde{X}, \tilde{Y})$, $K_t(\tilde{X} - \tilde{Y}, \tilde{X} + \tilde{Y})$ are supposed to be the "collisions distributions" between corresponding S'-valued processes. (The "collision distribution" between S'-valued processes can be defined similarly to the collision measure between measure-valued processes; here we use an intuitive concept of collision distribution for motivational purposes only and omit precise definitions.)

Let $\{\tilde{P}_t, t \geq 0\}$ be the set of one-dimensional distributions of $(\tilde{X}, \tilde{Y})$. Now define $H \equiv \tilde{X} + i\tilde{Y}$, $\overline{H} \equiv \tilde{X} - i\tilde{Y}$ and use Itô's formula to show that

$$
\begin{aligned}
&\tilde{P}_t[e^{-\langle\psi_1, H_t\rangle - \langle\psi_2, \overline{H}_t\rangle}]\\
&\quad = \tilde{P}_0[e^{-\langle\psi_1, \tilde{X}_0 + i\tilde{Y}_0\rangle - \langle\psi_2, \tilde{X}_0 - i\tilde{Y}_0\rangle}]\\
&\qquad + \int_0^t \tilde{P}_s\left[e^{-\tilde{X}_s(\psi_1)+i\tilde{Y}_s(\psi_2)}\left(-H_s\left(\frac{1}{2}\Delta\psi_1\right) - \overline{H}_s\left(\frac{1}{2}\Delta\psi_2\right) + 2\lambda\tilde{X}_s(\psi_1\psi_2)\right)\right] ds\\
&\qquad + \int_0^t \tilde{P}_s\left[e^{-\tilde{X}_s(\psi_1)+i\tilde{Y}_s(\psi_2)}\left(\frac{1}{2}K_s(\tilde{X}-\tilde{Y}, \tilde{X}+\tilde{Y})(\psi_1+\psi_2) + iK_s(\tilde{X}, \tilde{Y})(\psi_1-\psi_2)\right)\right] ds.
\end{aligned}
\tag{4.5}
$$

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Let $(\tilde{X}, \tilde{Y})$ be as in (4.4), independent of (X, Y) with $\tilde{X}_0 = \phi_1$, $\tilde{Y}_0 = \phi_2$ and $\phi = \phi_1 + i\phi_2$. Let us imagine for a moment that we can consider the collision distribution as a multiplicative operator (that is, $K_t(\tilde{X}, \tilde{Y})$ " = " $\tilde{X}_t\tilde{Y}_t$). Then applying the duality arguments from Chapter 4.4 of [10], one can conjecture that

$$P[e^{-\langle X_t^1, \phi\rangle - \langle X_t^2, \overline{\phi}\rangle}] = P \times \tilde{P}_t[e^{-\langle X_0^1, \tilde{X}_t + i\tilde{Y}_t\rangle - \langle X_0^2, \tilde{X}_t - i\tilde{Y}_t\rangle}], \tag{4.6}$$

and this implies that the original martingale problem has a unique solution. The "only" problem is the existence of such a process $(\tilde{X}, \tilde{Y})$. We will avoid this problem by constructing a sequence of processes that "should" approximate solution to (4.4). This sequence of processes determines the sequence of one-dimensional distributions $\{\tilde{P}_t^{(n)}, t \geq 0\}$ on $M_F \times S'$ which we will call *approximating sequence of dual (probability) measures.* For the sequence $\{\tilde{P}_t^{(n)}, t \geq 0\}$ we will establish the existence of a "limit point"—a set of limiting one dimensional distributions $\{\tilde{P}_t, t \geq 0\}$ which satisfies the conditions of Lemma 4.3. This will complete the proof of Theorem 2.6.

Remark 4.4 We will not prove the existence of $(\tilde{X}, \tilde{Y})$ which solves (4.4), though such a process is likely to exist.

The next two sections are crucial for our construction of the approximating sequence of dual measures.

5 Basic Evolution Equation

Given $1 \leq p \leq \infty$ and $B \in \mathcal{B}(\mathrm{R}^d)$, we define the space $L^p(B) \equiv L^p(B, dx)$ as the normed space of equivalent classes of measurable complex-valued functions with the finite norm

$$\|f\|_p \equiv \|f\|_{p,B} = \int_B |f(x)|^p \, dx, \quad 1 \leq p < \infty,$$
$$\|f\|_\infty \equiv \|f\|_{\infty,B} = \operatorname{ess\,sup} |f|.$$

We will suppress the subscript B in the notation of the norm if there is no ambiguity. $L^p_{\mathrm{R}}(B)$ stands for real-valued functions in $L^p(B)$. Let $C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)$ be the space of real-valued infinitely-differentiable functions with compact support in R^d.

For each $\phi \in \tilde{S}'$ define

$$S_t(\phi)(x) \equiv \langle \phi, p_t(x - \cdot)\rangle, \quad \forall\, t > 0, x \in \mathrm{R}^d.$$

This extends the domain of the semigroup S_t to the set of tempered distributions.

For each $\phi = \phi_1 + i\phi_2 \in \tilde{S}'$, $r < t$ and $\kappa \in L^\infty_{\mathrm{R}}\big([r, \infty)\big)_+$, let $V_{r,t}(\phi, \kappa) = V^1_{r,t} + iV^2_{r,t}$ denote a function-valued solution (if exists) of the following non-linear evolution equation

$$v(t) = S_{t-r}(\phi) - \int_r^t \kappa(s) S_{t-s}\big(v(s)^2\big)\, ds, \quad r < t, \tag{5.1}$$

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where $V^1 = \mathrm{Re}(V)$ and $V^2 = \mathrm{Im}(V)$ are, respectively, the real and imaginary parts of V. We say that $V_{r,t}(\phi, \kappa)$ is a strong solution to (5.1) if it satisfies

$$\begin{cases} \dfrac{\partial v(t)}{\partial t} = \dfrac{1}{2}\Delta v(t) - \kappa(t) v(t)^2, & r < t, \\ \lim\limits_{t \downarrow r} v(t) = \phi_1 + i\phi_2 \text{ in } \tilde{S}'. \end{cases} \tag{5.2}$$

We adopt the convention that $V_{t,t}(\phi, \kappa) \equiv \phi$ for all $t \geq 0$.

The equation (5.1) can be rewritten as a system of equations

$$\begin{cases} v_1(t) = S_{t-r}(\phi_1) - \displaystyle\int_r^t \kappa(s) S_{t-s}\big(v_1(s)^2 - v_2(s)^2\big)\, ds, & r < t, \\ v_2(t) = S_{t-r}(\phi_2) - \displaystyle\int_r^t 2\kappa(s) S_{t-s}\big(v_1(s) v_2(s)\big)\, ds, & r < t \end{cases} \tag{5.3}$$

with $V^1 = v_1$, $V^2 = v_2$. It is easy to check that

$$V^1_{r,t}(\phi_1 + i\phi_2, \kappa) = \frac{1}{2}\big(V_{r,t}(\phi_1 + i\phi_2, \kappa) + V_{r,t}(\phi_1 - i\phi_2, \kappa)\big), \tag{5.4}$$

$$V^2_{r,t}(\phi_1 + i\phi_2, \kappa) = -i\frac{1}{2}\big(V_{r,t}(\phi_1 + i\phi_2, \kappa) - V_{r,t}(\phi_1 - i\phi_2, \kappa)\big). \tag{5.5}$$

The existence and uniqueness of a solution to (5.1) for smooth initial conditions was proved in Lemma A1 of [2] (the coefficients there do not depend on t but the required extension to non-smooth coefficients is straightforward). The case of real measure-valued initial conditions ($\phi_2 = 0$ in our setting) has been investigated by several authors (*e.g.* [14], [4]), whereas equation (5.1) with complex $\tilde{S}'$-valued initial conditions does not seem to have been previously investigated.

We will need the following auxiliary lemmas.

Lemma 5.1

$$\|S_t(\phi)\|_q \leq \|\phi\|_q, \quad \forall \phi \in L^q(\mathrm{R}^d), \quad \forall 1 \leq q \leq \infty, \quad t \geq 0,$$
$$\|S_t(\phi)\|_q \leq \|p_{t-r}\|_{\frac{2q}{q+2}} \|\phi\|_2, \quad \forall \phi \in L^2(\mathrm{R}^d), \quad 2 \leq q \leq \infty, \quad \forall t > r$$

where p_t is the Brownian density.

Proof The result follows immediately from Young's inequality (see *e.g.* [1, 1.1.7]). ■

Lemma 5.2 *For each $\phi \in \mathcal{D}(\frac{1}{2}\Delta)_+$ and $\kappa \in L^\infty_{\mathrm{R}}\big([r,\infty)\big)_+$ we have*

$$|V_{r,t}(\phi, \kappa)(x)| \leq S_{t-r}(|\phi|)(x), \quad t \geq r \tag{5.6}$$

and, therefore,

$$\|V_{r,t}(\phi, \kappa)(\cdot)\|_q \leq \|\phi\|_q, \quad \forall 1 \leq q \leq \infty, \quad t \geq r, \tag{5.7}$$

$$\|V_{r,t}(\phi, \kappa)(\cdot)\|_q \leq \|p_{t-r}\|_{\frac{2q}{q+2}} \|\phi\|_2, \quad 2 \leq q \leq \infty, \quad \forall t > r \tag{5.8}$$

where p_t is the Brownian density.

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Proof The Feynman-Kac formula is used to get (5.6). The estimates (5.7), (5.8) follow from (5.6) and Lemma 5.1. ■

Let $(\phi,\psi)\in\tilde{S}'\times M_F$, and let $U_{r,t}\big(V_{r,\cdot}(\phi,\kappa),\psi\big)$ be a solution (if exists) of the following linear equation

$$u(t)=S_{t-r}(\psi)-\int_r^t 2\kappa(s)S_{t-s}\big(V_{r,s}(\phi,\kappa)u(s)\big)\,ds,\quad t>r. \tag{5.9}$$

We will adopt the convention that $U_{t,t}\big(V_{t,\cdot}(\phi,\kappa),\psi\big)=\psi$ for all $t\geq 0$. For the specific case when $\psi=\delta_x$, the corresponding solution to (5.9) (if exists) will be denoted by $U_{r,t}\big(V_{r,\cdot}(\phi,\kappa),x\big)$. That is, $U_{r,t}\big(V_{r,\cdot}(\phi,\kappa),x\big)$ solves

$$u(t,y)=p_{t-r}(x-y)-\int_r^t 2\kappa(s)S_{t-s}\big(V_{r,s}(\phi,\kappa)u(s)\big)(y)\,ds,\quad t>r,\quad y\in\mathrm{R}^d. \tag{5.10}$$

One can consider $U_{r,t}\big(V_{r,\cdot}(\phi,\kappa),x\big)$ as a "fundamental solution" to the equation (5.9). We set

$$U_{r,t}\big(V_{r,\cdot}(\phi,\kappa),x\big)(\nu)\equiv\int_{\mathrm{R}^d}U_{r,t}\big(V_{r,\cdot}(\phi,\kappa),x\big)(y)\,\nu(dy),\quad\forall\,\nu\in M_F.$$

It is easy to check that, for each $(\phi,\psi)\in\tilde{S}'\times M_F$,

$$U_{r,t}\big(V_{r,\cdot}(\overline{\phi},\kappa),\psi\big)=\overline{U}_{r,t}\big(V_{r,\cdot}(\phi,\kappa),\psi\big),\quad\forall\,r\leq t\leq T.$$

We will establish the existence, uniqueness and properties of solutions for (5.1) and (5.9) under certain regularity assumptions on the distribution valued boundary conditions. Let $\rho\geq 0$. We set

$$w(s,\mu)\equiv\|S_s\mu\|_2^2=\int_{\mathrm{R}^d}|(S_s\mu)(x)|^2\,dx,\quad\forall\,s>0,\quad\mu\in\tilde{S}',$$

$$\overline{w}_\rho(\delta,\mu)\equiv\sup_{s\leq\delta}s^\rho w(s,\mu),\quad\forall\,\delta>0,\quad\mu\in\tilde{S}',$$

$$\tilde{w}_\rho(s,\mu)\equiv\rho\int_0^s u^{\rho-1}w(u,\mu)\,du,\quad\forall\,s>0,\quad\mu\in\tilde{S}'.$$

In order to study $V_{r,t}$ and $U_{r,t}$ we need to introduce the following subsets of the spaces M_F, S' and $\tilde{S}'$

$$S'^{\rho}\equiv\{\mu\in S':\lim_{\delta\to 0}\overline{w}_\rho(\delta,\mu)=0\},$$

$$M_F^\rho\equiv\{\mu\in M_F:\lim_{\delta\to 0}\overline{w}_\rho(\delta,\mu)=0\},$$

$$\tilde{S}'^{\rho}\equiv\{\mu=\mu_1+i\mu_2\in\tilde{S}':\mu_1\in M_F^\rho,\mu_2\in S'^{\rho}\}.$$

We would like to introduce the following definition.

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Definition 5.3 Let $\{f^{(n)}\} \in \tilde{S}'^{\rho}$, $f^{(n)} \to f$ in $\tilde{S}'$ as $n \to \infty$, and

$$\lim_{h\downarrow 0} \limsup_{n\to\infty} \tilde{w}_\rho(h, f^{(n)}) = 0.$$

Then we say that $f^{(n)} \to f$ in $\tilde{S}'^{\rho}$ as $n \to \infty$. Let τ^ρ be the corresponding topology on $\tilde{S}'^{\rho}$.

Sometimes it will be convenient for us to consider $\tilde{S}'^{\rho}$ as a subspace of $\tilde{S}'$ with relative topology $\hat{\tau}^\rho$ induced on it. In this case we will use the notation $(\tilde{S}'^{\rho}, \hat{\tau}^\rho)$ to emphasize the fact of using of topology induced by $\tilde{S}'$. But if it is not stated otherwise we assume that topology on $\tilde{S}'^{\rho}$ is τ_ρ.

It should be also pointed out that the notation τ_ρ and $\hat{\tau}_\rho$ will have a double meaning throughout this paper. They will denote not only topologies on $\tilde{S}'^{\rho}$, but on $M_F^\rho \times S'^\rho$ as well. The correct meaning will be always obvious from the context.

Lemma 5.4 *For any $\phi \in \tilde{S}'^{\rho}$ there exists $\{\phi^{(n)}\}$ in $\tilde{S}(\mathrm{R}^d)$ such that $\lim_{n\to\infty} \phi^{(n)} = \phi$ in $\tilde{S}'^{\rho}$.*

Proof The proof is elementary. Take a function $\psi \in S(R^d)_+$ such that

$$\psi(x) = 1, \quad \text{for } |x| \le 1.$$

Define $\phi^{(n)}(x) = S_{1/n}(\phi)(x)\psi(x/n)$. Then it is easy to check that $\phi^{(n)} \in \tilde{S}(\mathrm{R}^d)$, $\phi^{(n)} \to \phi$ in $\tilde{S}'$ and

$$\lim_{h\downarrow 0} \limsup_{n\to\infty} \tilde{w}_\rho(h, \phi^{(n)}) = 0. \qquad \blacksquare$$

For each $\mu = \mu_1 + i\mu_2 \in \tilde{S}'$ such that $w(s,\mu) < \infty$, one can easily check, using the definition of the heat kernel and integration by parts formula, that $w(s,\mu)$ is differentiable in s and that

$$\begin{aligned} w'(s,\mu) &= \frac{\partial w(s,\mu)}{\partial s} \\ &= \sum_{k=1}^{2} \int_{\mathrm{R}^d} \frac{\partial}{\partial s}(S_s\mu_k)(y)^2\,dy \\ &= \sum_{k=1}^{2} \int_{\mathrm{R}^d} 2(S_s\mu_k)(y)\frac{1}{2}\Delta(S_s\mu_k)(y)\,dy \\ &= -\sum_{k=1}^{2} \int_{\mathrm{R}^d} \sum_{j=1}^{d} \left(\frac{\partial}{\partial y_j}(S_s\mu_k)(y)\right)^2 dy \le 0, \quad \forall\, s > 0. \end{aligned} \tag{5.11}$$

The following lemma will be frequently used.

Lemma 5.5 *Let $\rho > 0$, $\mu \in \tilde{S}'$. Suppose*

$$\tilde{w}_\rho(T,\mu) < \infty, \quad \forall\, T > 0. \tag{5.12}$$

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Then

(5.13) $$s^{\rho} w(s,\mu) \le \tilde{w}_{\rho}(s,\mu), \quad \forall\, s > 0,$$

and

(5.14) $$\lim_{\delta\to 0} \overline{w}_{\rho}(\delta,\mu) = 0,$$

that is, $\mu \in \tilde{S'}^{\rho}$.

Proof (5.14) follows immediately from (5.13). Let us show that $\lim_{s\downarrow 0} s^{\rho} w(s,\mu) = 0$. Since $w(s,\mu)$ is differentiable in s, for any $T > s$ we have

$$s^{\rho} w(s,\mu) = T^{\rho} w(T,\mu) - \rho \int_s^T t^{\rho-1} w(t,\mu)\, dt - \int_s^T t^{\rho} w'(t,\mu)\, dt,$$

where $w(t,\mu) \ge 0$, $w'(t,\mu) \le 0$; therefore, both integrals at the right side are monotone in s. Monotonicity of the integrals combined with the boundedness of $\int_0^T t^{\rho-1} w(t,\mu)\, dt$ imply existence of the limit on the left hand side, so there exists a such that $\lim_{s\downarrow 0} s^{\rho} w(s,\mu) = a$. Moreover, $a = 0$ since otherwise

$$\int_0^T t^{\rho-1} w(t,\mu)\, dt = \int_0^T \big(t^{\rho} w(t,\mu)\big) t^{-1}\, ds = \infty,$$

which contradicts (5.12).

Thus, $\lim_{s\downarrow 0} s^{\rho} w(s,\mu) = 0$ and, hence,

$$\begin{aligned} s^{\rho} w(s,\mu) &= \rho \int_0^s t^{\rho-1} w(t,\mu)\, dt + \int_0^s t^{\rho} w'(t,\mu)\, dt \\ &\le \rho \int_0^s t^{\rho-1} w(t,\mu)\, dt. \end{aligned}$$ ∎

For any $\rho > 0$ and $A \subset M_F^{\rho} \times S'^{\rho}$ let us define two relative topologies:

$$\tau_{\rho}^{A} \equiv \{B \cap A, B \in \tau_{\rho}\}$$
$$\hat{\tau}_{\rho}^{A} \equiv \{B \cap A, B \in \hat{\tau}_{\rho}\}.$$

Corollary 5.6 *Let* $A = \{(\mu_1,\mu_2) \in M_F \times S' : \sum_{l=1}^{2} \int_0^{\delta} t^{\rho-1} w(t,\mu_l)\, dt \le k\}$ *for some* $\rho, \delta, k > 0$.

(a) Then $A \subset M_F^{\rho} \times S'^{\rho}$ *and* A *is closed in* $M_F \times S'$.
(b) Suppose that in addition A *is compact in* $M_F \times S'$*. Then, for any* $\rho' > \rho$*,* A *is compact in* $(M_F^{\rho'} \times S'^{\rho'}, \tau^{\rho'})$ *and* $\tau_{\rho'}^{A} = \hat{\tau}_{\rho'}^{A}$.

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Proof (a) The fact that $A \subset M_F^\rho \times S'^\rho$ is an immediate consequence of the previous lemma. Let us check that A is closed. Let $(\mu_1^{(n)}, \mu_2^{(n)}) \to (\mu_1, \mu_2)$ in $M_F \times S'$ such that $(\mu_1^{(n)}, \mu_2^{(n)}) \in A$ for all n. Then

$$\int_0^\delta t^{\rho-1} \sum_{l=1}^2 w(t, \mu_l)\, dt = \int_0^\delta t^{\rho-1} \sum_{l=1}^2 w(t, \lim_{n\to\infty} \mu_l^{(n)})\, dt$$
$$\leq \liminf_{n\to\infty} \int_0^\delta t^{\rho-1} \sum_{l=1}^2 w(t, \mu_l^{(n)})\, dt \leq k$$

where the first inequality follows from Fatou's lemma. Therefore, $(\mu_1, \mu_2) \in A$ and we are done.

(b) Let $\{f^{(n)}\}$ be any sequence in A. Since A is compact in $M_F \times S'$, there exists subsequence $\{n'\}$ such that $\{f^{(n')}\}$ converges in $M_F \times S'$ to some f. Let us check that $\{f^{(n')}\}$ actually converges in $M_F^{\rho'} \times S'^{\rho'}$ for any $\rho' > \rho$. Without loss of generality we may assume that $\epsilon < \delta$ and then we have

$$\begin{aligned}
\tilde{w}_{\rho'}(\epsilon, f^{(n')}) &= \rho' \int_0^\epsilon s^{\rho'-1} w(s, f^{(n')})\, ds \\
&= \rho' \int_0^\epsilon s^{\rho'-\rho-1} s^\rho w(s, f^{(n')})\, ds \\
&\leq \rho' \int_0^\epsilon s^{\rho'-\rho-1} \tilde{w}_{\rho'}(s, f^{(n')})\, ds \\
&\leq \rho' \int_0^\epsilon s^{\rho'-\rho-1} k\, ds \\
&\to 0, \text{ as } \epsilon \downarrow 0,
\end{aligned}$$

uniformly in n'. The first inequality follows from Lemma 5.5 and the second inequality follows by our assumptions on A. Therefore $f^{(n')}$ converges in $M_F^{\rho'} \times S'^{\rho'}$, and hence A is compact in $M_F^{\rho'} \times S'^{\rho'}$. The same arguments will readily show that subsets of A which are closed in $(M_F^{\rho'} \times S'^{\rho'}, \hat{\tau}_{\rho'})$ concise with with the subsets of A which are closed in $(M_F^{\rho'} \times S'^{\rho'}, \tau_{\rho'})$ and therefore $\tau_{\rho'}^A = \hat{\tau}_{\rho'}^A$. ■

We will assume, unless stated otherwise, that ρ is a fixed number satisfying the condition:

$$\left(\frac{d}{2} - 1\right) \vee 0 < \rho < 1 \wedge \left(\frac{3}{2} - \frac{d}{4}\right). \tag{5.15}$$

With this fixed ρ in mind, we define another constant $\hat{\rho}$ which satisfies the following condition

$$0 < \hat{\rho} < \left(3 - \frac{d}{2} - 2\rho\right) \wedge (1 - \rho).$$

The measures in M_F^ρ that we have previously defined, satisfy some conditions in terms of capacities, that we introduce here. Following [1] we define

$$\hat{\nu}(ds,dy) \equiv s^{\hat{\rho}} p_1(dy) 1(0 \le s \le 1)\, ds\, dy,$$

$$\mathcal{G} f(x) \equiv \int_0^1 \int_{\mathrm{R}^d} f(s,y) p_s(x-y) \hat{\nu}\,(ds,dy).$$

For any function $f \in L^2\big(R^d \times \mathrm{R}_+, \hat{\nu}(ds,dy)\big)$ we denote

$$\|f\|_{2,\hat{\nu}}^2 = \int_0^1 \int_{\mathrm{R}^d} |f(s,y)|^2 \hat{\nu}\,(ds,dy).$$

Now we are ready to introduce the capacity

$$\mathcal{C}(B) \equiv \inf\{\|f\|_{2,\hat{\nu}} : \mathcal{G} f(x) \ge 1,\ \forall x \in B\}, \quad \forall B \in \mathcal{B}(\mathrm{R}^d).$$

A property that holds true for all x except those belonging to a set B with $\mathcal{C}(B) = 0$ is said to be true *quasi-everywhere*, abbreviated *q.e.*

Lemma 5.7

(i) *If $d \le 2$, then there are no non-empty sets of capacity zero.*
(ii) *If $\mu \in M_F^\rho$, then μ does not charge sets of capacity zero.*

Proof The result is an easy consequence of Theorem 2.5.1 and Proposition 2.6.1 of [1] and their proofs.

Now we are ready to present the theorems which are important for further proofs. They will be proved in the Appendix.

Theorem 5.8

(a) *For each $\mu \in \tilde{S}'^\rho$ and $\kappa \in L^\infty_{\mathrm{R}}\big([r,\infty)\big)_+$, there exists a unique solution $V_{r,t}(\mu,\kappa)$ for (5.1) such that*

$$V_{r,\cdot}(\mu,\kappa) \in L^2\big(\mathrm{R}^d \times (r,T]\big) \cap C\big(\mathrm{R}^d \times (r,T]\big)_+, \quad \forall T > r,$$

$$V_{r,\epsilon+\cdot}(\mu,\kappa) \in C\big(\mathrm{R}^d \times [r,T]\big)_+, \quad \forall T > r, \quad \epsilon > 0,$$

$$V_{r,t}(\mu,\kappa) \in L^q(\mathrm{R}^d)_+, \quad \forall t > r, \quad q \ge 2,$$

$$V^1_{r,t}(\mu,\kappa) \in L^1_{\mathrm{R}}(\mathrm{R}^d)_+, \quad \forall t > r.$$

If $\kappa \in C\big([r,\infty)\big)_+$, then $V_{r,t}(\mu,\kappa)$ is a strong solution for (5.1), that is, it satisfies (5.2).

(b) *Let $T > 0$, and let A, B be any compact subsets of $\tilde{S}'^\rho \times [0,T) \times \mathrm{R}^d$ and $\tilde{S}'^\rho \times [0,T)$ respectively. Let $\kappa^{(n)} \to \kappa$ weakly* in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$. Then*

$$\lim_{n\to\infty} \sup_{(\mu,t,x)\in A} |V_{t,T}(\mu,\kappa^{(n)})(x) - V_{t,T}(\mu,\kappa)(x)| = 0, \tag{5.16}$$

and

$$\sup_{(\mu,t,x)\in B\times\mathrm{R}^d} |V_{t,T}(\mu,\kappa^{(n)})(x)| < \infty. \tag{5.17}$$

(c) *Let* $T > 0$, $\psi \in S(R^d)$, *and let A be any compact subset of* $\tilde{S'}^\rho$ *and* $\kappa^{(n)} \to \kappa$ *weakly* in* $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$. *Then*

$$\lim_{\epsilon\downarrow 0} \sup_{|t-s|<\epsilon, s<t<T, \mu\in A} |\langle \psi, V_{s,t}(\mu,\kappa^{(n)})\rangle - \langle \psi,\mu\rangle| = 0$$

uniformly in n.

(d) *Let* $T > 0$. *The mapping*

$$(t,\mu,\kappa,x) \mapsto V_{t,T}(\mu,\kappa)(x)$$

of $[0,T) \times \tilde{S'}^\rho \times L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+ \times \mathrm{R}^d$ *into* C *is continuous on* $[0,T) \times \tilde{S'}^\rho \times L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+ \times \mathrm{R}^d$ *(where we induce the weak* topology on* $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$*).*

Theorem 5.9

(a) *For each* $\mu \in \tilde{S'}^\rho$, $r > 0$, $\kappa \in L^\infty_{\mathrm{R}}\big([r,\infty)\big)_+$ *and q.e. x there exists a unique solution* $U_{r,t}\big(V_{r,\cdot}(\mu,\kappa),x\big)$ *for (5.10) such that*

$$U_{r,\cdot}\big(V_{r,\cdot}(\mu,\kappa),x\big) \in C\big((r,\infty)\times \mathrm{R}^d\big)_+,$$
$$U_{r,t}\big(V_{r,\cdot}(\mu,\kappa),x\big) \in \overline{C}(\mathrm{R}^d), \quad \forall t > r.$$

For each $T > r$ *and* $y \in \mathrm{R}^d$ *the function* $U_{r,T}\big(V_{r,\cdot}(\mu,\kappa),\cdot\big)(y)$ *is quasicontinuous. For each* $0 \le r < T$ *there exists* $\mathcal{N} \subset \mathrm{R}^d$ *with* $\mathcal{C}(\mathcal{N}) = 0$ *such that*

$$\big|U_{r,T}\big(V_{r,\cdot}(\mu,\kappa),x\big)(y)\big| \le p_{T-r}(x-y), \quad \forall (y,x) \in \mathrm{R}^d \times (\mathrm{R}^d \setminus \mathcal{N}). \tag{5.18}$$

(b) *For each* $\nu \in M_F^\rho$,

$$U_{r,\cdot}\big(V_{r,\cdot}(\mu,\kappa),\nu\big) = \int_{\mathrm{R}^d} U_{r,\cdot}\big(V_{r,\cdot}(\mu,\kappa),x\big)\,\nu(dx),$$

that is, the solution for (5.9) is given as an integral of the fundamental solution with respect to the initial condition.

(c) *Let* $T > 0$ *and* $\psi_1,\psi_2 \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$. *For any* $\mu \in \tilde{S'}^\rho$ *let* $\mu_1 = \mathrm{Re}(\mu)$. *Then the mapping*

$$(t,\mu,\kappa) \mapsto \int_{\mathrm{R}^d} U_{t,T}\big(V_{t,\cdot}(\mu,\kappa),x\big)(\psi_1) U_{t,T}\big(V_{t,\cdot}(\overline{\mu},\kappa),x\big)(\psi_2)\,\mu_1(dx)$$

of $[0,T) \times \tilde{S'}^\rho \times L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$ *into* C *is continuous on* $[0,T) \times \tilde{S'}^\rho \times L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$.

(d) *Let* $T > 0$ *and* $\psi \in S(R^d)_+$. *Then the mapping*

$$(t,\mu,\kappa,x) \mapsto U_{t,T}\big(V_{t,\cdot}(\mu,\kappa),\psi\big)(x)$$

of $[0,T) \times \tilde{S'}^\rho \times L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+ \times \mathrm{R}^d$ *into* C *is continuous on* $[0,T) \times \tilde{S'}^\rho \times L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+ \times \mathrm{R}^d$.

6 A Certain $M_F \times S'$-Valued Process and Its Regularity Properties

Let $(\Omega, \mathcal{F}, P)$ be a probability space which is sufficiently rich to contain all the processes and random variables defined below. For any process $\{X_t, t \geq 0\}$ defined on $(\Omega, \mathcal{F}, P)$, let $\mathcal{F}_t^X \equiv \bigcap_{\epsilon>0} \sigma\{X_s : s \leq t+\epsilon\}$. Let $\mathcal{P}(\mathcal{F}_t)$ denote the predictable σ-algebra for the filtration $(\mathcal{F}_t)$.

We start this section with a reformulation of the result of [2] which establishes the existence and uniqueness of a solution for a certain martingale problem.

Lemma 6.1 *For each $r \geq 0$ and $\nu = P(X'_r, W'_r)^{-1} \in M_{1,c}(M_F \times S')$, there exists a unique solution $(X', W') \in C_{M_F \times S'}[r, \infty)$ of the following martingale problem*

$$M'_{r,\nu,\lambda} \begin{cases} \text{For all } \phi_1, \phi_2 \in \mathcal{D}\left(\frac{1}{2}\Delta\right), \\ X'_t(\phi_1) \equiv X'_r(\phi_1) + \int_r^t X'_s\left(\frac{1}{2}\Delta\phi_1\right) ds + Z_t^1(\phi_1), \quad t \geq r, \\ W'_t(\phi_2) \equiv W'_r(\phi_2) + Z_t^2(\phi_2), \quad t \geq r, \\ \text{where } Z^j(\phi_j) \text{ are continuous square integrable } \mathcal{F}_t\text{-martingales} \\ \quad \text{such that} \\ Z_r^j(\phi_j) = 0, \quad j = 1,2, \\ \langle Z^j(\phi_j), Z^k(\phi_k)\rangle_t = \delta_{jk} 2\lambda \int_r^t X'_s(\phi_1^2)\, ds, \quad j,k = 1,2 \end{cases}$$

with $\mathcal{F}_t \equiv \mathcal{F}_t^{(X',W')}$.

Proof A direct application of Itô's formula implies that each solution of the martingale problem $M'_{r,\nu,\lambda}$ is a solution of the martingale problem for (A,ν) on the time interval $[r,\infty)$ where

$$(6.1)\qquad A = \Big\{\exp\{-\mu_1(\phi_1) + i\mu_2(\phi_2)\}, \\ \exp\{-\mu_1(\phi_1) + i\mu_2(\phi_2)\}\frac{1}{2}\mu_1(-\Delta\phi_1 + 2\lambda\phi_1^2 - 2\lambda\phi_2^2) : \\ \phi_1 \in \mathcal{D}_{\mathrm{R}}\left(\frac{1}{2}\Delta\right)_+, \phi_2 \in S(R^d)\Big\}.$$

By Lemmas 4.10, 4.13, 4.18 from [2], each solution for (A,ν) is also a solution of $M'_{r,\nu,\lambda}$ and the two martingale problems are equivalent. The existence and uniqueness for (A,ν) established in Theorem 3.3 of [2] completes the proof. ■

One can extend Z^j, $(j = 1,2)$ to an orthogonal martingale measure $\{Z_t^j(\phi) : \phi \in B(\mathrm{R}^d), t \geq r\}$ and for each $\phi \in L_r^2(X', P)$ the stochastic integral $Z_t^j(\phi) = \int_r^t \int_{\mathrm{R}^d} \phi(s,\omega,x)\, Z^j(ds,dx)$ is well defined (see Remark 2.5).

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Corollary 6.2 *For each $r \geq 0$ and $\nu = P(X_0', Y_0')^{-1} \in M_{1,c}(M_F \times S')$, there exists a unique solution $(X', Y') \in C_{M_F \times S'}[r, \infty)$ of the following martingale problem*

$$M''_{r,\nu,\lambda}\begin{cases} \text{For all } \phi_1, \phi_2 \in \mathcal{D}\left(\frac{1}{2}\Delta\right) \\ X_t'(\phi_1) \equiv X_r'(\phi_1) + \int_r^t X_s'\left(\frac{1}{2}\Delta\phi_1\right) ds + Z_t^1(\phi_1), \quad t \geq r, \\ Y_t'(\phi_2) \equiv Y_r'(\phi_2) + \int_r^t Y_s'\left(\frac{1}{2}\Delta\phi_2\right) ds + Z_t^2(\phi_2), \quad t \geq r, \\ \text{where } Z^j(\phi_j) \text{ are continuous square integrable martingales such that} \\ Z_r^j(\phi_j) = 0, \quad j = 1, 2, \\ \langle Z^j(\phi_j), Z^k(\phi_k)\rangle_t = \delta_{jk} 2\lambda \int_r^t X_s'(\phi_j^2)\, ds, \quad j, k = 1, 2. \end{cases}$$

Proof Let (X', W') be as in the previous lemma with $W_r' = Y_r'$. Defining the S'-valued process Y' by

$$Y_t'(\phi) \equiv W_r'(S_{t-r}\phi) + \int_r^t \int_{\mathrm{R}^d} (S_{t-s}\phi)(x) Z^2\,(ds, dx), \quad t \geq r,$$

one can easily check that (X', Y') satisfies $M''_{r,\nu,\lambda}$ (see *e.g.* Theorem 5.1 [20]). For uniqueness one can check that

$$E[e^{-\langle X_t', \phi_1\rangle - i\langle Y_t', \phi_2\rangle}] = E[e^{-\langle X_r', u_{t-r}\rangle - i\langle Y_r', S_{t-r}(\phi_2)\rangle}],$$
$$\forall\, t > r, \quad \phi_1 \in \mathcal{D}\left(\frac{1}{2}\Delta\right)_+, \quad \phi_2 \in \mathcal{D}\left(\frac{1}{2}\Delta\right),$$

where u_t solves the following equation:

$$u(t) = S_t(\phi_1) - \int_0^t \lambda S_{t-s}\Big(u(s)^2 - \big(S_s(\phi_2)\big)^2\Big)\, ds, \quad t > 0,$$

and uniqueness will follow by standard arguments (see Theorems 4.4.2, 4.4.7 of [10]). ■

Remark 6.3 In the sequel, the law of the process (X', Y') which starts at $r \geq 0$ and satisfies $M''_{r,\nu,\lambda}$ will be denoted by $Q^1_{r,\nu,\lambda}$. With a slight abuse of notation, we set

$$Q^1_{r,(\mu_1,\mu_2),\lambda} \equiv Q^1_{r,\delta_{(\mu_1,\mu_2)},\lambda}, \quad \forall\,(\mu_1, \mu_2) \in M_F \times S',$$
$$M''_{r,(\mu_1,\mu_2),\lambda} \equiv M''_{r,\delta_{(\mu_1,\mu_2)},\lambda}, \quad \forall\,(\mu_1, \mu_2) \in M_F \times S'.$$

We will need the following equivalent representation of the martingale problem $M''_{r,\nu,\lambda}$:

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Lemma 6.4 *Suppose (X', Y') solves $M''_{r,\nu,\lambda}$ for some $r \geq 0$, $\nu \in M_{1,c}(M_F \times S')$. Then (X', Y') satisfies the following*

$$\tilde{M}^T_{r,\nu,\lambda}\begin{cases}\text{For all } \mu_1, \mu_2 \in M_F^\rho, \quad T > 0,\\ X'_t(S_{T-t}\mu_1) \equiv X'_r(S_{T-r}\mu_1) + Z_t^{1,T}(\mu_1), \quad r \leq t \leq T,\\ Y'_t(S_{T-t}\mu_2) \equiv Y'_r(S_{T-r}\mu_2) + Z_t^{2,T}(\mu_2), \quad r \leq t \leq T,\\ \text{where } Z^{j,T}(\mu_j) \text{ are continuous square integrable martingales on } [r, T]\\ \quad \text{such that}\\ Z_r^{j,T}(\mu_j) = 0, \quad j = 1, 2,\\ \langle Z^j(\mu_j), Z^k(\mu_k)\rangle_t = \delta_{jk} 2\lambda \int_r^t X'_s\big((S_{T-s}\mu_j)^2\big)\, ds, \quad j, k = 1, 2\end{cases}$$

where M_F^ρ and ρ are defined in Section 5.

Proof We identify $L^1_{\mathrm{R}}(\mathrm{R}^d)_+$ with M_F by the mapping $\phi(x) \mapsto \phi(x)\,dx$. For $\mu_1, \mu_2 \in \mathcal{D}(\frac{1}{2}\Delta) \cap L^1_{\mathrm{R}}(\mathrm{R}^d)_+$, the result follows from Itô's formula. For general $\mu_1, \mu_2 \in M_F^\rho$, one can choose sequences of smooth functions $\{\psi_1^{(n)}\}$, $\{\psi_2^{(n)}\}$ in $\mathcal{D}(\frac{1}{2}\Delta) \cap L^1_{\mathrm{R}}(\mathrm{R}^d)_+$ such that $\psi_j^{(n)} \Rightarrow \mu_j$ and

$$\lim_{n\to\infty} S_t\psi_j^{(n)} = S_t\mu_j \quad \text{in } L^2_{\mathrm{R}}\big((0, T] \times \mathrm{R}^d\big)_+, \quad j = 1, 2,$$

for all $T > 0$. The latter condition can be easily satisfied since $S_t\mu_j \in L^2_{\mathrm{R}}\big((0, T] \times \mathrm{R}^d\big)_+$ for $\mu_j \in M_F^\rho$, $j = 1, 2$. Further, $L^2_{\mathrm{R}}\big((0, T] \times \mathrm{R}^d\big)_+ \subset L^2_r(X', P)$ and $L^2_r(X', P)$ is complete, hence, the result follows immediately. ■

Given a bounded stopping time $\tau \geq r$, we will say that a pair of $M_F \times S'$-valued processes (X, Y) satisfies the stopped martingale problem $\tilde{M}^{T,\tau}_{u,\nu,\lambda}$ for $r \leq u < T$ if

$$\tilde{M}^{T,\tau}_{u,\nu,\lambda}\begin{cases}P(X_u, Y_u)^{-1} = \nu \in M_1(M_F \times S'),\\ X_t = X_u, Y_t = Y_u, \text{ for } u \leq t \leq T, \quad \tau < u.\\ \text{For all } \mu_1, \mu_2 \in M_F^\rho,\\ 1(\tau \geq u)X_t(S_{T-(t\wedge\tau)}\mu_1) \equiv 1(\tau \geq u)X_u(S_{T-u}\mu_1) + Z_t^{1,\tau,T}(\mu_1), \quad u \leq t \leq T,\\ 1(\tau \geq u)Y_t(S_{T-(t\wedge\tau)}\mu_2) \equiv 1(\tau \geq u)Y_u(S_{T-u}\mu_2) + Z_t^{2,\tau,T}(\mu_2), \quad u \leq t \leq T,\\ \text{where } Z^{j,\tau,T}(\mu_j) \text{ are continuous square integrable martingales on } [u, T]\\ \quad \text{such that}\\ Z_u^{j,\tau,T}(\mu_j) = 0, \quad j = 1, 2,\\ \langle Z^{j,\tau,T}(\mu_j), Z^{k,\tau,T}(\mu_k)\rangle_t = \delta_{jk} 2\lambda \int_u^t 1(\tau \geq s) X_s\big((S_{T-s}\mu_j)^2\big)\, ds, \quad j, k = 1, 2.\end{cases}$$

For any bounded stopping time $\tau \geq r$ the optional stopping theorem implies that if (X', Y') satisfy $\tilde{M}^T_{r,\nu,\lambda}$ on $[r, T]$ then $(X'_{\cdot\wedge\tau}, Y'_{\cdot\wedge\tau})$ satisfies $\tilde{M}^{T,\tau}_{u,\nu_u,\lambda}$, for any $r \leq u < T$ with $\nu_u = P(X'_{u\wedge\tau}, Y'_{u\wedge\tau})^{-1}$.

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In order to simplify the exposition, in the remainder of this section we will deal with the martingale problem $M''_{0,\nu,\lambda}$. (All the results hold in the general case $r > 0$ as well.)

Let us indicate several simple properties of (X', Y'). In what follows we will assume that

$$(X'_0, Y'_0) = (\mu_1, \mu_2) = \big(\phi_1(x)dx, \phi_2(x)dx\big),$$
$$\phi_1 \in L^1_{\mathrm{R}}(\mathrm{R}^d)_+ \cap L^2_{\mathrm{R}}(\mathrm{R}^d)_+, \quad \phi_2 \in L^2_{\mathrm{R}}(\mathrm{R}^d).$$

Some simple calculations give us the second moment formulae for X' and Y':

(6.2)
$$\begin{aligned} &P[\langle X'_t, \psi_1\rangle \langle X'_t, \psi_2\rangle] \\ &\quad = \langle \phi_1, S_t\psi_1\rangle \langle \phi_1, S_t\psi_2\rangle \\ &\qquad + 2\lambda P\Big[\int_0^t \int_{\mathrm{R}^d} S_{t-u}(\psi_1)(y) S_{t-u}(\psi_2)(y)(S_u\phi_1)(y)\, dy\, du\Big], \quad \forall \psi_1, \psi_2 \in S(R^d), \end{aligned}$$

(6.3)
$$\begin{aligned} &P[\langle Y'_t, \psi_1\rangle \langle Y'_t, \psi_2\rangle] \\ &\quad = \langle \phi_2, S_t\psi_1\rangle \langle \phi_2, S_t\psi_2\rangle \\ &\qquad + 2\lambda P\Big[\int_0^t \int_{\mathrm{R}^d} S_{t-u}(\psi_1)(y) S_{t-u}(\psi_2)(y)(S_u\phi_1)(y)\, dy\, du\Big], \quad \forall \psi_1, \psi_2 \in S(R^d). \end{aligned}$$

The next lemma establishes regularity properties of (X', Y') at a fixed time.

Lemma 6.5 *For all $t > 0$, P-a.s. $(X'_t, Y'_t) \in M_F^\beta \times S'^\beta$ for any $\beta > \frac{d}{2} - 1$.*

Proof By Lemma 5.5, it suffices to prove that

$$P\Big[\int_0^T s^{\beta-1} w(s, X'_t)\, ds\Big] < \infty, \quad P\Big[\int_0^T s^{\beta-1} w(s, Y'_t)\, ds\Big] < \infty, \quad \forall t \geq 0, \quad T > 0.$$

We will prove only the assertion about Y'_t; for X'_t the proof is the same. Since $w(s, Y'_t) \geq 0$, by Fubini's theorem and (6.3) we obtain

$$\begin{aligned} &P\Big[\int_0^T s^{\beta-1} w(s, Y'_t)\, ds\Big] \\ &\quad = \int_0^T \int_{\mathrm{R}^d} s^{\beta-1} P[S_s(Y'_t)(x)^2]\, dx\, ds \\ &\quad = \int_0^T \int_{\mathrm{R}^d} s^{\beta-1} (S_{t+s}\phi_2)(x)^2\, dx\, ds \\ &\qquad + 2\lambda \int_0^T s^{\beta-1} \int_{\mathrm{R}^d} \int_0^t \int_{\mathrm{R}^d} p_{t+s-u}(x-y)^2 (S_u\phi_1)(y)\, dy\, du\, dx\, ds \\ &\quad \leq \|\phi_2\|_2^2 \int_0^T s^{\beta-1}\, ds + 2\lambda C_d \|\phi_1\|_1 \int_0^T s^{\beta-1} \int_0^t (t+s-u)^{-d/2}\, du\, ds, \end{aligned}$$

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where the constant C_d depends on d. Both integrals are finite for any $T, t > 0$, $\beta > d/2-1$, therefore we are done. ■

7 Dual Probability Measures

In this section we will construct the approximating sequence of dual probability measures. (The motivation for our construction was discussed in Section 4.) We will also establish the existence of limiting dual probability measures satisfying some equation.

7.1 Construction of Approximating Sequence of Dual Measures

Let $\tilde{\Omega} = C_{M_F\times S'}[0,\infty)$ denote the space of continuous $M_F(\mathrm{R}^d)\times S'(\mathrm{R}^d)$-valued paths with the compact-open topology, and let $\tilde{\mathfrak{F}}$ denote its Borel σ-algebra. Let $(\tilde{\mathfrak{F}}_t)_{t\geq 0}$ denote the canonical right-continuous filtration on $(\tilde{\Omega},\tilde{\mathfrak{F}})$. $\tilde{X}_t(\omega) = \omega(t)$ will denote the coordinate mappings on $\tilde{\Omega}$. Let $\tilde{\mathfrak{F}}_{[r,t]} = \sigma(\tilde{X}_u : r \leq u \leq t)$. We wish to construct some sequence of probability measures $\tilde{P}^{(n)}$ on $(\tilde{\Omega},\tilde{\mathfrak{F}})$. Fix $n \geq 1$.

Let $Q^1_{r,(\mu,\nu)}$ be the probability law of the process (X', Y') defined in Remark 6.3. Before determining the probability law $Q^2_{r,(\mu,\nu)}$, we need to introduce further notation. Let

$$j^{(n)}(t) \equiv \begin{cases} 1, & \dfrac{2k}{n} \leq t \leq \dfrac{2k+1}{n}, k = 0,1,2,\dots, \\ 0, & \text{otherwise} \end{cases}$$

and

$$V^n_{r,t}(\mu_1 + i\mu_2) \equiv \begin{cases} V_{r,t}(\mu_1 + i\mu_2, j^{(n)}), & \text{if } (\mu_1,\mu_2) \in M^\rho_F(\mathrm{R}^d)\times S'^\rho(\mathrm{R}^d), \\ 0, & \text{otherwise,} \end{cases}$$

where $V_{r,t}$ is defined as in Section 5. Put $V^{1,n} = \mathrm{Re}(V^n)$, $V^{2,n} = \mathrm{Im}(V^n)$. For given $r \geq 0$ and $(\mu_1,\mu_2) \in M_F(\mathrm{R}^d)\times S'(\mathrm{R}^d)$ let $Q^2_{r,(\mu,\nu)}$ be the law of the deterministic process $(V^{1,n}_{r,t}, V^{2,n}_{r,t})_{t\geq r}$ starting at $\big(r,(\mu_1,\mu_2)\big)$.

Now we are ready to construct $\tilde{P}^{(n)}$ on $\tilde{\mathfrak{F}}_{l/n}$ by induction on l as follows. Fix arbitrary $(\phi_1,\phi_2) \in S(R^d)_+ \times S(R^d)$. Let

$$\tilde{P}^{(n)}|_{\tilde{\mathfrak{F}}_{1/n}} = Q^2_{0,(\phi_1,\phi_2)}|_{\tilde{\mathfrak{F}}_{1/n}}.$$

Since $(\phi_1,\phi_2) \in S(R^d)_+ \times S(R^d) \subset M^\rho_F(\mathrm{R}^d)\times S'^\rho(\mathrm{R}^d)$, we have

$$(\tilde{X}_t,\tilde{Y}_t) = \big(V^{1,n}_{0,t}(\phi_1 + i\phi_2), V^{2,n}_{0,t}(\phi_1 + i\phi_2)\big), \quad 0 \leq t \leq 1/n. \tag{7.1}$$

Let $Q^1_{1/n,(\tilde{X}_{1/n},\tilde{Y}_{1/n})}|_{\tilde{\mathfrak{F}}_{[1/n,2/n]}}$ be the regular conditional distribution of $\tilde{P}^{(n)}|_{\tilde{\mathfrak{F}}_{[1/n,2/n]}}$ given $\tilde{\mathfrak{F}}_{1/n}$. By Theorem 5.8(a) $(\tilde{X}_{1/n},\tilde{Y}_{1/n}) \in \big(L^1(\mathrm{R}^d)\cap L^2(\mathrm{R}^d)\big)_+ \times L^2(\mathrm{R}^d)_+$. Therefore by Lemma 6.5

$$(\tilde{X}_{2/n},\tilde{Y}_{2/n}) \in M^\rho_F(\mathrm{R}^d)\times S'^\rho(\mathrm{R}^d), \quad \tilde{P}^{(n)}\text{-a.s.} \tag{7.2}$$

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Let $Q^2_{2/n,(\tilde X_{2/n},\tilde Y_{2/n})}|_{\tilde{\mathcal F}_{[2/n,3/n]}}$ be the regular conditional distribution of $\tilde P^{(n)}|_{\tilde{\mathcal F}_{[2/n,3/n]}}$ given $\tilde{\mathcal F}_{2/n}$. By (7.2) we get

$$
\begin{aligned}
(\tilde X_t,\tilde Y_t) &= \big(V^{1,n}_{2/n,t}(\tilde X_{2/n}+i\tilde Y_{2/n}),V^{2,n}_{2/n,t}(\tilde X_{2/n}+i\tilde Y_{2/n})\big)\\
&= \big(V^{1}_{2/n,t}(\tilde X_{2/n}+i\tilde Y_{2/n},j^{(n)}),V^{2}_{2/n,t}(\tilde X_{2/n}+i\tilde Y_{2/n},j^{(n)})\big),\quad 2/n\le t\le 3/n,
\end{aligned}
\tag{7.3}
$$

$\tilde P^{(n)}$-a.s. Continuing in this way, we define $\tilde P^{(n)}$ on $\tilde{\mathcal F}$.

Roughly speaking, we defined the alternating $M_F\times S'$-valued process $(\tilde X,\tilde Y)$ which starts at (ϕ_1,ϕ_2), and evolves as $\big(V^{1,n}_{0,t}(\phi_1+i\phi_2),V^{2,n}_{0,t}(\phi_1+i\phi_2)\big)$ until time $t=1/n$. In the interval $[1/n,2/n]$ the process follows the paths of the (X',Y') processes constructed in the Section 6, starting at $\Big(\big(V^{1,n}_{0,1/n}(\phi_1+i\phi_2),V^{2,n}_{0,1/n}(\phi_1+i\phi_2)\big),1/n\Big)$. In the interval $[2/n,3/n]$ $(\tilde X,\tilde Y)$ evolves again as a solution to (5.3) starting at $\big((\tilde X_{2/n},\tilde Y_{2/n}),2/n\big)$, and the pattern of alternating deterministic and stochastic processes continues.

Let

$$\tilde P^{(n)}_t(B)=\tilde P^{(n)}\big((\tilde X_t,\tilde Y_t)\in B\big),\quad \forall B\in\mathcal B(M_F\times S'),\quad t\ge 0.$$

Let

$$j(t)\equiv 1/2,\quad \forall t\ge 0.$$

Define

$$V_t(\mu)\equiv\begin{cases}V_{0,t}(\mu,j), & \text{if } \mu\in\tilde{S'}^{\rho}(\mathrm R^d),\\ 0, & \text{otherwise,}\end{cases}$$

$$U^n_{r,t}(\mu,\cdot)\equiv\begin{cases}U_{r,t}\big(V^n_{r,\cdot}(\mu),\cdot\big), & \text{if } \mu\in\tilde{S'}^{\rho}(\mathrm R^d),\\ 0, & \text{otherwise,}\end{cases}$$

$$U_t(\mu,\cdot)\equiv\begin{cases}U_{0,t}\big(V_\cdot(\mu),\cdot\big), & \text{if } \mu\in\tilde{S'}^{\rho}(\mathrm R^d),\\ 0, & \text{otherwise.}\end{cases}$$

In order to simplify our notation, it is useful to define $\tilde{S'}^{\rho}$-valued processes $(H_t)_{t\ge 0}$ and $(\overline H_t)_{t\ge 0}$ by setting

$$H_t\equiv\tilde X_t+i\tilde Y_t,\quad \overline H_t\equiv\tilde X_t-i\tilde Y_t,\quad t\ge 0.$$

The rest of this section is devoted to the proof of the following two results.

Proposition 7.1 *For all $T>0$ and $(\psi_1,\psi_2)\in S(R^d)_+\times S(R^d)_+$,*

$$
\begin{aligned}
&\exp\{-\langle\psi_1,V^n_{t,T}(H_t)\rangle-\langle\psi_2,V^n_{t,T}(\overline H_t)\rangle\}\\
&\quad-\int_0^t \exp\{-\langle\psi_1,V^n_{s,T}(H_s)\rangle-\langle\psi_2,V^n_{s,T}(\overline H_s)\rangle\}\\
&\quad\times 4\lambda j^{(n)}\left(s+\frac1n\right)\int_{\mathrm R^d}U^n_{s,T}(H_s,x)(\psi_1)U^n_{s,T}(\overline H_s,x)(\psi_2)\tilde X_s\,(dx)\,ds
\end{aligned}
\tag{7.4}
$$

is a $\tilde P^{(n)}$-martingale on $0\le t\le T$.

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Theorem 7.2 *There exists* $\tilde{P}_t \in C_{M_1(M_F \times S')}[0,\infty)$ *which satisfies the following equation:*

(7.5)
$$\begin{aligned}\tilde{P}_t[\exp\{-\langle \nu_1, V_{T-t}(H_t)\rangle - \langle \nu_2, V_{T-t}(\overline{H}_t)\rangle\}]\\ = \exp\{-\langle \nu_1, V_T(H_0)\rangle - \langle \nu_2, V_T(\overline{H}_0)\rangle\}\\ + \int_0^t \tilde{P}_s\Big[\exp\{-\langle \nu_1, V_{T-s}(H_s)\rangle - \langle \nu_2 V_{T-s}(\overline{H}_s)\rangle\}\\ \times 2\lambda \int_{\mathrm{R}^d} U_{T-s}(H_s,x)(\nu_1)U_{T-s}(\overline{H}_s,x)(\nu_2)\tilde{X}_s(dx)\Big]\,ds, \quad 0 \le t < T,\end{aligned}$$

for all $T > 0$ *and* $(\nu_1,\nu_2) \in M_F^\rho \times M_F^\rho$*. Moreover* $\tilde{P}_t(M_F^\rho \times S'^\rho) = 1$*, for all* $t \ge 0$*.*

The above theorem is the key to proving our main uniqueness result in Section 8.

Remark 7.3 By our construction,

$$(\tilde{X}_t, \tilde{Y}_t) \in M_F^\rho \times S'^\rho \tag{7.6}$$

and, as a consequence, $H_t, \overline{H}_t \in \tilde{S}'^\rho$ $\tilde{P}^{(n)}$-a.s., for all $t \ge 0$. Therefore, everywhere throughout this section, we will treat $V^n_{t,T}(H_t)$ (resp. $U^n_{t,T}(H_t,\cdot)$) as $V_{t,T}(H_t, j^{(n)})$ (resp. $U_{t,T}\big(V^n_{t,\cdot},(H_t),\cdot\big)$).

7.2 Proof of Proposition 7.1

We will prove Proposition 7.1 via a series of lemmas. We start with two technical lemmas that will be extensively used in the proof.

Lemma 7.4 *For any* $T > 0$, $\mu \in M_F$ *there exists a superprocess* $\hat{X}$ *defined on* $[0,T]$ *such that*

$$\hat{X}_0 = \mu \tag{7.7}$$

$$\hat{P}[e^{-\langle \hat{X}_{T-t},\phi\rangle}] = e^{-\langle \mu, V^n_{t,T}(\phi)\rangle}, \quad \forall\, \phi \in \tilde{S}(\mathrm{R}^d). \tag{7.8}$$

For each $\phi \in \tilde{S}'^\rho$ *and for each t such that* $\frac{2m+1}{n} \vee 0 \le T - t \le \frac{2m+2}{n} \wedge T$*, we also have*

$$\hat{P}[e^{-\langle \hat{X}_{T-t},\phi\rangle}] = e^{-\langle \mu, V^n_{t,T}(\phi)\rangle}, \tag{7.9}$$

$$\hat{P}[e^{-\langle \hat{X}_{T-t},\phi\rangle}\hat{X}_{T-t}(x)] = e^{-\langle \mu, V^n_{t,T}(\phi)\rangle}\left\langle \mu,\, U^n_{t,T}(\phi,x)\right\rangle, \quad \textit{for q.e. } x \in \mathrm{R}^d. \tag{7.10}$$

Proof The existence of a superprocess $\hat{X}$ which satisfies (7.7)–(7.8) for $\phi \in S(R^d)_+$ follows from [9, Theorem 1.1] and then the extension to $\phi \in \tilde{S}(\mathrm{R}^d)$ is straightforward. Let us mention some simple properties of $\hat{X}$. Take any m such that $0 \le m \le Tn/2$. Then $\hat{X}$ evolves as a super-Brownian motion on the time interval $[T - \frac{2m+1}{n} \vee 0, T - \frac{2m}{n}]$ starting at $\hat{X}_{T-\frac{2m+1}{n}}$. On the time interval $[T - \frac{2m+2}{n} \vee 0, T - \frac{2m+1}{n}]$ $\hat{X}$ solves the heat equation

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starting at $\hat{X}_{T-\frac{2m+2}{n}}$. In particular, this implies that $\hat{X}$ is function-valued on the time interval $[T-\frac{2m+2}{n}\vee 0, T-\frac{2m+1}{n}]$ and, so, we can give the rigorous meaning to the left sides of (7.9) and (7.10).

For $\phi \in \tilde{S}'^{\rho}$ let $\{\phi^{(k)}\}$ be a sequence in $\tilde{S}(\mathrm{R}^d)$ such that $\lim_{k\to\infty}\phi^{(k)} = \phi$ in $\tilde{S}'^{\rho}$ (such sequence exists by Lemma 5.4). By (7.8) we have that $\hat{P}[\exp-\langle \hat{X}_{T-t},\phi^{(k)}\rangle] = \exp-\langle \mu, V^n_{t,T}(\phi^{(k)})\rangle$ for each k. Let $k\to\infty$ and apply Theorem 5.8 (d) to get (7.9).

Turning to (7.10), note that, for each $\phi\in\tilde{S}(\mathrm{R}^d)$ and $\psi\in S(R^d)_+$,

$$\begin{aligned}\hat{P}[e^{-\langle \hat{X}_{T-t},\phi\rangle}\langle \hat{X}_{T-t},\psi\rangle] &= -\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\hat{P}[e^{-\langle \hat{X}_{T-t},\phi+\epsilon\psi\rangle}-e^{-\langle \hat{X}_{T-t},\phi\rangle}]\\ &= -\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\big(e^{-\langle \mu, V^n_{t,T}(\phi+\epsilon\psi)\rangle}-e^{-\langle \mu, V^n_{t,T}(\phi)\rangle}\big)\\ &= e^{-\langle \mu, V^n_{t,T}(\phi)\rangle}\big\langle \mu,\, U^n_{t,T}(\phi,\psi)\big\rangle\end{aligned}$$

(see Section 6.3 of [7] for a similar result). For a general $\phi\in\tilde{S}'^{\rho}$ choose $\{\phi^{(k)}\}$ in $\tilde{S}(\mathrm{R}^d)$ such that $\lim_{k\to\infty}\phi^{(k)}=\phi$ in $\tilde{S}'^{\rho}$ and use Theorem 5.9(b), (d) to get

$$\lim_{k\to\infty} U^n_{t,T}(\phi^{(k)},\psi) = U^n_{t,T}(\phi,\psi) = \int_{\mathrm{R}^d} U^n_{t,T}(\phi,x)\psi(x)\,dx.$$

By Theorem 5.9 $U^n_{t,T}(\phi,\cdot)$ is quasicontinuous. Therefore Proposition 6.1.3 of [1] implies that $\int_{\mathrm{R}^d} U^n_{t,T}(\phi,x)\psi(y)\,dy \to U^n_{t,T}(\phi,x)$ as $\psi\to\delta_x$, for q.e. x, and (7.10) follows. ■

Lemma 7.5 *For all* $0<s<T$, $\nu_1,\nu_2\in M_F$,

$$\begin{aligned}&\int_{\mathrm{R}^d} |U^n_{s,T}(H_s,x)(\nu_1)U^n_{s,T}(\overline{H}_s,x)(\nu_2)|\tilde{X}_s\,(dx)\\ &\qquad\le \int_{\mathrm{R}^d} S_{T-s}(\nu_1)(x)S_{T-s}(\nu_2)(x)\tilde{X}_s\,(dx),\quad \tilde{P}^{(n)}\text{-a.s.}\end{aligned}$$

Proof By Theorem 5.9 $|U^n_{s,T}(\tilde{X}_s\pm\tilde{Y}_s,x)(\nu)|\le S_{T-s}(\nu)(x)$ for all $\nu\in M_F$ and q.e. $x\in\mathrm{R}^d$. By (7.6), $\tilde{X}_s\in M_F^{\rho}$ $\tilde{P}^{(n)}$-a.s. for all $s\ge 0$. Therefore, by Lemma 5.7, $\tilde{X}_s$ does not charge sets of nil capacity $\tilde{P}^{(n)}$-a.s., and the desired result follows. ■

In our proof of Proposition 7.1 we will use localization arguments. Let us define the stopped version of the canonical process $(\tilde{X},\tilde{Y})$. Let $\{B_k, k\ge 1\}$ be a sequence of open sets in $M_F^{\rho}\times S'^{\rho}$, such that $\lim_{k\to\infty}B_k = M_F^{\rho}\times S'^{\rho}$. We also assume that for each k there exists Γ_k—a compact set in $M_F^{\rho}\times S'^{\rho}$ such that $B_k\subset\Gamma_k$.

Let

$$\tau_k \equiv \inf\{t\ge 0 : (\tilde{X}_t,\tilde{Y}_t)\notin B_k\},$$

$$\tau^{(k)}(t)\equiv \tau_k\wedge t.$$

Then τ_k is an $\tilde{\mathfrak{F}}_t$-stopping time. Define

$$\tilde{X}_t^{(k)} \equiv \tilde{X}_{\tau^{(k)}(t)}, \quad \tilde{Y}_t^{(k)} \equiv \tilde{Y}_{\tau^{(k)}(t)},$$
$$H_t^{(k)} \equiv H_{\tau^{(k)}(t)}, \quad \overline{H}_t^{(k)} \equiv \overline{H}_{\tau^{(k)}(t)}.$$

The construction of $\tilde{P}^{(n)}$, and the fact that $\nu_m \equiv \tilde{P}^{(n)^{-1}}(\tilde{X}^{(k)}_{\frac{2m+1}{n}}, \tilde{Y}^{(k)}_{\frac{2m+1}{n}}) \in M_{1,c}(M_F \times S')$ for all $m \geq 0$, and the optional stopping theorem imply that for each $m \geq 0$ the process $(\tilde{X}_t^{(k)}, \tilde{Y}_t^{(k)})$ satisfies $\tilde{M}^{2(m+1)/n,\tau_k}_{\frac{2m+1}{n},\nu_m}$ on the interval $[\frac{2m+1}{n}, \frac{2m+2}{n}]$ (see Section 6 for the definition of $\tilde{M}^{2(m+1)/n,\tau_k}_{\frac{2m+1}{n},\nu_m}$).

Let us introduce further notation. For all $T_1, T_2 > 0$ and $(\psi_1, \psi_2) \in S(R^d) \times S(R^d)$, define

$$N^1_{t,T_1,T_2}(\psi_1,\psi_2) \equiv \left\langle \psi_1, V^n_{t,T_1}(H_t)\right\rangle + \left\langle \psi_2, V^n_{t,T_2}(\overline{H}_t)\right\rangle, \quad 0 \leq t \leq T_1 \wedge T_2,$$
$$N^2_{t,T_1,T_2}(\psi_1,\psi_2) \equiv \left\langle \psi_1, V^n_{t,T_1}(H_t)\right\rangle + \left\langle \psi_2, V^n_{t,T_2}(H_t)\right\rangle, \quad 0 \leq t \leq T_1 \wedge T_2.$$

Lemma 7.6 *For all $T_1, T_2 > 0$ and $(\psi_1, \psi_2) \in S(R^d)_+ \times S(R^d)_+$,*

$$\exp\{-N^1_{\tau^{(k)}(t),T_1,T_2}(\psi_1,\psi_2)\} - \int_0^{\tau^{(k)}(t)} \exp\{-N^1_{s,T_1,T_2}(\psi_1,\psi_2)\} \times 4\lambda j^{(n)}\left(s+\frac{1}{n}\right) \int_{\mathrm{R}^d} U^n_{s,T_1}(H_s,x)(\psi_1) U^n_{s,T_2}(\overline{H}_s,x)(\psi_2)\tilde{X}_s\,(dx)\,ds \tag{7.11}$$

and

$$\exp\{-N^2_{\tau^{(k)}(t),T_1,T_2}(\psi_1,\psi_2)\} \tag{7.12}$$

are $\tilde{P}^{(n)}$-martingales on $0 \leq t \leq T$, where $T = T_1 \wedge T_2$.

Proof We will prove the assertion only for (7.11) since the proof for (7.12) is the same.

Let $t \in [2m/n, (2m+1)/n]$ for some $m \geq 0$. Then the construction of $\tilde{P}^{(n)}$ and $\tilde{X}^{(k)}$ implies that $\tilde{X}_t^{(k)} \pm i\tilde{Y}_t^{(k)} = V^n_{\tau^{(k)}(2m/n),\tau^{(k)}(t)}(\tilde{X}^{(k)}_{2m/n} \pm i\tilde{Y}^{(k)}_{2m/n})$. Therefore, by the semigroup property of $V^n_{r,t}$ ($V^n_{r,t}V^n_{s,r} = V^n_{s,t}$ for $s \leq r \leq t$), we obtain that

$$V^n_{\tau^{(k)}(t),T_l}(\tilde{X}_t^{(k)} \pm i\tilde{Y}_t^{(k)}) = V^n_{\tau^{(k)}(2m/n),T_l}(\tilde{X}^{(k)}_{2m/n} \pm i\tilde{Y}^{(k)}_{2m/n}), \quad l = 1, 2,$$

is a constant on $\frac{2m}{n} \leq t \leq \frac{2m+1}{n}$. The fact that

$$\exp\{-\langle\psi_1, V^n_{\tau^{(k)}(t),T_1}(H_t^{(k)})\rangle - \langle\psi_2, V^n_{\tau^{(k)}(t),T_2}(\overline{H}_t^{(k)})\rangle\} \tag{7.13}$$

is a martingale on $[\frac{2m}{n}, \frac{2m+1}{n}]$ follows immediately.

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Let $t \in [\frac{2m+1}{n}, \frac{2m+2}{n}]$ for some $m \geq 0$. Let $(\hat{X}^1, \hat{X}^2)$ be a pair of independent superprocesses defined on $[0, T_1]$ and $[0, T_2]$ respectively, such that for all $f_l \in \tilde{S}(\mathrm{R}^d)$

$$\hat{P}^l[\exp\{-\langle \hat{X}^l_{T_l - t}, f_l\rangle\}] = \exp\{-\langle \psi_l, V^n_{t,T}(f_l)\rangle\}, \quad 0 \leq t \leq T_l, \quad l = 1, 2.$$

As we explained in Lemma 7.4, $\hat{X}^l$ $(l = 1, 2)$ solves the heat equation on the intervals $[T_l - \frac{2m+2}{n} \vee 0, T_l - \frac{2m+1}{n}]$ starting at $\hat{X}^l_{T_l - \frac{2m+2}{n}}$. By the way, this also implies that, for each $m \leq (T_l n - 1)/2$,

$$\hat{X}^l_{T_l - t} = S_{\frac{2m+2}{n} \wedge T_l - t} \hat{X}^l_{0 \vee T_l - \frac{2m+2}{n}}, \qquad \frac{2m+1}{n} \leq t \leq \frac{2m+2}{n} \wedge T, \quad l = 1, 2. \tag{7.14}$$

$t \in [\frac{2m+1}{n}, \frac{2m+2}{n}]$ and, in order to simplify the exposition, we will assume that $\frac{2m+2}{n} \leq T$ (the case $\frac{2m+1}{n} \leq T \leq \frac{2m+2}{n}$ can be treated in the same manner). Then we get

$$\begin{aligned} &\exp\{-\langle \psi_1, V^n_{\tau^{(k)}(t),T_1}(H^{(k)}_t)\rangle - \langle \psi_2, V^n_{\tau^{(k)}(t),T_2}(\overline{H}^{(k)}_t)\rangle\} \\ &\qquad = \hat{P}^1 \times \hat{P}^2[\exp\{-\langle \hat{X}^1_{T_1-\tau^{(k)}(t)}, H^{(k)}_t\rangle - \langle \hat{X}^2_{T_2-\tau^{(k)}(t)}, \overline{H}^{(k)}_t\rangle\}]. \end{aligned} \tag{7.15}$$

Since $\hat{X}^l_t \in M^\rho_F$ $\hat{P}^l$-a.s. for all $0 \leq t \leq T_l$, therefore all the variables above are well defined $\hat{P}^1 \times \hat{P}^2 \times \tilde{P}^{(n)}$-a.s.

By (7.14) we have that, for each $t \in [\frac{2m+1}{n}, \frac{2m+2}{n}]$,

$$\begin{aligned} &\langle \hat{X}^1_{T_1-\tau^{(k)}(t)} + \hat{X}^2_{T_2-\tau^{(k)}(t)}, \tilde{X}^{(k)}_t\rangle \\ &\qquad \equiv 1\big(\tau^{(k)}(t) < (2m+1)/n\big)\langle \hat{X}^1_{T_1-\tau^{(k)}(t)} + \hat{X}^2_{T_2-\tau^{(k)}(t)}, \tilde{X}^{(k)}_t\rangle \\ &\qquad\qquad + 1\big(\tau^{(k)}(t) \geq (2m+1)/n\big)\langle S_{\frac{2m+2}{n}-\tau^{(k)}(t)}(\hat{X}^1_{T_1-\frac{2m+2}{n}} + \hat{X}^2_{T_2-\frac{2m+2}{n}}), \tilde{X}^{(k)}_t\rangle, \end{aligned} \tag{7.16}$$

$$\begin{aligned} &\langle \hat{X}^1_{T_1-\tau^{(k)}(t)} - \hat{X}^2_{T_2-\tau^{(k)}(t)}, \tilde{Y}^{(k)}_t\rangle \\ &\qquad \equiv 1\big(\tau^{(k)}(t) < (2m+1)/n\big)\langle \hat{X}^1_{T_1-\tau^{(k)}(t)} - \hat{X}^2_{T_2-\tau^{(k)}(t)}, \tilde{Y}^{(k)}_t\rangle \\ &\qquad\qquad + 1\big(\tau^{(k)}(t) \geq (2m+1)/n\big)\langle S_{\frac{2m+2}{n}-\tau^{(k)}(t)}(\hat{X}^1_{T_1-\frac{2m+2}{n}} - \hat{X}^2_{T_2-\frac{2m+2}{n}}), \tilde{Y}^{(k)}_t\rangle. \end{aligned} \tag{7.17}$$

Since $(\tilde{X}^{(k)}, \tilde{Y}^{(k)})$ satisfies the martingale problem $\tilde{M}^{2(m+1)/n,\tau_k}_{\frac{2m+1}{n},\nu_m}$, the last terms in (7.16), (7.17) may be rewritten as

$$\begin{aligned} &1\big(\tau^{(k)}(t) \geq (2m+1)/n\big)\langle S_{\frac{2m+2}{n}-\tau^{(k)}(t)}(\hat{X}^1_{T_1-\frac{2m+2}{n}} + \hat{X}^2_{T_2-\frac{2m+2}{n}}), \tilde{X}^{(k)}_t\rangle \\ &\qquad = 1\big(\tau^{(k)}(t) \geq (2m+1)/n\big)\langle S_{\frac{1}{n}}(\hat{X}^1_{T_1-\frac{2m+2}{n}} + \hat{X}^2_{T_2-\frac{2m+2}{n}}), \tilde{X}^{(k)}_{\frac{2m+1}{n}}\rangle + Z^{m,1}_t, \end{aligned} \tag{7.18}$$

$$\begin{aligned} &1\big(\tau^{(k)}(t) \geq (2m+1)/n\big)\langle S_{\frac{2m+2}{n}-\tau^{(k)}(t)}(\hat{X}^1_{T_1-\frac{2m+2}{n}} - \hat{X}^2_{T_2-\frac{2m+2}{n}}), \tilde{Y}^{(k)}_t\rangle \\ &\qquad = 1\big(\tau^{(k)}(t) \geq (2m+1)/n\big)\langle S_{\frac{1}{n}}(\hat{X}^1_{T_1-\frac{2m+2}{n}} - \hat{X}^2_{T_2-\frac{2m+2}{n}}), \tilde{Y}^{(k)}_{\frac{2m+1}{n}}\rangle + Z^{m,2}_t, \end{aligned} \tag{7.19}$$

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where $Z^{m,1}, Z^{m,2}$ are continuous square integrable martingales on $[\frac{2m+1}{n}, \frac{2m+2}{n}]$ such that $Z^{m,1}_{\frac{2m+1}{n}} = Z^{m,2}_{\frac{2m+1}{n}} = 0$, $\langle Z^{m,1}, Z^{m,2}\rangle_t = 0$ and

$$\begin{aligned}\langle Z^{m,1}\rangle_t &= 2\lambda \int_{(2m+1)/n}^t 1(\tau^{(k)} \ge s)\big\langle \big(S_{\frac{2m+2}{n}-s}(\hat{X}^1_{T_1-\frac{2m+2}{n}} + \hat{X}^2_{T_2-\frac{2m+2}{n}})\big)^2, \tilde{X}^{(k)}_s\big\rangle\, ds \\ &= 2\lambda \int_{(2m+1)/n}^t 1(\tau^{(k)} \ge s)\big\langle \big(\hat{X}^1_{T_1-s} + \hat{X}^2_{T_2-s}\big)^2, \tilde{X}^{(k)}_s\big\rangle\, ds,\end{aligned} \tag{7.20}$$

$$\begin{aligned}\langle Z^{m,2}\rangle_t &= 2\lambda \int_{(2m+1)/n}^t 1(\tau^{(k)} \ge s)\big\langle \big(S_{\frac{2m+2}{n}-s}(\hat{X}^1_{T_1-\frac{2m+2}{n}} - \hat{X}^2_{T_2-\frac{2m+2}{n}})\big)^2, \tilde{X}^{(k)}_s\big\rangle\, ds \\ &= 2\lambda \int_{(2m+1)/n}^t 1(\tau^{(k)} \ge s)\langle (\hat{X}^1_{T_1-s} - \hat{X}^2_{T_2-s})^2, \tilde{X}^{(k)}_s\rangle\, ds,\end{aligned} \tag{7.21}$$

where for the second equalities in (7.20), (7.21) we use (7.14). Using (7.14) again and combining (7.16–7.19) we get

$$\begin{aligned}\langle \hat{X}^1_{T_1-\tau^{(k)}(t)} + \hat{X}^2_{T_2-\tau^{(k)}(t)}, \tilde{X}^{(k)}_t\rangle &= \langle \hat{X}^1_{T_1-\tau^{(k)}(\frac{2m+1}{n})} + \hat{X}^2_{T_2-\tau^{(k)}(\frac{2m+1}{n})}, \tilde{X}^{(k)}_{\frac{2m+1}{n}}\rangle + Z^{m,1}_t, \\ \langle \hat{X}^1_{T_1-\tau^{(k)}(t)} - \hat{X}^2_{T_2-\tau^{(k)}(t)}, \tilde{Y}^{(k)}_t\rangle &= \langle \hat{X}^1_{T_1-\tau^{(k)}(\frac{2m+1}{n})} - \hat{X}^2_{T_2-\tau^{(k)}(\frac{2m+1}{n})}, \tilde{Y}^{(k)}_{\frac{2m+1}{n}}\rangle + Z^{m,2}_t.\end{aligned}$$

We apply Itô's formula and obtain that

$$\begin{aligned}&e^{-\langle \hat{X}^1_{T_1-\tau^{(k)}(t)}, H^{(k)}_t\rangle - \langle \hat{X}^2_{T_2-\tau^{(k)}(t)}, \overline{H}^{(k)}_t\rangle} \\ &\quad - \int_{\frac{2m+1}{n}}^t 1\big(s \le \tau^{(k)}(t)\big) e^{-\langle \hat{X}^1_{T_1-\tau^{(k)}(s)}, H^{(k)}_s\rangle - \langle \hat{X}^2_{T_2-\tau^{(k)}(s)}, \overline{H}^{(k)}_s\rangle} 4\lambda \langle \hat{X}^1_{T_1-s}\hat{X}^2_{T_2-s}, \tilde{X}_s\rangle\, ds\end{aligned}$$

is a martingale on $[(2m+1)/n, (2m+2)/n]$ for $\hat{P}^1 \times \hat{P}^2$-a.s $(\hat{X}^1, \hat{X}^2)$. Now by checking the conditions of the stochastic Fubini theorem (*e.g.* [20, Theorem 2.6]) we may conclude that

$$\begin{aligned}&\hat{P}^1 \times \hat{P}^2[e^{-\langle \hat{X}^1_{T_1-\tau^{(k)}(t)}, H^{(k)}_t\rangle - \langle \hat{X}^2_{T_2-\tau^{(k)}(t)}, \overline{H}^{(k)}_t\rangle}] \\ &\quad - \hat{P}^1 \times \hat{P}^2\Big[\int_{\frac{2m+1}{n}}^{\frac{2m+1}{n}\vee\tau^{(k)}(t)} e^{-\langle \hat{X}^1_{T_1-\tau^{(k)}(s)}, H^{(k)}_s\rangle - \langle \hat{X}^2_{T_2-\tau^{(k)}(s)}, \overline{H}^{(k)}_s\rangle} 4\lambda\langle \hat{X}^1_{T_1-s}\hat{X}^2_{T_2-s}, \tilde{X}_s\rangle\, ds\Big]\end{aligned}$$

is a martingale on $[(2m+1)/n, (2m+2)/n]$. By (7.15) and by the ordinary Fubini theorem we obtain that

$$\begin{aligned}&e^{-\langle \psi_1, V^n_{\tau^{(k)}(t),T_1}(H^{(k)}_t)\rangle - \langle \psi_2, V^n_{\tau^{(k)}(t),T_2}(\overline{H}^{(k)}_t)\rangle} \\ &\quad - \int_{\frac{2m+1}{n}}^{\frac{2m+1}{n}\vee\tau^{(k)}(t)} \hat{P}^1 \times \hat{P}^2[e^{-\langle \hat{X}^1_{T_1-\tau^{(k)}(s)}, H^{(k)}_s\rangle - \langle \hat{X}^2_{T_2-\tau^{(k)}(s)}, \overline{H}^{(k)}_s\rangle} 4\lambda\langle \hat{X}^1_{T_1-s}\hat{X}^2_{T_2-s}, \tilde{X}_s\rangle]\, ds\end{aligned}$$

is a martingale on $[(2m+1)/n,(2m+2)/n]$. By Lemma 7.4 it is easy to integrate with respect to $\hat{P}^1\times\hat{P}^2$ inside the integral and to obtain that

$$
\begin{aligned}
&e^{-\langle\psi_1,V^n_{\tau^{(k)}(t),T_1}(H^{(k)}_t)\rangle-\langle\psi_2,V^n_{\tau^{(k)}(t),T_2}(\overline{H}^{(k)}_t)\rangle}\\
&\quad-\int_{\frac{2m+1}{n}}^{\frac{2m+1}{n}\vee\tau^{(k)}(t)} e^{-\langle\psi_1,V^n_{s,T_1}(H_s)\rangle-\langle\psi_2,V^n_{s,T_2}(\overline{H}_s)\rangle}\\
&\quad\times 4\lambda\int_{\mathrm{R}^d}U^n_{s,T_1}(H_s,x)(\psi_1)U^n_{s,T_2}(\overline{H}_s,x)(\psi_2)\tilde{X}_s(dx)\,ds
\end{aligned}
\tag{7.22}
$$

is a martingale on $[(2m+1)/n,(2m+2)/n]$. (7.11) follows by (7.13), (7.22) and the fact that m was arbitrary. ■

Lemma 7.7 *For each $\psi_1,\psi_2\in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+\times C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$, $N^1_{\tau^{(k)}(t),T_1,T_2}(\psi_1,\psi_2)$ is a martingale on $[0,T_1\wedge T_2]$ with quadratic variation given by*

$$\int_0^{\tau^{(k)}(t)}8\lambda j^{(n)}\left(s+\frac{1}{n}\right)\int_{\mathrm{R}^d}U^n_{s,T_1}(H_s,x)(\psi_1)U^n_{s,T_2}(\overline{H}_s,x)(\psi_2)\tilde{X}_s(dx)\,ds$$

and $N^2_{\tau^{(k)}(t),T_1,T_2}(\psi_1,\psi_2)$ is a martingale on $[0,T_1\wedge T_2]$ with quadratic variation equal to 0.

Proof Since by truncation the process $(\tilde{X}^{(k)},\tilde{Y}^{(k)})$ "lives" on a compact subset of $M^\rho_F\times S'^\rho$ it is easy to check that

$$
\begin{aligned}
&\lim_{\epsilon\downarrow 0}-\frac{1}{\epsilon}\left(\exp\{-\epsilon N^l_{\tau^{(k)}(t),T_1,T_2}(\psi_1,\psi_2)\}-1\right)=N^l_{\tau^{(k)}(t),T_1,T_2}(\psi_1,\psi_2),\\
&\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^{\tau^{(k)}(t)}\exp\{-\epsilon N^1_{s,T_1,T_2}(\psi_1,\psi_2)\}\\
&\qquad\times\epsilon^2 4\lambda j^{(n)}\left(s+\frac{1}{n}\right)\int_{\mathrm{R}^d}U^n_{s,T_1}(H_s,x)(\psi_1)U^n_{s,T}(\overline{H}_s,x)(\psi_2)\,\tilde{X}_s(dx)\,ds=0,
\end{aligned}
$$

$\tilde{P}^{(n)}$-a.s. and in L^1 for $l=1,2$. This together with Lemma 7.6 implies that $N^l_{\tau^{(k)}(t),T_1,T_2}(\psi_1,\psi_2)$ is a continuous martingale for $l=1,2$. Applying Itô's formula, we obtain that for $l=1,2$

$$e^{-N^l_{\tau^{(k)}(t),T_1,T_2}(\psi_1,\psi_2)}-\int_0^{\tau^{(k)}(t)}e^{-N^l_{s,T_1,T_2}(\psi_1,\psi_2)}\frac{1}{2}\,d\langle N^l_\cdot(\psi_1,\psi_2)\rangle_s$$

is a martingale and the result follows from Lemma 7.6. ■

Lemma 7.7 yields the following corollary.

Corollary 7.8 *For each* $(\psi_1, \psi_2) \in \overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+ \times S(R^d)_+$, $T \geq 0$,

$$\tilde{P}^{(n)}[\langle \psi_1, V^{1,n}_{t,T}(H_t)\rangle] = \langle \psi_1, V^{1,n}_{0,T}(H_0)\rangle \leq \langle \psi_1, S_T(|H_0|)\rangle, \quad \forall\, 0 \leq t \leq T, \tag{7.23}$$

$$\begin{aligned} &\tilde{P}^{(n)}[\langle \psi_1, V^{1,n}_{t,T}(H_t)\rangle^2] \\ &\quad \leq \langle \psi_1, S_T(|H_0|)\rangle^2 \\ &\qquad + \int_0^t 2\lambda j^{(n)}\left(s + \frac{1}{n}\right)\int_{\mathrm{R}^d} S_{T-s}(\psi_1)(x)^2 S_s(|H_0|)(x)\,dx\,ds, \quad \forall\, 0 \leq t \leq T, \end{aligned} \tag{7.24}$$

$$\tilde{P}^{(n)}[\langle \psi_2, \tilde{Y}_t\rangle] = \langle \psi_2, V^{2,n}_{0,t}(H_0)\rangle, \quad t \geq 0, \tag{7.25}$$

$$\begin{aligned} &\tilde{P}^{(n)}[\langle \psi_2, \tilde{Y}_t\rangle^2] \leq \langle \psi_2, S_t(|H_0|)\rangle^2 \\ &\qquad + \int_0^t 2\lambda j^{(n)}\left(s + \frac{1}{n}\right)\int_{\mathrm{R}^d} S_{t-s}(\psi_2)(x)^2 S_s(|H_0|)(x)\,dx\,ds, \quad t \geq 0, \end{aligned} \tag{7.26}$$

$$\begin{aligned} &\tilde{P}^{(n)}[\langle \psi_l, V^{l,n}_{t,T}(H_t)\rangle^2] \\ &\quad \leq \langle \psi_l, S_T(|H_0|)\rangle^2 \\ &\qquad + \int_0^t 2\lambda j^{(n)}\left(s + \frac{1}{n}\right)\int_{\mathrm{R}^d} S_{T-s}(\psi_l)(x)^2 S_s(|H_0|)(x)\,dx\,ds, \quad l = 1, 2, \quad 0 \leq t \leq T. \end{aligned} \tag{7.27}$$

Moreover, for each $\psi_1, \psi_2 \in S(R^d)_+$, $T_1, T_2 > 0$, $0 \leq t \leq T_1 \wedge T_2$ *and* $l = 1, 2$ *we have*

$$\begin{aligned} &\tilde{P}^{(n)}[\langle \psi_1, V^{1,n}_{t,T_1}(H_t)\rangle\langle \psi_2, V^{1,n}_{t,T_2}(H_t)\rangle] \\ &\quad \leq \langle \psi_1, S_{T_1}(|H_0|)\rangle\langle \psi_2, S_{T_2}(|H_0|)\rangle \\ &\qquad + \int_0^t 2\lambda\left(s + \frac{1}{n}\right)\int_{\mathrm{R}^d} S_{T_1-s}(\psi_1)(x) S_{T_2-s}(\psi_2)(x) S_s(|H_0|)(x)\,dx\,ds. \end{aligned} \tag{7.28}$$

Remark 7.9 *The analogues of (7.25), (7.26) for* $\tilde{X}$ *are immediate from (7.23), (7.24).*

Proof By (5.4), (5.5) and the definition of $N^1_{t,\cdot,\cdot}(\cdot,\cdot)$ we get

$$\begin{aligned} \langle \psi_1, V^{1,n}_{t,T}(H_t)\rangle &= \frac{1}{2}\big(\langle \psi_1, V^n_{t,T}(H_t)\rangle + \langle \psi_1, V^n_{t,T}(\overline{H}_t)\rangle\big) \\ &= N^1_{t,T,T}\left(\frac{1}{2}\psi_1, \frac{1}{2}\psi_1\right), \\ \langle \psi_2, V^{2,n}_{t,T}(H_t)\rangle &= -i\frac{1}{2}\big(\langle \psi_2, V^n_{t,T}(H_t)\rangle - \langle \psi_2, V^n_{t,T}(\overline{H}_t)\rangle\big) \\ &= iN^1_{t,T,T}\left(\frac{1}{2}\psi_2, -\frac{1}{2}\psi_2\right). \end{aligned}$$

A bit of calculation based on Lemma 7.7 yields $\langle iN^1_{\cdot,T,T}(\frac{1}{2}\psi_2, -\frac{1}{2}\psi_2)\rangle_t = \langle N^1_{\cdot,T,T}(\frac{1}{2}\psi_2, \frac{1}{2}\psi_2)\rangle_t$ for each $\psi_2 \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$. Hence, for each $(\psi_1, \psi_2) \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+ \times C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$, the corollary follows easily from Lemma 7.7 by passing to the limit as $k \to \infty$ and using bounds from Lemma 7.5 and Lemma 5.2. For $(\psi_1, \psi_2) \in \overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+ \times S(R^d)_+$ the result follows by an approximation of (ψ_1, ψ_2) with functions from $C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+ \times C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$. ■

Proof of Proposition 7.1 Take $T_1 = T_2 = T$ in (7.11) and let $k \to \infty$. $(\tilde{X}^{(k)}, \tilde{Y}^{(k)}) \to (\tilde{X}, \tilde{Y})$ and $\tau^{(k)}(t) \to t$ $\tilde{P}^{(n)}$-a.s. By Corollary 7.8 all the random variables in (7.11) are uniformly integrable. This immediately yields the result. ■

7.3 Regularity Properties of the Approximating Dual Measures

Lemma 7.10 *For each $t > 0$, $\delta > 0$, there exist constants $C_1(\rho, t, \delta), C_2(\rho, t, \delta)$ such that $\forall\, n \geq 1$*

$$\tilde{P}^{(n)}\left[\int_0^\delta s^{\rho-1} w(s, \tilde{X}_t)\, ds\right] \leq C_1(\rho, t, \delta),$$

$$\tilde{P}^{(n)}\left[\int_0^\delta s^{\rho-1} w(s, \tilde{Y}_t)\, ds\right] \leq C_2(\rho, t, \delta),$$

where

$$\lim_{\delta \downarrow 0} C_l(\rho, t, \delta) = 0, \quad l = 1, 2$$

uniformly on $0 \leq t \leq T$, $n \geq 1$, for all $T > 0$.

Proof By the Fubini theorem and (7.26) we get

$$\begin{aligned}
&\tilde{P}^{(n)}\left[\int_0^\delta s^{\rho-1} w(s, \tilde{Y}_t)\, ds\right] \\
&\quad = \int_0^\delta \tilde{P}^{(n)}[s^{\rho-1} w(s, \tilde{Y}_t)]\, ds \\
&\quad \leq \int_0^\delta s^{\rho-1} S_s\big(S_t(|H_0|)\big)(x)^2\, dx\, ds \\
&\qquad\quad + \int_0^\delta s^{\rho-1} \int_0^t 2\lambda j^{(n)}\left(u + \frac{1}{n}\right) \int_{\mathrm{R}^d} p_{t+s-u}(x-y)^2 S_u(|H_0|)(y)\, dy\, du\, dx\, ds \\
&\quad \leq \|H_0\|_\infty \|H_0\|_1 \frac{\delta^\rho}{\rho} + C_d \|H_0\|_1 \int_0^\delta s^{\rho-1} \int_0^t (t-u+s)^{-d/2}\, du\, ds \\
&\quad \equiv C_2(\rho, t, \delta),
\end{aligned}$$

and the result for $C_2(\rho, t, \delta)$ follows immediately for our choice of ρ. The proof for $C_1(\rho, t, \delta)$ is the same. ■

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Lemma 7.11 *For each $\epsilon > 0$, there exists $A_\epsilon \subset M_F^\rho \times S'^\rho$ such that A_ϵ is a closed subset of $M_F \times S'$ and*

$$\sup_{0\le t\le T} \tilde{P}_t^{(n)}[A_\epsilon^c] \le \epsilon$$

for all $n \ge 1$.

Proof By the previous lemma we can always find a sequence $\delta_k \downarrow 0$ such that

$$\sup_n \sup_{0\le t\le T} \tilde{P}_t^{(n)}\left[\int_0^{\delta_k} s^{\rho-1}\big(w(s,\tilde{X}_t) + w(s,\tilde{Y}_t)\big)\,ds\right] \le \frac{\epsilon}{k2^{k+1}}. \tag{7.29}$$

Define

$$A_\epsilon \equiv \left\{(\mu_1,\mu_2) \in M_F \times S'(\mathrm{R}^d) : \int_0^{\delta_k} s^{\rho-1}\big(w(s,\mu_1) + w(s,\mu_2)\big)\,ds \le \frac{1}{k}, \quad \forall\, k > 0\right\}.$$

By Corollary 5.6(a) A_ϵ is closed in $M_F \times S'$, and by (7.29) $\tilde{P}^{(n)}(A_\epsilon^c) \le \epsilon$ for all $n \ge 1$. ■

7.4 Proof of Theorem 7.2

In this subsection we will prove Theorem 7.2 and investigate the properties of the limiting dual measures.

In a moment we will formulate the lemma which gives the tightness of the measures $\{\tilde{P}_t^{(n)}, 0 \le t \le T, n \ge 1\}$ for any $T > 0$ (which sometimes is called the compact containment condition), where by tightness we mean the tightness of probability measures on $M_F \times S'$. However, we also need the *compact containment condition* (in the following we abbreviate it by CCC) on $M_F^\rho \times S'^\rho$ for this set of probability measures. Therefore the following definition is in order.

Definition 7.12 Let $\{\mu_t^{(n)}, t \in \mathrm{R}_+, n \ge 1\} \subset M_1(M_F \times S')$. Then we say that $\{\mu_t^{(n)}, t \in \mathrm{R}_+, n \ge 1\}$ satisfies CCC on $M_F^\rho \times S'^\rho$ if, for each $\epsilon > 0$, $T > 0$, there exists a compact set $B_{\epsilon,T}$ in $(M_F^\rho \times S'^\rho, \tau^\rho)$ such that

$$\sup_{n\ge 1} \sup_{0\le t\le T} \mu_t^{(n)}(B_\epsilon^c) \le \epsilon. \tag{7.30}$$

Recall that τ^ρ denotes the topology, corresponding to the convergence introduced in Definition 5.3.

Lemma 7.13 *For each $T > 0$, the set of probability measures $\{\tilde{P}_t^{(n)}, 0 \le t \le T, n \ge 1\}$ is tight in $M_1(M_F \times S')$.*

Proof By (7.24), and (7.26) we get

$$\sup_{t\le T, n\ge 1} \tilde{P}_t^{(n)}[\tilde{X}_t(\psi_1)^2] < \infty, \quad \forall\, T > 0, \quad \forall\, \psi_1 \in \overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+$$

$$\sup_{t\le T, n\ge 1} \tilde{P}_t^{(n)}[\tilde{Y}_t(\psi_2)^2] < \infty, \quad \forall\, T > 0, \quad \forall\, \psi_2 \in S(R^d)_+.$$

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This yields the tightness of $\{(\tilde{X}_t(\psi_1), \tilde{Y}_t(\psi_2)) : 0 \le t \le T, n \ge 1\}$ for each $(\psi_1, \psi_2) \in \overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+ \times S(R^d)_+$. By Mitoma's theorem [16] we get the tightness of our set of measures in $M_1(S'_+ \times S')$. (Our case is even simpler than the one covered by Mitoma's theorem which deals with probability measures on $D_{S'}[0,\infty)$, while our concern is about the probability measures on S'.) Since $\sup_{t\le T, n\ge 1} \tilde{P}^{(n)}_t[\tilde{X}_t(1)^2] < \infty$, then, in fact, this set of measures is tight in $M_1(M_F \times S')$ and we are done. ■

Corollary 7.14 *The set of probability measures $\{\tilde{P}^{(n)}_t, t \ge 0, n \ge 1\}$ satisfies CCC on $M^\rho_F \times S'^\rho$.*

Proof Fix arbitrary $\epsilon > 0$ and $\tilde{\rho}$ such that $(\frac{d}{2} - 1 \vee 0) < \tilde{\rho} < \rho$. (Recall (5.15)—the condition on ρ.) By Lemma 7.11, we can choose $A_\epsilon \subset M^{\tilde{\rho}}_F \times S'^{\tilde{\rho}}$ which is a closed subset in $M_F \times S'$ and $\inf_{0\le t\le T} \tilde{P}^{(n)}_t[A_\epsilon] \ge 1 - \epsilon/2$ for all n. By the previous lemma there exists B_ϵ, a compact subset of $M_F \times S'$, such that $\inf_{0\le t\le T} \tilde{P}^{(n)}_t[B_\epsilon] \ge 1 - \epsilon/2$ for all n. Then $E_\epsilon \equiv A_\epsilon \cap B_\epsilon$ is also a compact set in $M_F \times S'$, and by Corollary 5.6(b), E_ϵ is a compact set in $M^\rho_F \times S'^\rho$ since $\rho > \tilde{\rho}$. By our construction $\inf_{0\le t\le T} \tilde{P}^{(n)}_t[E_\epsilon] \ge 1 - \epsilon$ for all n, and we are done. ■

Remark 7.15 Let E_ϵ be as in the above proof. By Corollary 5.6(b), $\tau^{E_\epsilon}_\rho = \hat{\tau}^{E_\epsilon}_\rho$. Hence, any function f on E_ϵ which is continuous on $(E_\epsilon, \tau^{E_\epsilon}_\rho)$ (*i.e.*, $f \in C\big((E_\epsilon, \tau^{E_\epsilon}_\rho)\big)$) will be also continuous on $(E_\epsilon, \hat{\tau}^{E_\epsilon}_\rho)$ (*i.e.*, $f \in C\big((E_\epsilon, \hat{\tau}^{E_\epsilon}_\rho)\big)$), and we may just write $f \in C(E_\epsilon)$ without explicitly mentioning the topology on E_ϵ.

Lemma 7.16 *For each $\psi_1, \psi_2 \in S(R^d)_+$, the sequence of mappings*

$$t \mapsto \tilde{P}^{(n)}_t[\exp\{-\langle \psi_1, \tilde{X}_t + i\tilde{Y}_t\rangle - \langle \psi_2, \tilde{X}_t - i\tilde{Y}_t\rangle\}]$$

of R_+ *into* C *is relatively compact in* $C_{\mathrm{C}}[0,\infty)$.

Proof $\tilde{P}^{(n)}_t[\exp\{-\langle \psi_1, H_t\rangle - \langle \psi_2, \overline{H}_t\rangle\}]$ is bounded uniformly in n and t. As the proof relies on the Arzela-Ascoli theorem, we need to check that, for each $T > 0$,

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \sup_{|t-s|<\epsilon, s<t<T} |\tilde{P}^{(n)}_t[\exp\{-\langle \psi_1 H_t\rangle - \langle \psi_2, \overline{H}_t\rangle\}] - \tilde{P}^{(n)}_s[\exp\{-\langle \psi_1, H_s\rangle - \langle \psi_2, \overline{H}_s\rangle\}]| = 0.$$

For each $n \ge 1$ and $s < t \le T$,

$$\begin{aligned} &|\tilde{P}^{(n)}_t[\exp\{-\langle \psi_1, H_t\rangle - \langle \psi_2 \overline{H}_t\rangle\}] - \tilde{P}^{(n)}_s[\exp\{-\langle \psi_1, H_s\rangle - \langle \psi_2, \overline{H}_s\rangle\}]| \\ &\quad \le |\tilde{P}^{(n)}_t[\exp\{-\langle \psi_1, H_t\rangle - \langle \psi_2, \overline{H}_t\rangle\}] \\ &\qquad - \tilde{P}^{(n)}_s[\exp\{-\langle \psi_1, V^n_{s,t}(H_s)\rangle - \langle \psi_2, V^n_{s,t}(\overline{H}_s)\rangle\}]| \\ &\qquad + |\tilde{P}^{(n)}_s[\exp\{-\langle \psi_1, V^n_{s,t}(H_s)\rangle - \langle \psi_2, V^n_{s,t}(\overline{H}_s)\rangle\}] \\ &\qquad - \tilde{P}^{(n)}_s[\exp\{-\langle \psi_1, H_s\rangle - \langle \psi_2, \overline{H}_s\rangle\}]|. \end{aligned} \tag{7.31}$$

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The first term is bounded by

$$\begin{aligned}
&\Big| \int_s^t \tilde{P}^{(n)}_u \Big[\exp\{-\langle \psi_1, V^n_{u,t}(H_u)\rangle - \langle \psi_2, V^n_{u,t}(\overline{H}_u)\rangle\} \\
&\qquad \times 4\lambda j^{(n)}\Big(s+\frac{1}{n}\Big) \int_{\mathrm{R}^d} U^n_{u,t}(H_u,x)(\psi_1) U^n_{u,t}(\overline{H}_u,x)(\psi_2)\tilde{X}_u\,(dx)\Big]\,du\Big| \\
&\quad \le 4\lambda \|\psi_1\|_\infty \|\psi_2\|_1 \sup_{s\le T} \|S_s(|H_0|)\|_\infty |t-s| \\
&\quad \le 4\lambda \|\psi_1\|_\infty \|\psi_2\|_1 \|H_0\|_\infty |t-s|,
\end{aligned} \tag{7.32}$$

where the first expression follows from Proposition 7.1, and the latter inequalities follow from Lemma 7.5, inequality (7.23) and Lemma 5.1. From (7.32) we obtain that the first term in (7.31) approaches 0 as $\epsilon \to 0$ uniformly in n.

Let us treat the second term in (7.31). Corollary 7.14 implies that for each $\delta > 0$ there exists a compact set $B_{\delta,T} \subset \tilde{S'}^{\rho}$ such that

$$\sup_{n\ge 1} \sup_{0\le t\le T} \tilde{P}^{(n)}_t [B^c_{\delta,T}] \le \delta.$$

By Theorem 5.8

$$\lim_{\epsilon\downarrow 0} \sup_{|t-s|<\epsilon, s<t<T, \mu\in B_{\delta,T}} |\langle \psi, V^n_{s,t}(\mu)\rangle - \langle \psi, \mu\rangle| = 0$$

uniformly in n. δ was arbitrary, therefore the second term in (7.31) converges to 0 uniformly in n and this finishes the proof of the lemma. ■

Lemma 7.16 and Lemma 7.13 yield

Lemma 7.17 *There exist $\{n_k\}$ and $\tilde{P}_t \in C_{M_1(M_F\times S')}[0,\infty)$ such that, for each $f \in \overline{C}(M_F \times S')$,*

$$\tilde{P}^{(n_k)}_t(f) \to \tilde{P}_t(f) \quad \text{in} \quad C_{\mathrm{C}}[0,\infty) \quad \text{as} \quad n_k \to \infty$$

and

$$\tilde{P}_t(M^{\rho}_F \times S'^{\rho}) = 1, \quad \forall\, t \ge 0. \tag{7.33}$$

Proof (7.33) is immediate from Corollary 7.14.

Let $\{(\psi^{(m)}_1, \psi^{(m)}_2)m \ge 1\}$ be a dense subset of $\big(S(R^d)_+, S(R^d)_+\big)$. For any $(\psi_1,\psi_2) \in \big(S(R^d)_+, S(R^d)_+\big)$, let us define the function $e_{\psi_1,\psi_2} \in \overline{C}(M_F \times S')$ by

$$e_{\psi_1,\psi_2}(\mu_1,\mu_2) \equiv e^{-\langle \psi_1+\psi_2,\mu_1\rangle - i\langle \psi_1-\psi_2,\mu_2\rangle}.$$

Now use Lemma 7.16 and the Cantor diagonalization procedure to construct a subsequence $\tilde{P}^{(n_k)}_t$ such that $\tilde{P}^{(n_k)}_t(e_{\psi^{(m)}_1,\psi^{(m)}_2})$ converges in $C_{\mathrm{C}}[0,\infty)$ for each $m \ge 1$. The reader can easily check that the set of functions

$$F \equiv \text{ linear span } \{e_{\psi^{(m)}_1,\psi^{(m)}_2}, m \ge 1\}$$

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is bp-dense in $\tilde{L}$ (for the definition of $\tilde{L}$ see the proof of Lemma 4.2) and therefore is separating on $M_1(M_F \times S')$. This together with the relative compactness of $\{\tilde{P}_t^{(n)}, n \geq 1\}$ (for each $t \geq 0$) implies that, for each $t > 0$, there exists $\tilde{P}_t \in M_1(M_F \times S')$ such that $\tilde{P}_t^{(n_k)} \Rightarrow \tilde{P}_t$. Therefore, for each $t > 0$ and $f \in \overline{C}(M_F \times S')$, $\tilde{P}_t^{(n_k)}(f) \to \tilde{P}_t(f)$ (pointwise convergence). The fact that $\tilde{P}_t^{(n_k)}(f)$ converges to $\tilde{P}_t(f)$ in $C_{\mathrm{C}}[0,\infty)$ (uniformly on compact intervals in R_+) follows immediately from an appropriate approximation of f by functions from F. ■

In the following we assume that $\tilde{P}_t^{(n_k)}$ and $\tilde{P}_t$ are as in Lemma 7.17. The following corollary is immediate.

Corollary 7.18 *Let $f \in \overline{C}(\mathrm{R}_+ \times M_F \times S')$. Then $\tilde{P}_t^{(n_k)}\big(f(t,\cdot)\big) \to \tilde{P}_t\big(f(t,\cdot)\big)$ in $C_{\mathrm{C}}[0,\infty)$.*

Lemma 7.19 *Let I be any interval in R_+. Let $\{f^{(n)}\}$ be in $B(I \times M_F \times S')$. Assume that $\hat{f}^{(n)} \equiv f^{(n)} \restriction I \times M_F^{\rho} \times S'^{\rho} \in \overline{C}(I \times M_F^{\rho} \times S'^{\rho})$ is bounded uniformly in n and*

$$\hat{f}^{(n)} \to \hat{f}^{(0)} \quad \text{as} \quad n \to \infty$$

uniformly on compact subsets of $I \times M_F^{\rho} \times S'^{\rho}$. Define

$$f^{(0)}(s,\mu_1,\mu_2) \equiv \begin{cases} \hat{f}^{(0)}(s,\mu_1,\mu_2), & \text{if } (s,\mu_1,\mu_2) \in I \times M_F^{\rho} \times S'^{\rho}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\tilde{P}_t^{(n_k)}(f_t^{(n_k)}) \to \tilde{P}_t(f_t^0) \quad \text{in} \quad C_{\mathrm{C}}(I) \quad \text{as} \quad n \to \infty.$$

Before giving the proof we derive the following consequence.

Corollary 7.20 *Let $\{f^{(n)}\}$ be in $\mathcal{B}(I \times M_F \times S')$ and assume that $\hat{f}^{(n)} \equiv f^{(n)} \restriction I \times M_F^{\rho} \times S'^{\rho} \in C(I \times M_F^{\rho} \times S'^{\rho})$, $n \geq 1$. Suppose that*

$$\hat{f}^{(n)} \to \hat{f}^{(0)} \quad \text{as} \quad n \to \infty$$

uniformly on compact subsets of $I \times M_F^{\rho} \times S'^{\rho}$. Define $f^{(0)}$ as in the previous lemma.

(a) Assume the uniform integrability condition

$$\lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} \sup_{k \geq 0} \sup_{0 \leq t \leq T} \tilde{P}_t^{(n)}[|f_t^{(k)}| 1(|f_t^{(k)}| \geq \epsilon^{-1})] = 0, \quad \forall\, T > 0. \tag{7.34}$$

Then

$$\tilde{P}_t^{(n_k)}(f_t^{(n_k)}) \to \tilde{P}_t(f_t^0), \quad \text{in} \quad C_{\mathrm{C}}(I) \quad \text{as} \quad n \to \infty.$$

(b) If $f^{(n)} \in \mathcal{B}_{\mathrm{R}}(I \times M_F \times S')_+$ for each n, then

$$\tilde{P}_t(f_t^{(0)}) \leq \liminf_{n_k} \tilde{P}_t^{(n_k)}(f_t^{(n_k)}), \quad \forall\, t \in I.$$

Proof of Corollary 7.20 (a) The proof is immediate by the uniform integrability technique.

(b) Recall one of the properties of weak convergence (see *e.g.* the proof of Theorem 2.4.1 of [3]). If $\mu^{(n)} \Rightarrow \mu$, then for any nonnegative lower semicontinuous function h

$$\mu(h) \le \liminf_{n\to\infty} \mu^{(n)}(h). \tag{7.35}$$

By Lemma 7.19 we obtain that the sequence of probability measures $\{\tilde{P}_t^{(n_k)} f_t^{(n_k)^{-1}}, n_k \ge 1\}$ (which is in $M_1(\mathrm{C})$) converges weakly to $\tilde{P}_t f_t^{(0)^{-1}}$ for any $t \in I$. Therefore (b) follows immediately from (7.35).

Proof of Lemma 7.19 Fix any compact $\mathrm{T} \subset I$ and choose T such that $\mathrm{T} \subset [0, T]$. Fix arbitrary $\epsilon > 0$. By Corollary 7.14 there exists a closed compact set $B_{\epsilon,T} \subset M_F^\rho \times S'^\rho$ such that

$$\sup_{n\ge 1} \sup_{0\le t\le T} \tilde{P}_t^{(n)}[B^c_{\epsilon,T}] \le \epsilon. \tag{7.36}$$

For each k we have

$$\begin{aligned}
\tilde{P}_t^{(n_k)}(f_t^{(n_k)}) - \tilde{P}_t(f_t^{(0)}) &= \tilde{P}_t^{(n_k)}(f_t^{(n_k)} - f_t^{(0)}) + \tilde{P}_t^{(n_k)}(f_t^{(0)}) - \tilde{P}_t(f_t^{(0)})\\
&= \tilde{P}_t^{(n_k)}\big((f_t^{(n_k)} - f_t^{(0)})1_{B_{\epsilon,T}}\big) + \tilde{P}_t^{(n_k)}\big((f_t^{(n_k)} - f_t^{(0)})1_{B^c_{\epsilon,T}}\big)\\
&\quad + \big(\tilde{P}_t^{(n_k)}(f_t^{(0)}1_{B_{\epsilon,T}}) - \tilde{P}_t(f_t^{(0)}1_{B_{\epsilon,T}})\big)\\
&\quad + \big(\tilde{P}_t^{(n_k)}(f_t^{(0)}1_{B^c_{\epsilon,T}}) - \tilde{P}_t(f_t^{(0)}1_{B^c_{\epsilon,T}})\big).
\end{aligned} \tag{7.37}$$

The first term in (7.37) approaches 0 uniformly in $t \in \mathrm{T}$ and $\lim_{n_k\to\infty} f^{(n_k)} = f$ uniformly on $I \times B_{\epsilon,T}$. Hence by the compact containment condition (7.36) the second and the fourth terms may be made arbitrarily small uniformly on $t \in \mathrm{T}$ by fixing ϵ sufficiently small. Let us treat the third term $\tilde{P}_t^{(n_k)}(f_t^{(0)}1_{B_{\epsilon,T}}) - \tilde{P}_t(f_t^{(0)}1_{B_{\epsilon,T}})$. $\mathrm{T} \times B_{\epsilon,T}$ is a closed compact set in $I \times M_F^\rho \times S'^\rho$ and by our assumptions (see also Remark 7.15)

$$f^{(0)} \restriction \mathrm{T} \times B_{\epsilon,T} = \hat{f}^{(0)} \restriction \mathrm{T} \times B_{\epsilon,T} \in \overline{C}(\mathrm{T} \times B_{\epsilon,T}).$$

By Tietze extension theorem there is a function $\tilde{f} \in \overline{C}(I \times M_F \times S')$ such that

$$\begin{aligned}
\tilde{f}(t,\mu_1,\mu_2) &= f^{(0)}(t,\mu_1,\mu_2), \quad \forall (t,\mu_1,\mu_2) \in \mathrm{T} \times B_{\epsilon,T},\\
\|\tilde{f}\|_{\infty, I\times M_F\times S'} &= \|f^{(0)}\|_{\infty,\mathrm{T}\times B_{\epsilon,T}} \le \|f^{(0)}\|_{\infty, I\times M_F\times S'} \equiv \|f^{(0)}\|.
\end{aligned} \tag{7.38}$$

By Corollary 7.18 we obtain

$$\lim_{k\to\infty} \sup_{t\in\mathrm{T}} |\tilde{P}_t^{(n_k)}(\tilde{f}_t) - \tilde{P}_t(\tilde{f}_t)| = 0.$$

Therefore

$$\begin{aligned}
&\lim_{k\to\infty}\sup_{t\in \mathrm{T}}|\tilde{P}_t^{(n_k)}(f_t^{(0)}1_{B_{\epsilon,T}})-\tilde{P}_t(f_t^{(0)}1_{B_{\epsilon,T}})|\\
&\quad\le \lim_{k\to\infty}\sup_{t\in \mathrm{T}}|\tilde{P}_t^{(n_k)}(f_t^{(0)}1_{B_{\epsilon,T}}-\tilde{f}_t)-\tilde{P}_t(f_t^{(0)}1_{B_{\epsilon,T}}-\tilde{f}_t)|\\
&\qquad+\lim_{k\to\infty}\sup_{t\in \mathrm{T}}|\tilde{P}_t^{(n_k)}(\tilde{f})-\tilde{P}_t(\tilde{f})|\\
&\quad=\lim_{k\to\infty}\sup_{t\in \mathrm{T}}|\tilde{P}_t^{(n_k)}(\tilde{f}_t1_{B^c_{\epsilon,T}})-\tilde{P}_t(\tilde{f}_t1_{B^c_{\epsilon,T}})|\\
&\quad\le 2\epsilon\|f^{(0)}\|
\end{aligned}$$

where the last inequality follows by the choice of $B_{\epsilon,T}$ and (7.38). So, it is clear that by first choosing ϵ sufficiently small and then n_k sufficiently large we can make the third term in (7.37) arbitrarily small uniformly in t and the proof is complete. ■

In what follows, given an interval $I\subset \mathrm{R}_+$ and a function $h\in\mathcal{B}(I\times M_F\times S')$, set $\hat{h}\equiv h\restriction I\times M_F^\rho\times S'^\rho$.

The next lemma establishes the simple properties of the limiting measures.

Lemma 7.21 *For each* $\psi\in\overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+$, $T\ge 0$,

$$\tilde{P}_t[\langle\psi,V^1_{T-t}(H_t)\rangle]\le\langle\psi,S_T(|H_0|)\rangle,\quad 0\le t\le T, \tag{7.39}$$

$$\tilde{P}_t[\langle\psi,\tilde{X}_t\rangle^2]\le\langle\psi,S_t(|H_0|)\rangle^2+\int_0^t\lambda\int_{\mathrm{R}^d}S_{t-s}(\psi)(x)^2S_s(|H_0|)(x)\,dx\,ds,\quad\forall t\ge 0, \tag{7.40}$$

$$\begin{aligned}&\tilde{P}_t[\langle\psi,V^l_{T-t}(H_t)\rangle^2]\\&\quad\le\langle\psi,S_T(|H_0|)\rangle^2\\&\qquad+\int_0^t\lambda\int_{\mathrm{R}^d}S_{T-s}(\psi)(x)^2S_s(|H_0|)(x)\,dx\,ds,\quad l=1,2,\quad 0\le t<T.\end{aligned}\tag{7.41}$$

Moreover, for each $\psi_1,\psi_2\in\overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+$, $T_1\ne T_2>0$, $0\le t\le T_1\wedge T_2$ *and* $l=1,2$ *we have*

$$\begin{aligned}&\tilde{P}_t[\langle\psi_1,V^1_{T_1-t}(H_t)\rangle\langle\psi_2,V^1_{T_2-t}(H_t)\rangle]\\&\quad\le\langle\psi_1,S_{T_1}(|H_0|)\rangle\langle\psi_2,S_{T_2}(|H_0|)\rangle\\&\qquad+\int_0^t\lambda\int_{\mathrm{R}^d}S_{T_1-s}(\psi_1)(x)S_{T_2-s}(\psi_2)(x)S_s(|H_0|)(x)\,dx\,ds.\end{aligned}\tag{7.42}$$

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Proof First we will prove the lemma for $\psi, \psi_1, \psi_2 \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$. We pass to the limit in the corresponding inequalities in Corollary 7.8. The right sides of (7.39)–(7.42) follow immediately since

$$j^{(n)} \to \frac{1}{2} \tag{7.43}$$

weakly* in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)$ as $n \to \infty$. Let us treat the left sides of (7.39)–(7.42). (7.40) follows from (7.35). By Theorem 5.8, for each function $\psi \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$ and each $T > 0$,

$$\langle \psi, \widehat{V^n_{\cdot,T}(\cdot)} \rangle \to \langle \psi, \widehat{V_{T-\cdot}(\cdot)} \rangle \quad \text{as} \quad n \to \infty \tag{7.44}$$

uniformly on compact subsets of $[0, T) \times \tilde{S}'^{\rho}$. Recall that $\tilde{P}^{(n)}_t(M^\rho_F \times S'^\rho) = 1$ for all $n \geq 1$, $t \geq 0$ and therefore Corollary 7.20(b) yields the result for $\psi, \psi_1, \psi_2 \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$. For arbitrary $\psi, \psi_1, \psi_2 \in \overline{C}_{\mathrm{R}}(\mathrm{R}^d)_+$ just approximate them by functions from $C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$ and use Fatou's lemma. ■

Corollary 7.22 *For all* $0 < s < T$, $\nu_1, \nu_2 \in M_F$,

$$\begin{aligned} &\int_{\mathrm{R}^d} |U_{T-s}(\tilde{X}_s + i\tilde{Y}_s, x)(\nu_1) U_{T-s}(\tilde{X}_s - i\tilde{Y}_s, x)(\nu_2)|\, \tilde{X}_s(dx) \\ &\qquad \leq \int_{\mathrm{R}^d} S_{T-s}(\nu_1)(x) S_{T-s}(\nu_2)(x)\, \tilde{X}_s(dx), \quad \tilde{P}_s\text{-a.s.}, \end{aligned} \tag{7.45}$$

and

$$\begin{aligned} &\tilde{P}\left[\int_{\mathrm{R}^d} |U_{T-s}(\tilde{X}_s + i\tilde{Y}_s, x)(\nu_1) U_{T-s}(\tilde{X}_s - i\tilde{Y}_s, x)(\nu_2)|\, \tilde{X}_s(dx)\right] \\ &\qquad \leq \int_{\mathrm{R}^d} S_{T-s}(\nu_1)(x) S_{T-s}(\nu_2)(x) S_s(|H_0|)(x)\, dx. \end{aligned} \tag{7.46}$$

Proof By (7.39), (7.46) is immediate from (7.45).

By Lemma 7.17 $\tilde{X}_s \in M^\rho_F$ $\tilde{P}_s$-a.s. for each $s \geq 0$ and the proof of (7.45) goes in the same way as the proof of Lemma 7.5. ■

Corollary 7.23 *For each* $T > 0$,

$$\sup_{\epsilon > 0, 0 \leq t \leq T} \tilde{P}_t[\langle 1, V^1_\epsilon(H_t) \rangle^2] < \infty.$$

Proof By (7.41)

$$\tilde{P}_t[\langle \psi, V^1_\epsilon(H_t) \rangle^2] \leq \|H_0\|_1^2 + t\lambda \|H_0\|_1.$$

and we are done by the assumptions on $(\tilde{X}_0, \tilde{Y}_0)$.

Proof of Theorem 7.2 Fix $T > 0$. For fixed $(\nu_1, \nu_2) \in M_F^\rho \times M_F^\rho$ let us define the functions $f^{(n)}_{\mu_1,\mu_2}, g^{(n)}_{\nu_1,\nu_2}, f^{(0)}_{\nu_1,\nu_2}, g^{(0)}_{\nu_1,\nu_2} \in \mathcal{B}\big([0,T) \times M_F \times S'\big)$ by

$$f^{(n)}_{\nu_1,\nu_2}(t,\mu_1,\mu_2) = \exp\{-\langle \nu_1, V^n_{t,T}(\mu_1 + i\mu_2)\rangle - \langle \nu_2, V^n_{t,T}(\mu_1 - i\mu_2)\rangle\},$$

$$g^{(n)}_{\nu_1,\nu_2}(t,\mu_1,\mu_2) = \exp\{-\langle \nu_1, V^n_{t,T}(\mu_1 + i\mu_2)\rangle - \langle \nu_2, V^n_{t,T}(\mu_1 - i\mu_2)\rangle\} \times 4\lambda \int_{\mathrm{R}^d} U^n_{t,T}(\mu_1 + i\mu_2, x)(\nu_1) U^n_{t,T}(\mu_1 - i\mu_2, x)(\nu_2)\,\mu_1(dx),$$

$$f^{(0)}_{\nu_1,\nu_2}(t,\mu_1,\mu_2) = \exp\{-\langle \nu_1, V_{T-t}(\mu_1 + i\mu_2)\rangle - \langle \nu_2, V_{T-t}(\mu_1 - i\mu_2)\rangle\},$$

$$g^{(0)}_{\nu_1,\nu_2}(t,\mu_1,\mu_2) = \exp\{-\langle \nu_1, V_{T-t}(\mu_1 + i\mu_2)\rangle - \langle \nu_2, V_{T-t}(\mu_1 - i\mu_2)\rangle\} \times 4\lambda \int_{\mathrm{R}^d} U_{T-t}(\mu_1 + i\mu_2, x)(\nu_1) U_{T-t}(\mu_1 - i\mu_2, x)(\nu_2)\,\mu_1(dx).$$

Fix arbitrary $\psi_1, \psi_2 \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$ and let us first get (7.5) for $(\nu_1, \nu_2) = (\psi_1 dx, \psi_2 dx)$.

Recall that for any function $h \in \mathcal{B}\big([0,T) \times M_F \times S'\big)$ we set $\hat{h} \equiv h \restriction [0,T) \times M_F^\rho \times S'^\rho$. By Theorems 5.8, 5.9 it is easy to see that

$$\hat{f}^{(n)}_{\psi_1,\psi_2}, \hat{g}^{(n)}_{\psi_1,\psi_2} \in C\big([0,T) \times M_F^\rho \times S'^\rho\big), \quad \forall n,$$

and

$$\hat{f}^{(n)}_{\psi_1,\psi_2} \to \hat{f}^{(0)}_{\psi_1,\psi_2}, \quad \hat{g}^{(n)}_{\psi_1,\psi_2} \to \hat{g}^{(0)}_{\psi_1,\psi_2} \quad \text{as} \quad n \to \infty$$

uniformly on compact subsets of $[0,T) \times M_F^\rho \times S'^\rho$. The functions $\{f^{(n)}_{\psi_1,\psi_2}, n \geq 1\}$ are bounded; by Corollary 7.8 and Lemma 7.5 functions $\{g^{(n)}_{\psi_1,\psi_2}, n \geq 1\}$ are uniformly integrable (in the sense of (7.34)). Fix any $t < T$. By Corollary 7.20 we obtain that

$$\tilde{P}^{(n_k)}_t\big(f^{(n_k)}_{\psi_1,\psi_2}(t,\cdot,\cdot)\big) \to \tilde{P}_t\big(f^0_{\psi_1,\psi_2}(t,\cdot,\cdot)\big) \quad \text{as} \quad n \to \infty, \tag{7.47}$$

$$\tilde{P}^{(n_k)}_s\big(g^{(n_k)}_{\psi_1,\psi_2}(s,\cdot,\cdot)\big) \to \tilde{P}_s\big(g^0_{\psi_1,\psi_2}(s,\cdot,\cdot)\big), \quad \text{in} \quad C_{\mathrm{C}}([0,t]) \quad \text{as} \quad n \to \infty. \tag{7.48}$$

(7.43), (7.47), (7.48) give (7.5) for any $(\nu_1, \nu_2) = (\psi_1 dx, \psi_2 dx)$ with $\psi_1, \psi_2 \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$.

For arbitrary $(\nu_1, \nu_2) \in M_F^\rho \times M_F^\rho$ take the sequence $(\psi_1^{(n)}, \psi_2^{(n)}) \Rightarrow (\nu_1, \nu_2)$ in $M_F^\rho \times M_F^\rho$ with $\psi_1^{(n)}, \psi_2^{(n)} \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$ for each n. It is easy to check that

$$f^{(0)}_{\psi_1^{(n)},\psi_2^{(n)}}(\mu_1,\mu_2) \to f^{(0)}_{\nu_1,\nu_2}(\mu_1,\mu_2), \quad g^{(0)}_{\psi_1^{(n)},\psi_2^{(n)}}(\mu_1,\mu_2) \to g^{(0)}_{\nu_1,\nu_2}(\mu_1,\mu_2) \tag{7.49}$$

for every $(\mu_1, \mu_2) \in M_F^\rho \times M_F^\rho$. Lemma 7.5 and the uniform integrability condition (7.40) give (7.5) for any $(\nu_1, \nu_2) \in M_F^\rho \times M_F^\rho$ and we are done.

8 Uniqueness

In this section we will finish the proof of Theorem 2.6 and the proof will rely on checking the conditions of Lemma 4.3. Fix arbitrary $\nu \in M_1^*(M_{F,w} \times M_{F,w})$. Throughout this section we assume that (X^1, X^2) is an arbitrary solution of the martingale problem M^λ with $P(X^1_0, X^2_0)^{-1} = \nu$.

First, we define the martingales that arise from the martingale problem M^λ for the process (X^1, X^2).

Lemma 8.1 *Suppose $\mu \in \tilde{S'}^\rho$ and $T > 0$. Then*

(a)

$$
\begin{aligned}
&e^{-\langle X_t^1, V_{T-t}(\mu)\rangle - \langle X_t^2, V_{T-t}(\overline{\mu})\rangle} \\
&\quad = e^{-\langle X_0^1, V_T(\mu)\rangle - \langle X_0^2, V_T(\overline{\mu})\rangle} \\
&\qquad + \int_0^t e^{-\langle X_s^1, V_{T-s}(\mu)\rangle - \langle X_s^2, V_{T-s}(\overline{\mu})\rangle} 2\lambda K_s(X^1, X^2)\big(V_{T-s}^1(\mu)\big)\, ds \\
&\qquad + \int_0^t \int_{\mathrm{R}^d} e^{-\langle X_s^1, V_{T-s}(\mu)\rangle - \langle X_s^2, V_{T-s}(\overline{\mu})\rangle} \\
&\qquad \times \big(V_{T-s}(\mu)(x)\, M^1(ds, dx) + V_{T-s}(\overline{\mu})(x)\, M^2(ds, dx)\big), \quad 0 \le t < T,
\end{aligned}
$$

where $M^l(ds, dx)$ $(l = 1, 2)$ are the martingale measures defined in Remark 2.5.

(b) For each $x \in \mathrm{R}^d$, $T_1, T_2 > 0$,

$$
\begin{aligned}
&e^{-\langle X_t^1, V_{T_1-t}(\mu)\rangle - \langle X_t^2, V_{T_1-t}(\overline{\mu})\rangle} S_{T_2-t}(X_t^1)(x) S_{T_2-t}(X_t^2)(x) \\
&\quad - \int_0^t e^{-\langle X_u^1, V_{T_1-u}(\mu)\rangle - \langle X_u^2, V_{T_1-u}(\overline{\mu})\rangle} \\
&\qquad \times \Big\{ 2\lambda \langle K_u(X^1, X^2), V_{T_1-u}^1(\mu)\rangle S_{T_2-u}(X_u^1)(x) S_{T_2-u}(X_u^2)(x) \\
&\qquad\quad + \lambda\big(S_{T_2-u}(X_u^1)(x) + S_{T_2-u}(X_u^2)(x)\big) S_{T_2-u}\big(K_u(X^1, X^2)\big)(x) \\
&\qquad\quad + \left(\int_{\mathrm{R}^d} V_{T_1-u}(\mu)(y) p_{T_2-u}(x-y) X_u^1\,(dy)\right) S_{T_2-u}(X_u^2)(x) \\
&\qquad\quad + \left(\int_{\mathrm{R}^d} V_{T_1-u}(\mu)(y) p_{T_2-u}(x-y) X_u^2(dy)\right) S_{T_2-u}(X_u^1)(x) \Big\}\, du
\end{aligned}
$$

is a martingale on $[0, T)$ where $T = T_1 \wedge T_2$.

Proof By routine arguments (see *e.g.* Exercise 5.1 in [20], or calculations around (4.14) in [12] for similar results) we get that for each $\psi \in \overline{C}^{1,2}(\mathrm{R}_+ \times \mathrm{R}^d)$

$$
\begin{aligned}
X_t^j(\psi_t) \equiv X_0^j(\psi_0) &+ \int_0^t X_s^j \left(\frac{1}{2}\Delta\psi_s + \frac{\partial}{\partial s}\psi_s\right) ds - \lambda \int_0^t \int_{\mathrm{R}^d} \psi_s(x) L(X^1, X^2)\,(dx, ds) \\
&+ \int_0^t \int_{\mathrm{R}^d} \psi_s(x) M^k\,(dx, ds), \quad j = 1, 2.
\end{aligned}
$$

By choosing functions $\psi_t^1 = V_{T-t}(\mu)$, $\psi_t^2 = V_{T-t}(\overline{\mu})$ in (a) and $\psi_t^1 = V_{T_1-t}(\mu)$, $\psi_t^2 = V_{T_1-t}(\overline{\mu})$, $\psi_t^3 = p_{T_2-t}(x - \cdot)$, $\psi_t^4 = p_{T_2-t}(x - \cdot)$ in (b), and then applying Itô's formula on the interval $[0, T)$ one can readily complete the proof of the lemma. ■

In the following two lemmas we establish some simple properties of the process (X^1, X^2) that we will use later. Let $L(X^1, X^2)$ denote the random measure on $\mathcal{B}(\mathrm{R}_+ \times \mathrm{R}^d)$ given by $L(X^1, X^2)([0,t] \times B) = L_t(X^1, X^2)(B)$.

Lemma 8.2 *Let ψ be any bounded random function which is in $\overline{C}(\mathrm{R}_+ \times \mathrm{R}^d)$ P-a.s. Then for each $s < t$*

$$\int_s^t \int_{\mathrm{R}^d} S_\epsilon(X_u^1)(x) S_\epsilon(X^2)(x) \psi(u,x)\, dx\, du \stackrel{\mathrm{P}}{\to} \int_s^t \int_{\mathrm{R}^d} \psi(u,x) L(X^1, X^2)\,(du, dx) \tag{8.1}$$

and in L^1 as $\epsilon \downarrow 0$.

Proof Our definition of $L_t(X^1, X^2)$ and $L(X^1, X^2)$ implies that

$$S_\epsilon(X_u^1)(x) S_\epsilon(X^2)(x)\, dx\, du \Rightarrow L(X^1, X^2)\,(du, dx)$$

in probability and therefore convergence in probability in (8.1) is immediate. The L^1 convergence follows from uniform integrability condition which one can check easily. ∎

Lemma 8.3 *For each $t > 0$, P-a.s. $(X_t^1, X_t^2) \in M_F^\rho \times M_F^\rho$.*

Proof By the domination property, it suffices to prove that for each $t > 0$ the dominating superprocesses Y_t^l $(l = 1, 2)$ are in M_F^ρ P-a.s. But this follows immediately from Lemma 6.5 (X' in Lemma 6.5 is a superprocess). ∎

In the following let $\tilde{P}_t$ be as in Lemma 7.17.

Lemma 8.4 *For any $t, \epsilon > 0$*

$$\begin{aligned}
P \times \tilde{P}_0 &[\exp\{-\langle X_t^1, V_\epsilon(H_0)\rangle - \langle X_t^2, V_\epsilon(\overline{H}_0)\rangle\}] \\
&\quad - P \times \tilde{P}_t[\exp\{-\langle X_0^1, V_\epsilon(H_t)\rangle - \langle X_0^2, V_\epsilon(\overline{H}_t)\rangle\}] \\
&= 2\lambda \int_0^t P \times \tilde{P}_{t-s}\Big[\exp\{-\langle X_s^1, V_\epsilon(H_{t-s})\rangle - \langle X_s^2, V_\epsilon(\overline{H}_{t-s})\rangle\} \\
&\qquad \times \Big\{\langle K_s(X^1, X^2), V_\epsilon^1(H_{t-s})\rangle \\
&\qquad\qquad - \int_{\mathrm{R}^d} U_\epsilon(H_{t-s}, x)(X_s^1) U_\epsilon(\overline{H}_{t-s}, x)(X_s^2) \tilde{X}_{t-s}\,(dx)\Big\}\Big]\, ds.
\end{aligned}$$

Proof Fix any $T > 0$ and define three functions: h_1, h_2, f by

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$$f(t,s) = P \times \tilde{P}_s[\exp\{-\langle X_t^1, V_{T-t-s}(H_s)\rangle - \langle X_t^2, V_{T-t-s}(\overline{H}_s)\rangle\}]$$

$$h_1(t,s) = 2\lambda P \times \tilde{P}_s[\exp\{-\langle X_t^1, V_{T-t-s}(H_s)\rangle - \langle X_t^2, V_{T-t-s}(\overline{H}_s)\rangle\} \times \langle K_t(X^1,X^2), V^1_{T-t-s}(H_s)\rangle]$$

$$h_2(t,s) = 2\lambda P \times \tilde{P}_s\Big[\exp\{-\langle X_t^1, V_{T-t-s}(H_s)\rangle - \langle X_t^2, V_{T-t-s}(\overline{H}_s)\rangle\} \times \int_{\mathrm{R}^d} U_{T-t-s}(H_s,x)(X_t^1) U_{T-t-s}(\overline{H}_s,x)(X_t^2)\,\tilde{X}_s(dx)\Big]$$

for $0 \le s+t < T$. By Lemma 8.1 we have

$$f(t,s) = f(0,s) + \int_0^t h_1(u,s)\,du, \quad \forall t,s \ge 0 : t+s < T.$$

By Lemma 8.3 and Theorem 7.2 we obtain

$$f(t,s) = f(t,0) + \int_0^s h_2(t,u)\,du, \quad \forall t,s \ge 0 : t+s < T.$$

From Lemma 4.4.10 of [10] (see *e.g.* [17, Lemma 4.17]) it follows that

$$f(t,0) - f(0,t) = \int_0^t h_1(s,t-s) - h_2(s,t-s)\,ds \tag{8.2}$$

for almost every t, $0 \le t < T$. We leave to the reader to check that the right side of (8.2) is continuous on $[0,T)$ (this requires only continuity of $\tilde{P}_t, X_t^1, X_t^2, V_t$ and U_t). Using the continuity of the right side of (8.2) we show that the equality in (8.2) is satisfied for each $0 < t < T$. Take $T = t + \epsilon$ and the proof is complete. ■

Our main goal now is to prove the following lemma.

Lemma 8.5

$$\lim_{\epsilon \downarrow 0}\{P \times \tilde{P}_t[\exp\{-\langle X_t^1, V_\epsilon(H_0)\rangle - \langle X_t^2, V_\epsilon(\overline{H}_0)\rangle\}] - P \times \tilde{P}_t[\exp\{-\langle X_0^1, V_\epsilon(H_t)\rangle - \langle X_0^2, V_\epsilon(\overline{H}_t)\rangle\}]\} = 0.$$

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The rest of this section is devoted to the proof of this lemma. The main idea of the proof is based on applying Itô's formula to functions of X_s^1, X_s^2 while considering $\tilde{X}_{t-s}$ and $\tilde{Y}_{t-s}$ to be fixed.

Let P be any probability measure on M_F. If the measure $\hat{\mu} \in M_F$ defined by $\hat{\mu}(A) = \int_{M_F} \mu(A) P(d\mu)$ has a density, this density, with a slight abuse of notation, will be denoted by $P[\mu(x)]$.

Fix δ' such that $0 < \delta' < t/2$. Now rewrite the result of the previous lemma in the following way:

(8.3)

$$\begin{aligned}
&|P \times \tilde{P}_t[\exp\{-\langle X_t^1, V_\epsilon(H_0)\rangle - \langle X_t^2, V_\epsilon(\overline{H}_0)\rangle\}] \\
&\qquad - P \times \tilde{P}_t[\exp\{-\langle X_0^1, V_\epsilon(H_t)\rangle - \langle X_0^2, V_\epsilon(\overline{H}_t)\rangle\}]| \\
&\leq \Big| 2\lambda \int_0^{\delta'} \tilde{P}_{t-s} \times P \Big[\exp\{-\langle X_s^1, V_\epsilon(H_{t-s})\rangle - \langle X_s^2, V_\epsilon(\overline{H}_{t-s})\rangle\} \\
&\qquad \times \Big\{ \langle K_s(X^1, X^2), V_\epsilon^1(H_{t-s})\rangle \\
&\qquad - \int_{\mathrm{R}^d} U_\epsilon(H_{t-s}, x)(X_s^1) U_\epsilon(\overline{H}_{t-s}, x)(X_s^2)\, \tilde{X}_{t-s}(dx) \Big\} \Big]\, ds \Big| \\
&\quad + 2\lambda \Big| \lim_{\epsilon' \downarrow 0} \int_{\delta'}^t \tilde{P}_{t-s} \times P \Big[\exp\{-\langle X_s^1, V_\epsilon(H_{t-s})\rangle - \langle X_s^2, V_\epsilon(\overline{H}_{t-s})\rangle\} \\
&\qquad \times \Big\{ \langle S_{\epsilon'}(X_s^1) S_{\epsilon'}(X_s^2), V_\epsilon^1(H_{t-s})\rangle \\
&\qquad - \int_{\mathrm{R}^d} S_\epsilon(X_s^1)(x) S_\epsilon(X_s^2)(x)\, \tilde{X}_{t-s}(dx) \Big\} \Big]\, ds \Big| \\
&\quad + 2\lambda \Big| \int_{\delta'}^t \tilde{P}_{t-s} \times P \Big[\exp\{-\langle X_s^1, V_\epsilon(H_{t-s})\rangle - \langle X_s^2, V_\epsilon(\overline{H}_{t-s})\rangle\} \\
&\qquad \times \Big\{ \int_{\mathrm{R}^d} \int_0^\epsilon \langle S_{\epsilon-u}\big(U_u(\overline{H}_{t-s}, x)\big), X_s^2\rangle S_{\epsilon-u} \\
&\qquad \times \langle S_{\epsilon-u}\big(U_u(H_{t-s}, x) V_u(H_{t-s})\big), X_s^1\rangle\, du\, \tilde{X}_{t-s}(dx) \\
&\qquad + \int_{\mathrm{R}^d} \int_0^\epsilon S_{\epsilon-u}\big(U_u(H_{t-s}, x)\big)(X_s^1) \\
&\qquad \times \langle S_{\epsilon-u}\big(U_u(\overline{H}_{t-s}, x) V_u(\overline{H}_{t-s})\big), X_s^2\rangle\, du\, \tilde{X}_{t-s}(dx) \Big\} \Big] \Big| \\
&\equiv I_{\epsilon,\delta'} + II_{\epsilon,\delta'} + III_{\epsilon,\delta'},
\end{aligned}$$

where for the inequality we use Lemma 8.2, the definition of U_ϵ as a solution of the evolution equation and integration by parts formula.

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We have to show that $I_{\epsilon,\delta'}$ approach 0 as $\delta' \downarrow 0$ uniformly in ϵ, and $II_{\epsilon,\delta'}$, $III_{\epsilon,\delta'}$ approach 0 as $\epsilon \downarrow 0$ for each $\delta' > 0$.

We start with $II_{\epsilon,\delta'}$. For each $\epsilon, \epsilon' > 0$, $\epsilon'' \geq 0$, $\delta < \delta'$, we define

$$\begin{aligned}
&M_s^1(\delta,\epsilon,\epsilon',\epsilon'') \\
&\quad \equiv 2\lambda \int_{s-\delta}^{s} e^{-\langle X_u^1, V_{s+\epsilon-u}(H_{t-s})\rangle - \langle X_u^2, V_{s+\epsilon-u}(\overline{H}_{t-s})\rangle} \\
&\qquad \times \langle K_u(X^1,X^2), 2V^1_{s-u+\epsilon}(H_{t-s})\rangle \langle S_{s+\epsilon'-u}(X_u^1) S_{s+\epsilon'-u}(X_u^2), V^1_{\epsilon''}(H_{t-s})\rangle \, du \\
&M_s^2(\delta,\epsilon,\epsilon',\epsilon'') \\
&\quad \equiv \lambda \int_{s-\delta}^{s} e^{-\langle X_u^1, V_{s+\epsilon-u}(H_{t-s})\rangle - \langle X_u^2, V_{s+\epsilon-u}(\overline{H}_{t-s})\rangle} \\
&\qquad \times \langle \big(S_{s+\epsilon'-u}(X_u^1) + S_{s+\epsilon'-u}(X_u^2)\big) S_{s+\epsilon'-u}\big(K_u(X^1,X^2)\big), V^1_{\epsilon''}(H_{t-s})\rangle \, du \\
&M_s^3(\delta,\epsilon,\epsilon',\epsilon'') \\
&\quad \equiv \int_{s-\delta}^{s} e^{-\langle X_u^1, V_{s+\epsilon-u}(H_{t-s})\rangle - \langle X_u^2, V_{s+\epsilon-u}(\overline{H}_{t-s})\rangle} \\
&\qquad \times \int_{\mathrm{R}^d} \left(\int_{\mathrm{R}^d} V_{s+\epsilon-u}(H_{t-s})(y) p_{s+\epsilon'-u}(x-y)\, X_u^1(dy) \right) \\
&\qquad \times S_{s+\epsilon'-u}(X_u^2)(x) V^1_{\epsilon''}(H_{t-s})(x)\, dx\, du, \\
&M_s^4(\delta,\epsilon,\epsilon',\epsilon'') \\
&\quad \equiv \int_{s-\delta}^{s} e^{-\langle X_u^1, V_{s+\epsilon-u}(H_{t-s})\rangle - \langle X_u^2, V_{s+\epsilon-u}(\overline{H}_{t-s})\rangle} \\
&\qquad \times \int_{\mathrm{R}^d} \left(\int_{\mathrm{R}^d} V_{s+\epsilon-u}(\overline{H}_{t-s})(y) p_{s+\epsilon'-u}(x-y)\, X_u^2(dy) \right) \\
&\qquad \times S_{s+\epsilon'-u}(X_u^1)(x) V^1_{\epsilon''}(H_{t-s})(x)\, dx\, du.
\end{aligned}$$

From Lemma 8.1(b) it follows that

$$\begin{aligned}
&II_{\epsilon,\delta'} \\
&= 2\lambda \Big| \lim_{\epsilon'\downarrow 0} \int_{\delta'}^{t} \tilde{P}_{t-s} \times P\Big[\exp\{-\langle X^1_{s-\delta}, V_{\epsilon+\delta}(H_{t-s})\rangle - \langle X^2_{s-\delta}, V_{\epsilon+\delta}(\overline{H}_{t-s})\rangle\} \\
&\times \Big\{ \langle S_{\epsilon'+\delta}(X^1_{s-\delta}) S_{\epsilon'+\delta}(X^2_{s-\delta}), V^1_{\epsilon}(H_{t-s})\rangle - \int_{\mathrm{R}^d} S_{\epsilon+\delta}(X^1_{s-\delta})(x) S_{\epsilon+\delta}(X^2_{s-\delta})(x)\, \tilde{X}_{t-s}(dx) \Big\} \\
&\quad + M_s^1(\delta,\epsilon,\epsilon',\epsilon) + M_s^2(\delta,\epsilon,\epsilon',\epsilon) + M_s^3(\delta,\epsilon,\epsilon',\epsilon) \\
&\quad + M_s^4(\delta,\epsilon,\epsilon',\epsilon) - M_s^1(\delta,\epsilon,\epsilon,0) - M_s^2(\delta,\epsilon,\epsilon,0) - M_s^3(\delta,\epsilon,\epsilon,0) - M_s^4(\delta,\epsilon,\epsilon,0) \Big]\, ds \Big|.
\end{aligned}$$

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Recall that if $\mu = \mu_1 + i\mu_2 \in \tilde{S}'^{\rho}$, then by Theorem 5.8 $V_{\cdot}(\mu)(\cdot) \in \overline{C}([\delta, T] \times \mathrm{R}^d)$ for each $0 < \delta < T$, and $V^1_\epsilon(\mu) \Rightarrow \mu_1$ as $\epsilon \downarrow 0$. Therefore

$$
\begin{aligned}
&\lim_{\epsilon\downarrow 0}\lim_{\epsilon'\downarrow 0} \exp\{-\langle X^1_{s-\delta}, V_{\epsilon+\delta}(H_{t-s})\rangle - \langle X^2_{s-\delta}, V_{\epsilon+\delta}(\overline{H}_{t-s})\rangle\} \\
&\qquad \times \Big\{ \langle S_{\epsilon'+\delta}(X^1_{s-\delta}) S_{\epsilon'+\delta}(X^2_{s-\delta}), V^1_\epsilon(H_{t-s})\rangle \\
&\qquad\qquad - \int_{\mathrm{R}^d} S_{\epsilon+\delta}(X^1_{s-\delta})(x) S_{\epsilon+\delta}(X^2_{s-\delta})(x) \tilde{X}_{t-s}(dx) \Big\} = 0
\end{aligned} \tag{8.4}
$$

$\tilde{P}_{t-s} \times P$-a.s. for each $\delta' \le s \le t$. Furthermore,

$$
\begin{aligned}
&\tilde{P}_{t-s} \times P[|\langle S_{\epsilon+\delta}(X^1_{s-\delta}) S_{\epsilon+\delta}(X^2_{s-\delta}), V^1_{\epsilon''}(H_{t-s})\rangle|^2] \\
&\qquad \le p_\delta(0)^2 \tilde{P}_{t-s}[|\langle 1, V^1_{\epsilon''}(H_{t-s})\rangle|^2] \times P[X^1_{s-\delta}(1)^2 X^2_{s-\delta}(1)^2], \quad \epsilon'' \ge 0.
\end{aligned} \tag{8.5}
$$

By (8.5), Lemma 3.1 and Corollary 7.23 the second moment of $\langle S_{\epsilon+\delta}(X^1_{s-\delta}) S_{\epsilon+\delta}(X^2_{s-\delta}), V^1_{\epsilon''}(H_{t-s})\rangle$ is bounded uniformly in ϵ, ϵ'' and $\delta' \le s \le t$. This gives the uniform integrability condition for $\{\langle S_{\epsilon+\delta}(X^1_{s-\delta}) S_{\epsilon+\delta}(X^2_{s-\delta}), V^1_{\epsilon''}(H_{t-s})\rangle, \epsilon > 0, \epsilon'' \ge 0, \delta' \le s \le t\}$, which together with (8.4) shows that

(8.6)

$$
\begin{aligned}
\lim_{\epsilon\downarrow 0} II_{\epsilon,\delta'} = 2\lambda \lim_{\epsilon\downarrow 0} \Big| \lim_{\epsilon'\downarrow 0} \int_{\delta'}^{t} \tilde{P}_{t-s} \times P[&M^1_s(\delta,\epsilon,\epsilon',\epsilon) + M^2_s(\delta,\epsilon,\epsilon',\epsilon) \\
&+ M^3_s(\delta,\epsilon,\epsilon',\epsilon) + M^4_s(\delta,\epsilon,\epsilon',\epsilon) - M^1_s(\delta,\epsilon,\epsilon,0) \\
&- M^2_s(\delta,\epsilon,\epsilon,0) - M^3_s(\delta,\epsilon,\epsilon,0) - M^4_s(\delta,\epsilon,\epsilon,0)]\, ds \Big|.
\end{aligned}
$$

To prove that $\lim_{\epsilon\downarrow 0} II_{\epsilon,\delta'} = 0$ it suffices to show that for each $k = 1,2,3,4$ $\lim_{\delta\downarrow 0} \tilde{P}_{t-s} \times P[M^k_s(\delta,\epsilon,\epsilon',\epsilon'')] = 0$ uniformly in ϵ, ϵ', ϵ'' and $s \in [\delta', t]$. Without loss of generality, we will, henceforth, assume that

$$0 \le \epsilon + \epsilon' + \epsilon'' \le 1.$$

We introduce several functions and constants which will be frequently used throughout the remainder of this section.

$$c^0 \equiv P\big[X^1_0(1)X^2_0(1) + X^1_0(1)^2 X^2_0(1)^2 + X^1_0(1)X^2_0(1)\big(X^1_0(1) + X^2_0(1)\big)\big],$$

$$c_t^1 \equiv \sup_{s \geq t} \big(p_s(0) + p_s(0)^2\big) = p_t(0) + p_t(0)^2, \quad \forall t > 0,$$

$$c^2 \equiv \|H_0\|_\infty + \|H_0\|_\infty^2,$$

$$c^{2,1} \equiv \|H_0\|_1 + \|H_0\|_2^2,$$

$$c_t^3 \equiv \int_0^t \int_0^t (v_1 + v_2)^{-d/2}\, dv_1\, dv_2,$$

$$\kappa_d(u) \equiv \begin{cases} u^{-1/2}, & d = 3 \\ -\ln(u), & d = 2, \\ u^{1/2}, & d = 1, \end{cases}$$

and let $\tilde{c}$ be a constant that does not depend on $s, t, \delta, \delta', \epsilon, \epsilon', \epsilon''$ and may change from line to line.

Lemma 8.6

$$\lim_{\delta \downarrow 0} P \times \tilde{P}_{t-s}[M_s^1(\delta, \epsilon, \epsilon', \epsilon'')] = 0$$

uniformly in $\epsilon, \epsilon', \epsilon''$ *and* $s \in [\delta', t]$.

Proof It is easy to check that

$$\begin{aligned} &|P \times \tilde{P}_{t-s}[M_s^1(\delta, \epsilon, \epsilon', \epsilon'')]| \\ &\quad \leq \int_{s-\delta}^{s} P \times \tilde{P}_{t-s}[\langle K_u(X^1, X^2), 2V^1_{s-u+\epsilon}(H_{t-s})\rangle \\ &\qquad\qquad \times \langle S_{s+\epsilon'-u}(X_u^1) S_{s+\epsilon'-u}(X_u^2), V^1_{\epsilon''}(H_{t-s})\rangle]\, du. \end{aligned} \tag{8.7}$$

By (7.42) (Lemma 7.21)

$$\begin{aligned} &\tilde{P}_{t-s}[V^1_{s-u+\epsilon}(H_{t-s})(x) V^1_{\epsilon''}(H_{t-s})(x_1)] \\ &\quad \leq S_{t-u+\epsilon}(|H_0|)(x) S_{t-s+\epsilon''}(|H_0|)(x_1) \\ &\qquad + \lambda \int_0^{t-s} \int_{\mathbb{R}^d} p_{t-u+\epsilon-v}(x-y) p_{t-s-v+\epsilon''}(x_1 - y) S_v(|H_0|)(y)\, dy\, dv \\ &\quad \leq \|H_0\|_\infty^2 + \tilde{c}\|H_0\|_\infty \int_0^{t-s} p_{2t-u-s+\epsilon+\epsilon''-2v}(x - x_1)\, dv \\ &\quad \leq c^2 \left(1 + \tilde{c} \int_0^{t-s} (2t - u - s - 2v)^{-d/2}\, dv\right) \\ &\quad \leq c^2 \Big(1 + \tilde{c}\big(|\kappa(s-u)| + |\kappa(2t-u-s)|\big)\Big), \quad \text{a.e.-}(x, x_1). \end{aligned} \tag{8.8}$$

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$X^j \le Y^j$, $j = 1, 2$, where (Y^1, Y^2) is a dominating pair of superprocesses. Therefore we have

$$
\begin{aligned}
P[X_u^j&(x) S_{s+\epsilon'-u}(X_u^j)(x_1)] \\
&\le P[Y_u^j(x) S_{s+\epsilon'-u}(Y_u^j)(x_1)] \\
&= P\Big[S_u(X_0^j)(x) S_{s+\epsilon'}(X_0^j)(x_1) \\
&\qquad + \int_0^u \int_{\mathrm{R}^d} S_{v_j}(X_0^j)(y_j) p_{s+\epsilon'-v_j}(x_1 - y_j) p_{u-v_j}(x - y_j)\, dy_j\, dv_j \Big], \quad j = 1, 2.
\end{aligned}
$$

Recall that

$$
\sup_{\delta \le u} \|S_u(\mu)\|_\infty \le \mu(1) \sup_{\delta \le u} c_u^1 \le \mu(1) c_\delta^1, \quad \forall \mu \in M_F, \quad \delta > 0. \tag{8.9}
$$

Therefore by (3.6) (Lemma 3.4) we get

$$
\begin{aligned}
P\Big[&\int_{\mathrm{R}^d} \int_{\mathrm{R}^d} K_u(X^1, X^2)(x) S_{s+\epsilon'-u}(X_u^1)(x_1) S_{s+\epsilon'-u}(X_u^2)(x_1)\, dx\, dx_1 \Big] \\
&\le P\Big[\int_{\mathrm{R}^d} \int_{\mathrm{R}^d} \Big(S_u(X_0^1)(x) S_{s+\epsilon'}(X_0^1)(x_1) \\
&\qquad\qquad + \int_0^u \int_{\mathrm{R}^d} S_{v_1}(X_0^1)(y_1) p_{s+\epsilon'-v_1}(x_1 - y_1) p_{u-v_1}(x - y_1)\, dy_1\, dv_1 \Big) \\
&\qquad \times \Big(S_u(X_0^2)(x) S_{s+\epsilon'}(X_0^2)(x_1) \\
&\qquad\qquad + \int_0^u \int_{\mathrm{R}^d} S_{v_2}(X_0^2)(y_2) p_{s+\epsilon'-v_2}(x_1 - y_2) p_{u-v_2}(x - y_2)\, dy_2\, dv_2 \Big)\, dx_1\, dx \Big] \\
&\le c_{\delta'-\delta}^1 P[X_0^1(1)^2 X_0^2(1)^2] \\
&\quad + c_{\delta'-\delta}^1 P\Big[X_0^1(1) X_0^2(1) \int_0^u X_0^2(1)\, dv_1 + X_0^1(1) X_0^2(1) \int_0^u X_0^1(1)\, dv_2 \Big] \\
&\quad + P\Big[\int_0^u \int_0^u \int_{\mathrm{R}^d} \int_{\mathrm{R}^d} S_{v_1}(X_0^1)(y_1) p_{2s+2\epsilon'-v_1-v_2}(y_1 - y_2) \\
&\qquad\qquad \times S_{v_2}(X_0^2)(y_2) p_{2u-v_1-v_2}(y_1 - y_2)\, dy_1\, dy_2\, dv_1\, dv_2 \Big] \\
&\le c_{\delta'-\delta}^1 c^0 (1 + t) \\
&\quad + P\Big[\tilde{c} \int_0^u \int_0^u (2s + 2\epsilon' - v_1 - v_2)^{-d/2} \\
&\qquad\qquad \times \int_{\mathrm{R}^d} \int_{\mathrm{R}^d} S_{v_1}(X_0^1)(y_1) S_{v_2}(X_0^2)(y_2) p_{2u-v_1-v_2}(y_1 - y_2)\, dy_1\, dy_2\, dv_1\, dv_2 \Big]
\end{aligned}
$$

where the second inequality follows from (8.9). The last term is bounded by

$$P\Big[\tilde{c}\int_0^s\int_0^s (2s-v_1-v_2)^{-d/2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_{2u}(z_1-z_2)X_0^1(dz_1)X_0^2(dz_2)\,dv_1\,dv_2\Big]$$
$$\le c^1_{2\delta'-2\delta}P[X_0^1(1)X_0^2(1)]\int_0^s\int_0^s (v_1+v_2)^{-d/2}\,dv_1\,dv_2$$
$$\le c^1_{2\delta'-2\delta}c^0c^3_t \le c^1_{\delta'-\delta}c^0c^3_t,$$

where, again, the first inequality follows from (8.9) and the last inequality follows from monotonicity of c^1_t. This yields

$$P\Big[\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} K_u(X^1,X^2)(x)S_{s+\epsilon'-u}(X^1_u)(x_1)S_{s+\epsilon'-u}(X^2_u)(x_1)\,dx\,dx_1\Big] \le c^1_{\delta'-\delta}c^0(1+t+c^3_t)$$

for all $s-\delta \le u \le s \le t$. Combining the last bound with (8.7) and (8.8) we get

$$|P\times\tilde{P}_{t-s}[M^1_s(\delta,\epsilon,\epsilon',\epsilon'')]|$$
$$\le \int_{s-\delta}^s P\times\tilde{P}_{t-s}[\langle K_u(X^1,X^2), 2V^1_{s-u+\epsilon}(H_{t-s})\rangle$$
$$\times\langle S_{s+\epsilon'-u}(X^1_u)S_{s+\epsilon'-u}(X^2_u), V^1_{\epsilon''}(H_{t-s})\rangle]\,du$$
$$\le c^1_{\delta'-\delta}c^0(1+t+c^3_t)\int_{s-\delta}^s\big(|\kappa(s-u)|+|\kappa(2t-u-s)|\big)\,du \to 0,$$

as $\delta\downarrow 0$ uniformly in $s\in[\delta',t]$. The last bound does not depend on ϵ, ϵ', ϵ'', and the lemma follows. ∎

Lemma 8.7

$$\lim_{\delta\downarrow 0} P\times\tilde{P}_{t-s}[M^2_s(\delta,\epsilon,\epsilon',\epsilon'')] = 0$$

uniformly in ϵ, ϵ', ϵ'' *and* $s\in[\delta',t]$.

Proof As in the previous lemma, we have

$$|P\times\tilde{P}_{t-s}[M^2_s(\delta,\epsilon,\epsilon',\epsilon'')]|$$
$$\le \int_{s-\delta}^s P\times\tilde{P}_{t-s}\big[\big\langle\big(S_{s+\epsilon'-u}(X^1_u)+S_{s+\epsilon'-u}(X^2_u)\big) \tag{8.10}$$
$$\times S_{s+\epsilon'-u}\big(K_u(X^1,X^2)\big), V^1_{\epsilon''}(H_{t-s})\big\rangle\big]\,du.$$

By (7.39) we have

$$\tilde{P}_{t-s}[V^1_{\epsilon''}(H_{t-s})(x)\,dx] \le \|H_0\|_\infty\,dx \le c^2\,dx.$$

Combining this with (3.5) (Lemma 3.4) we obtain

$$
\begin{aligned}
&P \times \tilde{P}_{t-s}\big[\langle S_{s+\epsilon-u}(X_u^j) S_{s+\epsilon'-u}\big(K_u(X^1,X^2)\big), V^1_{\epsilon''}(H_{t-s})\rangle\big] \\
&\le c^2 P\bigg[\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} S_{s+\epsilon'-u}(S_u X_0^j)(x_1) p_{s+\epsilon'-u}(x_1-x) S_u(X_0^1)(x) S_u(X_0^2)(x)\,dx\,dx_1 \\
&\qquad + \int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\int_0^u S_{v_1}(X_0^j)(y_1) p_{s+\epsilon'-v_1}(x_1-y_1) p_{u-v_1}(x-y_1) \\
&\qquad \times p_{s+\epsilon'-u}(x_1-x) S_u(X_0^k)(x)\,dx\,dx_1\,dy_1\,dv_1\bigg] \qquad (8.11)\\
&\equiv c^2(I+II),
\end{aligned}
$$

where $j=1, k=2$ or $j=2, k=1$. Consider I first.

$$
\begin{aligned}
I &\le P\bigg[\int_{\mathrm{R}^d} S_{2s+2\epsilon'-2u}(X_0^j)(x) S_u(X_0^1)(x) S_u(X_0^2)(x)\,dx\bigg] \\
&\le c^1_{\delta'-\delta} P\bigg[X_0^1(1) X_0^2(1) \int_{\mathrm{R}^d} S_{2s+2\epsilon'-2u}(X_0^j)(x)\,dx\bigg] \qquad (8.12)\\
&\le c^1_{\delta'-\delta} P[X_0^1(1) X_0^2(1) X_0^j(1)] \le c^1_{\delta'-\delta} c^0.
\end{aligned}
$$

Now consider II in (8.11). Integrating with respect to x_1 and proceeding with simple calculations, we obtain

$$
\begin{aligned}
II &= P\bigg[\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\int_0^u S_{v_1}(X_0^j)(y_1) p_{2s+2\epsilon'-u-v_1}(x-y_1) p_{u-v_1}(x-y_1) S_u(X_0^k)(x)\,dx\,dy_1,\,dv_1\bigg] \\
&\le \tilde{c} P\bigg[\int_0^u (2s+2\epsilon'-u-v_1)^{-d/2} \int_{\mathrm{R}^d}\int_{\mathrm{R}^d} S_{v_1}(X_0^j)(y_1) p_{u-v_1}(x-y_1) S_u(X_0^k)(x)\,dx\,dy_1,\,dv_1\bigg] \\
&\le \tilde{c}\int_0^u (2s-u-v_1)^{-d/2} P\bigg[\int_{\mathrm{R}^d} S_u(X_0^j)(x) S_u(X_0^k)(x)\,dx\,dv_1\bigg] \\
&\le \tilde{c}\big(|\kappa(2s-2u)| + |\kappa(2s-u)|\big) P\bigg[\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_{2u}(z_1-z_2)\, X_0^1(dz_1)\, X_0^2(dz_2)\bigg] \\
&\le \tilde{c}\big(|\kappa(2s-2u)| + |\kappa(2s-u)|\big) P[X_0^1(1) X_0^2(1)] \sup_{2\delta'\le 2u\le 2t} p_{2u}(0) \\
&\le \tilde{c} c^1_{\delta'-\delta} c^0 \big(|\kappa(2s-2u)| + |\kappa(2s-u)|\big).
\end{aligned}
$$

The last bound and equations (8.10)–(8.12) imply that

$$
|P \times \tilde{P}_{t-s}[M_s^2(\delta,\epsilon,\epsilon',\epsilon'')]| \le c^2 c^1_{\delta'-\delta} c^0 \int_{s-\delta}^s 1 + \tilde{c}\big(|\kappa(2s-2u)| + |\kappa(2s-u)|\big)\,du \to 0,
$$

as $\delta \downarrow 0$ uniformly in $s \in [\delta', t]$. The last bound does not depend on ϵ, ϵ', ϵ'' and the lemma follows. ■

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Lemma 8.8 *Define*

$$M_s^5(\delta,\epsilon,\epsilon',\epsilon'') \equiv \int_{s-\delta}^{s} \tilde{P}_{t-s}\Big[\int_{\mathrm{R}^d} |V_{s+\epsilon-u}(H_{t-s})(y)| S_{s+\epsilon'-u}\big(V^1_{\epsilon''}(H_{t-s})\big)(y)\,dy\Big]\,du.$$

Then

$$M_s^5(\delta,\epsilon,\epsilon',\epsilon'') \le 2c^{2,1}\int_{s-\delta}^{s} 1+\tilde{c}\big(|\kappa(s-u)|+|\kappa(t-u)|\big)\,du, \quad \forall\,\epsilon,\epsilon',\epsilon'', \tag{8.13}$$

and

$$\lim_{\delta\downarrow 0} M_s^5(\delta,\epsilon,\epsilon',\epsilon'') = 0 \tag{8.14}$$

uniformly in $s\in[\delta',t]$ *and* ϵ, ϵ', ϵ''.

Proof (8.14) is immediate from (8.13). Now consider (8.13). Use Hölder inequality to bound $M_s^5(\delta,\epsilon,\epsilon',\epsilon'')$ by

(8.15)
$$\sqrt{\int_{s-\delta}^{s} \tilde{P}_{t-s}\Big[\int_{\mathrm{R}^d} |V_{s+\epsilon-u}(H_{t-s})(y)|^2\,dy\Big]\,du}\sqrt{\int_{s-\delta}^{s} \tilde{P}_{t-s}\Big[\int_{\mathrm{R}^d} S_{s+\epsilon'-u}\big(V^1_{\epsilon''}(H_{t-s})\big)(y)^2\,dy\Big]\,du.}$$

Recall that $c^{2,1}=\|H_0\|_1+\|H_0\|_2^2$. Use (7.41) (take $\psi=\delta_y$) to see that

$$\begin{aligned}
&\int_{\mathrm{R}^d} \tilde{P}_{t-s}[|V_{s+\epsilon-u}(H_{t-s})(y)|^2]\,dy\\
&\quad\le 2\int_{\mathrm{R}^d} S_{t+\epsilon-u}(|H_0|)(y)^2\,dy\\
&\qquad + 2\lambda\int_0^{t-s}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_{t+\epsilon-u-v}(x-y)^2 S_v(|H_0|)(x)\,dx\,dy\,dv\\
&\quad\le 2\|H_0\|_2^2+\tilde{c}\int_0^{t-s}\|H_0\|_1(t+\epsilon-u-v)^{-d/2}\,dv\\
&\quad\le 2c^{2,1}\Big(1+\tilde{c}\int_0^{t-s}(t-u-v)^{-d/2}\,dv\Big)\\
&\quad\le 2c^{2,1}\Big(1+\tilde{c}\big(|\kappa(s-u)|+|\kappa(t-u)|\big)\Big).
\end{aligned} \tag{8.16}$$

Turning to $\tilde{P}_{t-s}\big[\int_{\mathrm{R}^d} S_{s+\epsilon'-u}\big(V^1_{\epsilon''}(H_{t-s})\big)(y)^2\,dy\big]$, we may use (7.41) again to bound it by

$$\begin{aligned}
&\int_{\mathrm{R}^d} S_{s+\epsilon'-u}\big(S_{t-s+\epsilon''}(|H_0|)\big)(y)^2\,dy\\
&\qquad + \lambda\int_0^{t-s}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_{t+\epsilon+\epsilon'-u-v}(x-y)^2 S_v(|H_0|)(x)\,dx\,dy\,dv.
\end{aligned}$$

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Argue as in (8.16) to bound the above by

$$c^{2,1}\Big(1+\tilde{c}\big(|\kappa(s-u)|+|\kappa(t-u)|\big)\Big). \tag{8.17}$$

Combining (8.15), (8.16) and (8.17) we are done. ∎

Lemma 8.9 *For $k=3,4$,*

$$\lim_{\delta\downarrow 0} P\times\tilde{P}_{t-s}[M_s^k(\delta,\epsilon,\epsilon',\epsilon'')]=0$$

uniformly in ϵ, ϵ', ϵ'' and $s\in[\delta',t]$.

Proof We will only prove the assertion about $M_s^3(\delta,\epsilon,\epsilon',\epsilon'')$ since the proof of the assertion about $M_s^4(\delta,\epsilon,\epsilon',\epsilon'')$ is the same. Routine arguments similar to those used in the proof of Lemma 8.6 and Lemma 8.7 show that

$$\begin{aligned}
&|P\times\tilde{P}_{t-s}[M_s^3(\delta,\epsilon,\epsilon',\epsilon'')]|\\
&\le\int_{s-\delta}^{s}\tilde{P}_{t-s}\Big[\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|V_{s+\epsilon-u}(H_{t-s})(y)|p_{s+\epsilon'-u}(x-y)\\
&\qquad\times P[X_u^1(dy)S_{s+\epsilon'-u}(X_u^2)(x)]V^1_{\epsilon''}(H_{t-s})(x)\,dx\Big]\,du\\
&\le\int_{s-\delta}^{s}\tilde{P}_{t-s}\Big[\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|V_{s+\epsilon-u}(H_{t-s})(y)|p_{s+\epsilon'-u}(x-y)\\
&\qquad\times P\big[S_u(X_0^1)(y)S_{s+\epsilon'-u}\big(S_u(X_0^2)\big)(x)\big]\times V^1_{\epsilon''}(H_{t-s})(x)\,dy\,dx\Big]\,du\\
&\le c^1_{\delta'-\delta}P[X_0^1(1)X_0^2(1)]\int_{s-\delta}^{s}\tilde{P}_{t-s}\Big[\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|V_{s+\epsilon-u}(H_{t-s})(y)|p_{s+\epsilon'-u}(x-y)\\
&\qquad\times V^1_{\epsilon''}(H_{t-s})(x)\,dy\,dx\Big]\,du\\
&\le c^1_{\delta'-\delta}c^0\int_{s-\delta}^{s}\tilde{P}_{t-s}\Big[\int_{\mathrm{R}^d}|V_{s+\epsilon-u}(H_{t-s})(y)|S_{s+\epsilon'-u}\big(V^1_{\epsilon''}(H_{t-s})\big)(y)\,dy\Big]\,du\\
&=c^1_{\delta'-\delta}c^0M_s^5(\delta,\epsilon,\epsilon',\epsilon'').
\end{aligned}$$

Now we are done by Lemma 8.8. ∎

Lemma 8.10 *For each $\delta'>0$, $\lim_{\epsilon\downarrow 0}II_{\epsilon,\delta'}=0$.*

Proof Recall that by (8.6)

$$\lim_{\epsilon\downarrow 0}II_{\epsilon,\delta'}=2\lambda\lim_{\epsilon\downarrow 0}\Big|\lim_{\epsilon'\downarrow 0}\sum_{k=1}^{4}\int_{\delta'}^{t}\tilde{P}_{t-s}\times P[M_s^k(\delta,\epsilon,\epsilon',\epsilon)-M_s^k(\delta,\epsilon,\epsilon,0)]\,ds\Big| \tag{8.18}$$

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for each $\delta < \delta' < s$. Applying Lemmas 8.6, 8.7, 8.9 it is easy to make

$$\left|\sum_{k=1}^{4} \int_{\delta'}^{t} \tilde{P}_{t-s} \times P[M_s^k(\delta, \epsilon, \epsilon', \epsilon) - M_s^k(\delta, \epsilon, \epsilon, 0)]\, ds\right|$$

arbitrarily small uniformly in ϵ, ϵ' by choosing δ sufficiently small. This finishes the proof of the lemma. ∎

Lemma 8.11 $\lim_{\delta' \downarrow 0} I_{\epsilon,\delta'} = 0$ *uniformly in* $0 < \epsilon \leq 1$.

Proof

(8.19)
$$\begin{aligned} I_{\epsilon,\delta'} \leq \Bigg| 2\lambda \int_0^{\delta'} \tilde{P}_{t-s} \times P\Big[&\langle K_s(X^1, X^2), V_\epsilon^1(H_{t-s})\rangle \\ &+ \int_{\mathrm{R}^d} |U_\epsilon(H_{t-s}, x)(X_s^1) U_\epsilon(\overline{H}_{t-s}, x)(X_s^2)|\, \tilde{X}_{t-s}(dx)\Big]\, ds \Bigg|. \end{aligned}$$

By (7.39) we obtain

(8.20)
$$\begin{aligned} &\int_0^{\delta'} \tilde{P}_{t-s} \times P[\langle K_s(X^1, X^2), V_\epsilon^1(H_{t-s})\rangle]\, ds \\ &\qquad \leq \int_0^{\delta'} P\left[\int_{\mathrm{R}^d} S_s(X_0^1)(x) S_s(X_0^2)(x)\right] S_{t-s+\epsilon}(|H_0|)(x)\, dx\, ds \\ &\qquad \leq c^2 \int_0^{\delta'} P\left[\int_{\mathrm{R}^d} p_{2s}(x-y)\, X_0^1(dx)\, X_0^2(dy)\right] ds. \end{aligned}$$

The last integral is finite (and therefore approaches 0 as $\delta' \downarrow 0$) since $P(X_0^1, X_0^2)^{-1} \in M_1^*(M_{F,w} \times M_{F,w})$. Turning to the second term in (8.19), we may use Corollary 7.22 to see that

$$\begin{aligned} &\int_0^{\delta'} \tilde{P}_{t-s} \times P\left[\int_{\mathrm{R}^d} |U_\epsilon(H_{t-s}, x)(X_s^1) U_\epsilon(\overline{H}_{t-s}, x)(X_s^2)|\, \tilde{X}_{t-s}(dx)\right] ds \\ &\quad \leq \int_0^{\delta'} \tilde{P}_{t-s} \times P\left[\int_{\mathrm{R}^d} S_\epsilon(X_s^1)(x) S_\epsilon(X_s^2)(x) S_{t-s}(|H_0|)(x)\, dx\right] ds \\ &\quad \leq c^2 \int_0^{\delta'} P\left[\int_{\mathrm{R}^d} p_{2s+2\epsilon}(x-y)\, X_0^1(dx)\, X_0^2(dy)\, dx\, dy\right] ds \\ &\quad \leq c^2 \int_\epsilon^{\epsilon+\delta'} P\left[\int_{\mathrm{R}^d} p_{2s}(x-y)\, X_0^1(dx)\, X_0^2(dy)\, dx\, dy\right] ds. \end{aligned}$$

As in (8.20), the last expression approaches 0 as $\delta' \downarrow 0$ uniformly in $0 < \epsilon \leq 1$ and the lemma follows. ∎

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Lemma 8.12 *For each $\delta' > 0$, $\lim_{\epsilon\downarrow 0} III_{\epsilon,\delta'} = 0$.*

Proof

$$\begin{aligned} III_{\epsilon,\delta'} \le 2\lambda \int_{\delta'}^{t} \tilde{P}_{t-s} \times P\Big[\int_{\mathrm{R}^d}\int_0^{\epsilon} &\langle S_{\epsilon-u}\big(|U_u(\overline{H}_{t-s},x)|\big), X_s^2\rangle \\ &\times \langle S_{\epsilon-u}\big(|U_u(H_{t-s},x)|\,|V_u(H_{t-s})|\big), X_s^1\rangle\, du\, \tilde{X}_{t-s}(dx)\Big]\, ds \\ + 2\lambda \int_{\delta'}^{t} \tilde{P}_{t-s} \times P\Big[\int_{\mathrm{R}^d}\int_0^{\epsilon} &\langle S_{\epsilon-u}\big(|U_u(H_{t-s},x)|\big), X_s^1\rangle \\ &\times \langle S_{\epsilon-u}\big(|U_u(\overline{H}_{t-s},x)|\,|V_u(\overline{H}_{t-s})|\big), X_s^2\rangle\, du\, \tilde{X}_{t-s}(dx)\Big]\, ds. \end{aligned}$$

We will only prove that the first term converges to 0 as $\delta' \downarrow 0$ since the proof of the convergence for the second term is the same.

$$\begin{aligned} &\int_{\delta'}^{t} \tilde{P}_{t-s} \times P\Big[\int_{\mathrm{R}^d}\int_0^{\epsilon} \langle S_{\epsilon-u}\big(|U_u(\overline{H}_{t-s},x)|\big) X_s^2\rangle \\ &\qquad \times \langle S_{\epsilon-u}\big(|U_u(H_{t-s},x)|\,|V_u(H_{t-s})|\big), X_s^1\rangle\, du\, \tilde{X}_{t-s}(dx)\Big]\, ds \\ &\le \int_{\delta'}^{t} \tilde{P}_{t-s} \times P\Big[\int_{\mathrm{R}^d}\int_0^{\epsilon} \langle S_{\epsilon-u}\big(p_u(x-\cdot)\big), X_s^2\rangle \\ &\qquad \times \langle S_{\epsilon-u}\big(p_u(x-\cdot)|V_u(H_{t-s})|\big), X_s^1\rangle\, du\, \tilde{X}_{t-s}(dx)\Big]\, ds \\ &\le \int_{\delta'}^{t} \tilde{P}_{t-s} \times P\Big[\int_{\mathrm{R}^d}\int_0^{\epsilon} S_{\epsilon}(X_s^2)(x) \\ &\qquad \times \langle p_u(x-\cdot)|V_u(H_{t-s})|, S_{\epsilon-u}X_s^1\rangle\, du\, \tilde{X}_{t-s}(dx)\Big]\, ds \\ &\le \int_{\delta'}^{t} \tilde{P}_{t-s} \times P\Big[\int_0^{\epsilon} \|S_{\epsilon+s}(X_0^2)\|_\infty\, \|S_{\epsilon+s-u}(X_0^1)\|_\infty \\ &\qquad \times \int_{\mathrm{R}^d} |V_u(H_{t-s})|(y) S_u(\tilde{X}_{t-s})(y)\, dy\, du\Big]\, ds \\ &\le c^1_{\delta'} c^0 \int_{\delta'}^{t} \tilde{P}_{t-s}\Big[\int_s^{s+\epsilon}\int_{\mathrm{R}^d} |V_{s+\epsilon-u}(H_{t-s})|(y) S_{s+\epsilon-u}(\tilde{X}_{t-s})(y)\, dy\, du\Big]\, ds \\ &= c^1_{\delta'} c^0 \int_{\delta'}^{t} M^5_{s+\epsilon}(\epsilon,0,0,0)\, ds. \end{aligned}$$

The first inequality follows from (5.18) and the fact that $\tilde{X}_s \in M_F^{\rho}$ (and hence does not charge sets of nil capacity). The derivation of the other inequalities is straightforward. Now apply Lemma 8.8 to complete the proof. ■

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Recall we are proving Lemma 8.5.

Proof of Lemma 8.5

$$\lim_{\epsilon\downarrow 0}\{P\times\tilde{P}_t[\exp\{-\langle X_t^1, V_\epsilon(H_0)\rangle - \langle X_t^2, V_\epsilon(\overline{H}_0)\rangle\}]$$
$$- P\times\tilde{P}_t[\exp\{-\langle X_0^1, V_\epsilon(H_t)\rangle - \langle X_0^2, V_\epsilon(\overline{H}_t)\rangle\}]\} = 0.$$

The limit equals to

$$\lim_{\epsilon\downarrow 0}(I_{\epsilon,\delta'} + II_{\epsilon,\delta'} + III_{\epsilon,\delta'})$$

for each $\delta' > 0$. By Lemma 8.11 we can make $I_{\epsilon,\delta'}$ arbitrarily small uniformly in ϵ by fixing δ' sufficiently small. By Lemma 8.10 and Lemma 8.12 $II_{\epsilon,\delta'}$ and $III_{\epsilon,\delta'}$ approach 0 as $\epsilon\downarrow 0$ for the chosen fixed δ' and this finishes the proof of the lemma. ■

Corollary 8.13

$$P[\exp\{-\langle X_t^1,\phi\rangle - \langle X_t^2,\overline{\phi}\rangle\}] = \lim_{\epsilon\downarrow 0} P\times\tilde{P}_t[\exp\{-\langle X_0^1, V_\epsilon(H_t)\rangle - \langle X_0^2, V_\epsilon(\overline{H}_t)\rangle\}],$$

where $\phi = H_0$.

Proof Since $\phi,\overline{\phi}\in\tilde{S}(\mathrm{R}^d)$, we have that $\lim_{\epsilon\downarrow 0}\|V_\epsilon(\phi)-\phi\|_\infty = 0$, $\lim_{\epsilon\downarrow 0}\|V_\epsilon(\overline{\phi})-\overline{\phi}\|_\infty = 0$. Now we are done by Lemma 8.5. ■

Remark 8.14 $\nu = P(X_0^1, X_0^2)^{-1}\in M_1^*(M_{F,w}\times M_{F,w})$ and $\phi\in\tilde{S}(\mathrm{R}^d)$ were arbitrary. Therefore by Corollary 8.13 and Lemma 4.3 the proof of Theorem 2.6 is now finished.

A Appendix

Proof of Theorems 5.8, 5.9

A.1 Proof of Theorem 5.8

We start with some notation. For any interval $I\in\mathrm{R}_+$ and any measure $\nu\in M_F(\mathrm{R}_+\times\mathrm{R}^d)$ (the set of finite measures on $\mathrm{R}_+\times\mathrm{R}^d$) let $\|\cdot\|_{p,I}$ and $\|\cdot\|_{p,\nu}$ be the norms on the spaces $L^p(I\times\mathrm{R}^d)$ and $L^p\big(\mathrm{R}_+\times\mathrm{R}^d,\nu(ds,dx)\big)$ respectively.

Let us introduce two more spaces:

$$L^2_{\mathrm{loc}}\equiv\{f\in L^2\big((0,T]\times\mathrm{R}^d\big),\quad\forall T>0\},$$
$$L^{2,p}_{c,\mathrm{loc}}\equiv\Big\{f\in C\big((0,\infty)\times\mathrm{R}^d\big): \|f\|^2_{2,(0,T],p}$$
$$\equiv\int_{0+}^T s^{-p}\|f(s,\cdot)\|_2^2\,ds<\infty,\quad\forall T>0\Big\},\quad p\geq 0.$$

In what follows, in order to simplify notation, we will write $\int_0^{\cdot}$ instead of $\int_{0+}^{\cdot}$.

It is clear that if $f\in L^{2,p}_{c,\mathrm{loc}}$ for some $p>0$, then $f\in L^{2,q}_{c,\mathrm{loc}}$ for any $q<p$ and in particular $f\in L^2_{\mathrm{loc}}$.

Let $p>0$. We say that $f^{(n)}\to f$ in $L^{2,p}_{c,\mathrm{loc}}$ if

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(i) $f^{(n)} \to f$ uniformly on compact subsets of $(0,\infty) \times \mathrm{R}^d$,
(ii) $\lim_{\epsilon\downarrow 0} \limsup_n \|f^{(n)}\|_{2,(0,\epsilon],p} = 0$,
(iii) $\limsup_{n\to\infty} \sup_{\epsilon\le s\le T} \|f^{(n)}(s,\cdot)\|_\infty < \infty, \forall\, 0<\epsilon<T$.

Recall that ρ, $\hat\rho$ are constants that satisfy

$$\left(\frac{d}{2} - 1 \vee 0\right) < \rho < \left(1 \wedge \frac{3}{2} - \frac{d}{4}\right), \tag{A.1}$$

$$0 < \hat\rho < \left(3 - \frac{d}{2} - 2\rho \wedge 1 - \rho\right). \tag{A.2}$$

Recall that for any $r > 0$, $\kappa \in C_{\mathrm{R}}(\mathrm{R}_+)_+$ and $f \in \tilde S(\mathrm{R}^d)$ $V_{r,t}(f,\kappa)$ denotes the solution of the following evolution equation:

$$v_t = S_{t-r} f + \int_r^t S_{t-s}(v_s^2)\kappa(s)\,ds, \quad t > r. \tag{A.3}$$

Given $T > 0$, $V_{\cdot,T}(\cdot,\cdot)(\cdot)$ may be considered as a mapping

$$[0,T) \times \tilde S(\mathrm{R}^d) \times C_{\mathrm{R}}\big([0,T)\big)_+ \times \mathrm{R}^d \mapsto \mathrm{C}.$$

Our main concern in this subsection is to prove that $V_{\cdot,T}(\cdot,\cdot)$ may be extended to the mapping $[0,T) \times \tilde S'^{\rho} \times L^\infty_{\mathrm{R}}\big([0,T)\big)_+ \times \mathrm{R}^d \mapsto \mathrm{C}$ (where we induce weak* topology on $L^\infty_{\mathrm{R}}\big([0,T)\big)_+$) and this mapping is continuous. For the definition of $\tilde S'^{\rho}$ and the definitions and basic properties of $w(s,\mu)$, $\overline{w}_\rho(\delta,\mu)$, $\tilde w_\rho(s,\mu)$ (with $\mu \in \tilde S'$) the reader is referred to Section 5.

The following lemma will be extensively used.

Lemma A.1 *Let $f^{(n)} \to f$ in $\tilde S'^{\rho}$. Then*

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \int_0^\epsilon t^{-\hat\rho}\big(w(t,f^{(n)}) + \tilde w_\rho(t,f^{(n)})^2 t^{2-\frac{d}{2}-2\rho}\big)\,dt = 0, \tag{A.4}$$

$$\begin{aligned}\lim_{n\to\infty} &\int_0^T t^{-\hat\rho}\big(w(t,f^{(n)}) + \tilde w_\rho(t,f^{(n)})^2 t^{2-\frac{d}{2}-2\rho}\big)\,dt \\ &= \int_0^T t^{-\hat\rho}\big(w(t,f) + \tilde w_\rho(t,f)^2 t^{2-\frac{d}{2}-2\rho}\big)\,dt < \infty, \quad \forall\, T>0.\end{aligned} \tag{A.5}$$

Proof (A.5) is an easy consequence of (A.4). The derivation of (A.4) is straightforward from our assumptions on ρ and $\hat\rho$ and the definition of convergence in $\tilde S'^{\rho}$. The details are left to the reader. ■

Observe that for each $r > 0$, $V_{r,r+t}(f,\kappa) = V_{0,t}\big(f,\kappa(r+\cdot)\big)$, therefore many properties of $V_{r,t}$ may be expressed via the properties of $V_{0,t}$. For simplicity (but with a slight abuse of notation), set $V_t(f,\kappa) \equiv V_{0,t}(f,\kappa)$, that is, $V_t(f,\kappa)$ solves the following evolution equation

$$v_t = S_t f + \int_0^t S_{t-s}(v_s^2)\kappa(s)\,ds, \quad t > 0. \tag{A.6}$$

In the sequel for given f, κ, if we consider $V_\cdot(f,\kappa)(\cdot)$ as a function defined on $(0,\infty)\times \mathrm{R}^d$, then this function will be denoted by $V(f,\kappa)$. As we will see later, Theorem 5.8 is an easy consequence of the following proposition.

Proposition A.2

(a) For each $\mu \in \tilde{S}'^{\rho}$ and $\kappa \in L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$, there exists a unique solution $V_t(\mu,\kappa)$ for (A.6) such that

$$V(\mu,\kappa) \in L^{2,\hat\rho}_{c,\mathrm{loc}},$$

$$V_{\epsilon+\cdot}(\mu,\kappa) \in \overline{C}\big([0,T]\times \mathrm{R}^d\big)_+, \quad \forall\, T>0, \quad \epsilon>0,$$

$$V_t(\mu,\kappa) \in L^q(\mathrm{R}^d)_+, \quad \forall\, t>0, \quad q \geq 2,$$

$$V_t^1(\mu,\kappa) \in L^1_{\mathrm{R}}(\mathrm{R}^d)_+, \quad \forall\, t>0.$$

If $\kappa \in \overline{C}_{\mathrm{R}}(\mathrm{R}_+)_+$, then $V_t(\mu,\kappa)$ is a strong solution for (A.6).

(b) Let $\kappa^{(n)} \to \kappa$ weakly in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$ and $\mu^{(n)} \to \mu$ in $\tilde{S}'^{\rho}$. Then*

$$V(\mu^{(n)},\kappa^{(n)}) \to V(\mu,\kappa) \quad \text{in } L^{2,\hat\rho}_{c,\mathrm{loc}},$$

as $n\to\infty$.

(c) Let $\psi \in S(R^d)$, and let A be any compact subset of $\big(\tilde{S}'^{\rho} \times L^\infty_{\mathrm{R}}(\mathrm{R}_+)\big)_+$ (as usual, the topology on $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$ is weak). Then*

$$\lim_{\epsilon\downarrow 0} \sup_{s<\epsilon,(\mu,\kappa)\in A} |\langle \psi, V_s(\mu,\kappa)\rangle - \langle \psi,\mu\rangle| = 0.$$

We will prove this proposition via a series of lemmas.

Lemma A.3 *Let $\psi^{(n)} \in \overline{C}([\epsilon,T]\times \mathrm{R}^d)$ for some $0<\epsilon<T$. Suppose that*

$$\sup_n \|\psi^{(n)}\|_{\infty,[\epsilon,T]} < \infty$$

and $\psi^{(n)} \to \psi$ uniformly on compact subsets of $[\epsilon,T]\times\mathrm{R}^d$. Then, for all $(t,x)\in[\epsilon,T]\times\mathrm{R}^d$, $S_{t-\cdot}\big(\psi^{(n)}(\cdot,\cdot)\big)(x) \to S_{t-\cdot}\big(\psi(\cdot,\cdot)\big)(x)$ uniformly on $[\epsilon,t]$.

Proof Let $\{s_n\}$ be a sequence such that $s_n \to s$ in $[\epsilon,t]$. We need to show that

$$\lim_{n\to\infty} \left| \int_{y\in\mathrm{R}^d} p_{t-s_n}(x-y)\psi^{(n)}(s_n,y) - p_{t-s}(x-y)\psi(s,y)\,dy \right| = 0.$$

But the sequence of measures $p_{t-s_n}(x-y)\,dy$ converges weakly to the measure $p_{t-s}(x-y)\,dy$ (or δ_x in the special case $s=t$). By our assumptions on $\psi^{(n)}$ the result follows from standard theorems on weak convergence. ■

Lemma A.4 *Let* $\psi^{(n)} \in C\big((0,\infty)\times \mathrm{R}^d\big)$. *Define*

$$G(\psi^{(n)})(t,x) = \int_0^t S_{t-s}\big(\psi^{(n)}(s,\cdot)\big)(x)\,ds, \quad \forall\,(t,x)\in(0,\infty)\times\mathrm{R}^d.$$

Suppose that

$$\lim_{\epsilon\downarrow 0}\limsup_n \|\psi^{(n)}\|_{1,(0,\epsilon]} = 0, \tag{A.7}$$

and, for each $T > 0$,

$$\limsup_n \|\psi^{(n)}\|_{\infty,[\epsilon,T]} < \infty, \quad \forall\, 0<\epsilon<T. \tag{A.8}$$

Then $G(\psi^{(n)})$ *is relatively compact in* $C\big((0,\infty)\times\mathrm{R}^d\big)$, *that is, for each subsequence of* $G(\psi^{(n)})$, *there is a further subsequence that converges uniformly on compact subsets of* $(0,\infty)\times\mathrm{R}^d$.

Proof Let (t,x) be an arbitrary point in $(0,\infty)\times\mathrm{R}^d$. Choose $\epsilon>0$ such that $\epsilon<t$. Then

$$\begin{aligned}|G(\psi^{(n)})(t,x)| &\le \int_0^\epsilon\int_{\mathrm{R}^d}|p_{t-u}(x-z)|\,|\psi^{(n)}(u,z)|\,dz\,du \\ &\quad + \int_\epsilon^t\int_{\mathrm{R}^d}|p_{t-u}(x-z)|\,|\psi^{(n)}(u,z)|\,dz\,du \\ &\le \|p_{t-\cdot}(x-\cdot)\|_{\infty,(0,\epsilon]}\|\psi^{(n)}\|_{1,(0,\epsilon]} + |t-\epsilon|\,\|\psi^{(n)}\|_{\infty,[\epsilon,t]} < \infty\end{aligned} \tag{A.9}$$

uniformly in n. Let us check equicontinuity condition. For any $(t,x),(s,y)\in(0,\infty)\times\mathrm{R}^d$, $\epsilon>0$ (without loss of generality, we assume that $s\le t$, $0<\epsilon<s$)

$$\begin{aligned}&|G(\psi^{(n)})(t,x) - G(\psi^{(n)})(s,y)| \\ &\quad\le \int_0^\epsilon\int_{\mathrm{R}^d}|p_{t-u}(x-z)-p_{s-u}(y-z)|\,|\psi^{(n)}(u,z)|\,dz\,du \\ &\qquad + \int_\epsilon^s\int_{\mathrm{R}^d}|p_{t-u}(x-z)-p_{s-u}(y-z)|\,|\psi^{(n)}(u,z)|\,dz\,du \\ &\qquad + \left|\int_s^t S_{t-u}(\psi^{(n)})(u,x)\,du\right| \\ &\quad\le \|p_{t-\cdot}(x-\cdot)-p_{s-\cdot}(y-\cdot)\|_{\infty,(0,\epsilon]}\|\psi^{(n)}\|_{1,(0,\epsilon]} \\ &\qquad + \|\psi^{(n)}\|_{\infty,[\epsilon,s]}\|p_{t-\cdot}(x-\cdot)-p_{s-\cdot}(y-\cdot)\|_{1,[\epsilon,s]} + |t-s|\,\|\psi^{(n)}\|_{\infty,[s,t]}.\end{aligned}$$

Using (A.7), it is easy to make the first term arbitrarily small by fixing ϵ sufficiently small. Observe that $p_{t-\cdot}(x-\cdot)$ converges to $p_{s-\cdot}(y-\cdot)$ in $L^1([\epsilon,s]\times\mathrm{R}^d)$ as $(t,x)\to(s,y)$. This together with (A.8) implies that the second term may be made arbitrarily small uniformly in n for all (t,x) sufficiently close to (s,y). The same happens to the third term. Therefore $\{G(\psi^{(n)})n\ge 1\}$ is equicontinuous at (s,y) (for $s>t$ the arguments are the same). By (A.9) it is also bounded at (s,y). Since (s,y) was an arbitrary point in $(0,\infty)\times\mathrm{R}^d$, we are done by Arzela-Ascoli theorem. ■

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Lemma A.5 *Let $\mu^{(n)} \to \mu$ in S'. Then, for any $T > \epsilon > 0$ and any compact set $\Gamma \subset \mathrm{R}^d$,*

$$\lim_{n\to\infty} \sup_{\epsilon \le t \le T, x\in\Gamma} |S_t\mu^{(n)}(x) - S_t\mu(x)| = 0.$$

Proof By definition, $S_t\mu^{(n)}(x) = \langle p_t(x-\cdot), \mu^{(n)}\rangle$. The set of functions $\{p_t(x-\cdot) : \epsilon \le t \le T, x \in \Gamma\}$ is a bounded compact set in $S(R^d)$, and, as $\mu^{(n)} \to \mu$ in the strong topology of S', we are done. ∎

Lemma A.6 *Let $\Gamma \subset S'$ be compact. Then for any $\psi \in S(R^d)$*

$$\lim_{\epsilon\to\infty} \sup_{\mu\in\Gamma} |\langle \psi, S_\epsilon\mu\rangle - \langle \psi, \mu\rangle| = 0.$$

Proof Since $\langle \psi, S_\epsilon\mu\rangle = \langle S_\epsilon\psi, \mu\rangle$, $S_\epsilon\psi \to \psi$ in $S(R^d)$ as $\epsilon \to 0$, and Γ is a compact set of linear continuous functionals on $S(R^d)$, the result follows. ∎

Lemma A.7 *Let $f \in \tilde{S}(\mathrm{R}^d)$, $\kappa \in \overline{C}_{\mathrm{R}}(\mathrm{R}_+)_+$. Denote $v_{t,T} = V_{t,T}(f,\kappa)$. Then*

$$|v_{t,T}(x)| \le |S_{T-t}(f)(x)| + \|\kappa\|_\infty \int_t^T \int_{\mathrm{R}^d} p_{s-t}(x-y)|S_{T-s}(f)(y)|^2\,dy\,ds, \quad \forall\, 0 \le t < T, \quad x \in \mathrm{R}^d.$$

Proof The author suspects that the lemma may be proved by using tools from PDE-s. But here we will apply probability methods.

For $\epsilon > 0$ let X_t^ϵ be a super-Brownian motion started at $\epsilon\delta_x$ defined on the time interval $[0,T]$ such that

$$e^{-\epsilon v_{t,T}(x)} = P_{\epsilon\delta_x}[e^{-X^\epsilon_{T-t}(f)}], \quad 0 \le t < T.$$

By Taylor expansion we have

$$e^{-X_t^\epsilon(f)} = 1 - X_t^\epsilon(f) + \int_0^1 e^{-\theta X_t^\epsilon(f)} X_t^\epsilon(f)^2(1-\theta)\,d\theta,$$

where $0 \le \theta \le 1$ is some random number. Therefore

$$\begin{aligned}
\left|\frac{1}{\epsilon}(1 - e^{-\epsilon v_{T-t,T}(x)})\right| &= \left|\frac{1}{\epsilon}P_{\epsilon\delta_x}[X_t^\epsilon(f)] - \frac{1}{\epsilon}P_{\epsilon\delta_x}\left[\int_0^1 e^{-\theta X_t^\epsilon(f)} X_t^\epsilon(f)^2(1-\theta)\,d\theta\right]\right| \\
&\le |S_t(f)(x)| + \frac{1}{\epsilon}P_{\epsilon\delta_x}\left[|X_t^\epsilon(f)|^2 \int_0^1 \theta\,d\theta\right] \\
&= |S_t(f)(x)| + \frac{1}{2\epsilon}P_{\epsilon\delta_x}[X_t^\epsilon(f)\overline{X_t^\epsilon(f)}] \\
&= |S_t(f)(x)| + \frac{\epsilon}{2}|S_t(f)(x)|^2 \\
&\qquad + \frac{1}{2}\int_0^t \int_{\mathrm{R}^d} p_s(x-y)|S_{t-s}(f)(y)|^2\,dy\,2\kappa(T-s)\,ds,
\end{aligned}$$

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$$\leq |S_t(f)(x)| + \frac{\epsilon}{2}|S_t(f)(x)|^2 + \|\kappa\|_\infty \int_0^t \int_{\mathrm{R}^d} p_s(x-y)|S_{t-s}(f)(y)|^2 \, dy \, ds$$

where the last equality follows from the moment formula for superprocess.

Letting $\epsilon \to 0$ and using a simple change of variables, we are done. ■

The proof of the uniqueness of solution to (A.3) is based on the following lemma.

Lemma A.8 *Let $\upsilon(t)$ satisfy the following evolution equation*

(A.10)
$$\upsilon(t) = -\int_0^t S_{t-s}\big(\upsilon(s)u(s)\big)\kappa(s)\, ds,$$
$$\upsilon(s,\cdot) \in \overline{C}(\mathrm{R}^d), \quad \forall s > 0,$$
$$u(s,\cdot) \in \overline{C}(\mathrm{R}^d)_+, \quad \forall s > 0,$$
$$\kappa \in L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+.$$

Then, for all $t > 0$, $\upsilon(t) = 0$.

Proof For each $x \in \mathrm{R}^d$,

$$\int_0^t S_{t-s}\big(\upsilon(s)u(s)\big)(x)\kappa(s)\, ds = \int_0^t \int_{\mathrm{R}^d} p_{t-s}(x-y)\upsilon(s,y)u(s,y)\kappa(s)\, dy\, ds$$

is well defined, therefore

$$\int_0^t \int_{\mathrm{R}^d} p_{t-s}(x-y)|\upsilon(s,y)u(s,y)|\,|\kappa(s)|\, dy\, ds < \infty$$

for all $x \in \mathrm{R}^d$. Hence, for all $t > 0$,

$$\lim_{s\downarrow 0} \int_0^s S_{t-z}\big(|\upsilon(z)u(z)|\big)(x)|\kappa(z)|\, dz = 0.$$

Combining this with (A.10) we obtain

(A.11)
$$\lim_{s\downarrow 0} S_{t-s}\big(|\upsilon(s)|\big)(x) \leq \int_0^s S_{t-z}\big(|\upsilon(z)u(z)|\big)(x)|\kappa(z)|\, dz = 0, \quad \forall t > 0, \quad x \in \mathrm{R}^d.$$

By the semigroup property, $\upsilon(t)$ can be represented as

$$\upsilon(t) = S_{t-s}\upsilon(s) - \int_s^t S_{t-z}\big(\upsilon(z)u(z)\big)\kappa(z)\, dz,$$

for each $0 < s < t$. Noting that $\upsilon(s), u(s) \in \overline{C}(\mathrm{R}^d)$ for each $s > 0$, we use Feynman-Kac formula to obtain

$$\upsilon(t,x) = E_x\big[\upsilon\big(s, B(t-s)\big)e^{-\int_s^t u(z,B(t-z))\kappa(z)\,dz}\big], \quad \forall\, t > s,$$

where B is a Brownian motion starting at x. Since $\mathrm{Re}(u_s) \geq 0$, we obtain

$$|\upsilon(t,x)| \leq E_x\big[|\upsilon\big(s, B(t-s)\big)|\big] = S_{t-s}\big(|\upsilon(s)|\big)(x), \quad \forall\, t > s.$$

Therefore by (A.11) we get

$$|\upsilon(t,x)| \leq \lim_{s\downarrow 0} S_{t-s}\big(|\upsilon(s)|\big)(x) = 0,$$

and we are done since $(t,x) \in (0,\infty) \times \mathrm{R}^d$ was arbitrary. ■

Lemma A.9 *For each $f \in \tilde{S}'$ and $\kappa \in L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$, (A.6) has at most one solution v such that $v(s,\cdot) \in \overline{C}(\mathrm{R}^d)_+$ for all $s > 0$.*

Proof For any two solutions v^1, v^2 of (A.6) we define $v_t = v^1_t - v^2_t$ and $u_t = v^1_t + v^2_t$. Then it is easy to check that v satisfies (A.10). Therefore, by Lemma A.8, $v_t = 0$ for each $t > 0$, and the result follows. ■

Assumptions and Notation

(i) Fix arbitrary $f \in \tilde{S}'^{\rho}$, $\kappa \in L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$. Let $\{f^{(n)} = f^{(n)}_1 + i f^{(n)}_2\} \in \tilde{S}(\mathrm{R}^d)$, $\{\kappa^{(n)}\} \in \overline{C}_{\mathrm{R}}(\mathrm{R}_+)_+$, and $f^{(n)} \to f$ in $\tilde{S}'$, $\kappa^{(n)} \to \kappa$ weakly* in L^∞ as $n \to \infty$. Denote

$$v^{(n)}_t \equiv V_t(f^{(n)}, \kappa^{(n)}), \quad t > 0.$$

(ii) $\lim_{h\downarrow 0} \limsup_{n\to\infty} \int_0^h s^{\rho-1} w(s, f^{(n)})\,ds = 0$;
(iii) $\lim_{h\downarrow 0} \limsup_{n\to\infty} \overline{w}_\rho(h, f^{(n)}) = 0$.

Remark A.10 (iii) follows from (ii) by Lemma 5.5.

Remark A.11 Assumptions (i)–(iii) clearly imply that $f^{(n)} \to f$ in $\tilde{S}'^{\rho}$. We introduced assumptions (ii)–(iii) (instead of just saying $f^{(n)} \to f$ in $\tilde{S}'^{\rho}$) with the only purpose of making the future references more convenient.

Lemma A.12 *Let $f^{(n)}$, f, $v^{(n)}_t$ satisfy (i)–(iii). Then for all $n \geq 1$*

(A.12)
$$\|v^{(n)}_t\|_2^2 \leq C(d,\rho,\kappa^{(n)})\big(w(t, f^{(n)}) + \tilde{w}_\rho(t, f^{(n)})^2 t^{2-\frac{d}{2}-2\rho}\big), \quad \forall\, t > 0,$$

(A.13)
$$\|v^{(n)}\|^2_{2,(0,T],\hat{\rho}} \leq C(d,\rho,\kappa^{(n)}) \int_0^T t^{-\hat{\rho}}\big(w(t, f^{(n)}) + \tilde{w}_\rho(t, f^{(n)})^2 t^{2-\frac{d}{2}-2\rho}\big)\,dt, \quad \forall\, T > 0,$$

where $C(d,\rho,\kappa^{(n)})$ is the constant that depends only on d, ρ, $\kappa^{(n)}$.

Proof (A.13) is immediate from (A.12).

Let us check (A.12). By Lemma A.7 we obtain

$$\int_{\mathrm{R}^d} |v_t^{(n)}(y)|^2\,dy$$

$$\text{(A.14)}\qquad \le 2\int_{\mathrm{R}^d} |S_t(f^{(n)})(y)|^2\,dy + 2\|\kappa^{(n)}\|_\infty \int_{\mathrm{R}^d} \left| \int_0^t \int_{\mathrm{R}^d} p_s(y-z)|S_{t-s}(f^{(n)})(z)|^2\,dz\,ds \right|^2 dy, \quad \forall\, t>0.$$

Consider the second term in (A.14). For each $j=1,2$ we have

$$\begin{aligned}
&\int_{\mathrm{R}^d} \left(\int_0^t \int_{\mathrm{R}^d} p_s(y-z)\big(S_{t-s}(f_j^{(n)})(z)\big)^2\,dz\,ds \right)^2 dy \\
&= \int_0^t\int_0^t\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_s(y-z)S_{t-s}(f_j^{(n)})(z)^2 \\
&\qquad \times p_{s_1}(y-z_1)S_{t-s_1}(f_j^{(n)})(z_1)^2\,dy\,dz\,dz_1\,ds\,ds_1 \\
&= \int_0^t\int_0^t\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} p_{s+s_1}(z_1-z)\big(S_{t-s}(f_j^{(n)})(z)\big)^2\big(S_{t-s_1}(f_j^{(n)})(z_1)\big)^2\,dz\,dz_1\,ds\,ds_1 \\
&\le C_d \int_0^t\int_0^t (s+s_1)^{-d/2} w(t-s,f_j^{(n)})w(t-s_1,f_j^{(n)})\,ds\,ds_1 \\
&\le C_d \overline{w}_\rho(t,f_j^{(n)})^2 \int_0^t\int_0^t (s+s_1)^{-d/2}(t-s)^{-\rho}(t-s_1)^{-\rho}\,ds\,ds_1 \\
&= C_d \overline{w}_\rho(t,f_j^{(n)})^2 t^{2-\frac{d}{2}-2\rho} \int_0^1\int_0^1 (u+u_1)^{-d/2}(1-u)^{-\rho}(1-u_1)^{-\rho}\,du\,du_1 \\
&\le C_{d,\rho}\tilde{w}_\rho(t,f_j^{(n)})^2 t^{2-\frac{d}{2}-2\rho},
\end{aligned}$$

where $C_{d,\rho} = C_d\int_0^1\int_0^1 (u+u_1)^{-d/2}(1-u)^{-\rho}(1-u_1)^{-\rho}\,du\,du_1$ and the last inequality follows from Lemma 5.5. $C_{d,\rho}$ is finite since $\rho<1$, $d\le 3$. Since, by definition, we also have $\int_{\mathrm{R}^d}|S_t(f^{(n)})(y)|^2\,dy = w(t,f^{(n)})$, (A.14) shows that

$$\|v_t^{(n)}\|_2^2 \le 2w(t,f^{(n)}) + 2\|\kappa^{(n)}\|_\infty C_{d,\rho}\tilde{w}_\rho(t,f_j^{(n)})^2 t^{2-\frac{d}{2}-2\rho}, \quad \forall\, t>0,$$

and (A.12) follows. ■

Corollary A.13 *Let $f^{(n)}$, f, $v_t^{(n)}$ be as in Lemma A.12. Then*

$$\text{(A.15)}\qquad \limsup_{n\to\infty}\sup_{\epsilon\le s\le T}\|v_s^{(n)}\|_2 < \infty, \quad \forall\, 0<\epsilon<T,$$

$$\text{(A.16)}\qquad \lim_{\epsilon\downarrow 0}\sup_n \|v_\cdot^{(n)}\|_{2,(0,\epsilon]} = 0,$$

$$\text{(A.17)}\qquad \sup_n \|v_\cdot^{(n)}\|_{2,(0,T]}^2 < \infty, \quad \forall\, T>0.$$

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Proof (A.15) is immediate from Assumptions (ii), (iii) and the previous lemma. By (A.1), $\int_0^\epsilon t^{2-\frac{d}{2}-2\rho}\,dt$ is finite and this together with (A.12) and Assumptions (i)–(iii) yields (A.16). (A.17) is an easy consequence of (A.15), (A.16). ■

Lemma A.14 *Let $f^{(n)}$, f, $v_t^{(n)}$ and v_t be as in Lemma A.12. Then, for each $0 < \epsilon < T$,*

(A.18)
$$\sup_n \sup_{\epsilon \le t \le T} \|v_t^{(n)}\|_q < \infty, \quad \forall\, 2 \le q \le \infty,$$

(A.19)
$$\sup_{\epsilon \le t \le T} \|v_t^{(n)}\|_q \le \sqrt{C(d,\rho,\kappa^{(n)})}\, \|p_{\epsilon/4}\|_{\frac{2q}{q+2}} \times \sqrt{w(\epsilon/4, f^{(n)}) + \tilde{w}_\rho(\epsilon/4, f^{(n)})^2(\epsilon/4)^{2-\frac{d}{2}-2\rho}}, \quad \forall\, 2 \le q \le \infty, \quad n \ge 1.$$

Proof By the semigroup property of $V_{\cdot,\cdot}$

$$v_t^{(n)} = V_t(f^{(n)}, \kappa^{(n)}) = V_{\epsilon/2,t}\big(V_{0,\epsilon/2}(f^{(n)}, \kappa^{(n)}), \kappa^{(n)}\big)$$

and, so, by Lemma 5.2 we have

$$\|v_t^{(n)}\|_q \le \|V_{0,\epsilon/2}(f^{(n)}, \kappa^{(n)})\|_q \equiv \|V_{\epsilon/2}(f^{(n)}, \kappa^{(n)})\|_q, \quad \forall\, \epsilon \le t \le T.$$

Again using the semigroup property of $V_{\cdot,\cdot}$ and applying Lemma 5.2 we obtain

$$\begin{aligned}\|V_{\epsilon/2}(f^{(n)}, \kappa^{(n)})\|_q &\le \|V_{\epsilon/4,\epsilon/2}\big(V_{0,\epsilon/4}(f^{(n)}, \kappa^{(n)}), \kappa^{(n)}\big)\|_q \\ &\le \|p_{\epsilon/4}\|_{\frac{2q}{q+2}} \|V_{\epsilon/4}(f^{(n)}, \kappa^{(n)})\|_2.\end{aligned}$$

By Corollary A.13 $\|V_{\epsilon/4}(f^{(n)}, \kappa^{(n)})\|_2$ is bounded uniformly in n and (A.18) follows. (A.19) is also immediate from Lemma A.12. ■

Lemma A.15 *$\{v^{(n)}, n \ge 1\}$ is relatively compact in $C\big((0,\infty) \times \mathrm{R}^d\big)$.*

Proof Let $\psi^{(n)}(t,x) \equiv v_t^{(n)}(x)^2 \kappa^{(n)}(t)$. Then

$$v_t^{(n)} = S_t(f^{(n)}) + G(\psi^{(n)})(t,\cdot), \quad \forall\, t \in (0,\infty),$$

where $G(\psi^{(n)})$ is defined as in Lemma A.4. By Lemma A.5, $S_\cdot(f^{(n)})(\cdot) \to S_\cdot(f)(\cdot)$ uniformly on compact subsets of $(0,\infty) \times \mathrm{R}^d$ as $n \to \infty$. By Corollary A.13, Lemma A.14 and Lemma A.4, $G(\psi^{(n)})$ is relatively compact in $C\big((0,\infty) \times \mathrm{R}^d\big)$ and we are done. ■

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Corollary A.16 *Let v be a limit point of $v^{(n)}$. Then*

(A.20) $$v_t = V_t(f,\kappa), \quad \forall t > 0,$$

(A.21)
$$\sup_{\epsilon\le t\le T} \|V_t(f,\kappa)\|_q \le \sqrt{C(d,\rho,\kappa)}\,\|p_{\epsilon/4}\|_{\frac{2q}{q+2}} \times \sqrt{w(\epsilon/4,f)+\tilde{w}_\rho(\epsilon/4,f)^2(\epsilon/4)^{2-\frac{d}{2}-2\rho}}, \quad \forall 2\le q\le\infty, \quad \forall 0<\epsilon<T,$$

(A.22)
$$\|V(f,\kappa)\|^2_{2,(0,T],\hat\rho} \le C(d,\rho,\kappa^{(n)}) \times \int_0^T t^{-\hat\rho}\big(w(t,f)+\tilde{w}_\rho(t,f)^2 t^{2-\frac{d}{2}-2\rho}\big)\,dt < \infty, \quad \forall T>0,$$

(A.23) $$V(f,\kappa)\in L^{2,\hat\rho}_{c,\mathrm{loc},+},$$

(A.24) $$V_{\epsilon+\cdot}(\mu,\kappa)\in \overline{C}\big([0,T]\times \mathrm{R}^d\big)_+, \quad \forall T>0, \quad \epsilon>0,$$

(A.25) $$V_t(\mu,\kappa)\in L^q(\mathrm{R}^d)_+, \quad \forall t>0, \quad q\ge 2,$$

(A.26) $$V_t^1(\mu,\kappa)\in L^1_{\mathrm{R}}(\mathrm{R}^d)_+, \quad \forall t>0,$$

(A.27) $$\lim_{n\to\infty} v^{(n)} = V(f,\kappa), \quad in\, C\big((0,\infty)\times \mathrm{R}^d\big).$$

Proof Let $v^{(n_k)}$ be a subsequence of $v^{(n)}$ which converges to v. In order to prove that v satisfies (A.6) (and this means (A.20)), we need to show that

$$\lim_{n_k\to\infty}\Big(S_t(f^{(n_k)})(x)+\int_0^t\int_{\mathrm{R}^d} p_{t-s}(x-y)v_s^{(n_k)}(y)^2\kappa^{(n_k)}(s)\,dy\,ds\Big) = S_t(f)(x)+\int_0^t\int_{\mathrm{R}^d} p_{t-s}(x-y)v_s(y)^2\kappa(s)\,dy\,ds, \quad \forall t>0.$$

Convergence of the first term follows immediately from Lemma A.5. Consider the second term. By Lemma A.15, $v^{(n_k)}$ converges to v uniformly on compact subsets of $(0,\infty)\times\mathrm{R}^d$ and by Lemma A.14 $\sup_n \|v^{(n_k)}\|_{\infty,[\epsilon,T]} < \infty$ for all $0<\epsilon<T$. Now apply Lemma A.3 to see that, for all $0<\epsilon<t$,

(A.28) $$\lim_{n_k\to\infty}\sup_{\epsilon\le s\le t}\big|S_{t-s}\big((v_s^{(n_k)})^2\big)(x)-S_{t-s}(v_s^2)(x)\big| = 0.$$

Since $\kappa_{n_k}\to\kappa$ weakly* in $L^\infty(\mathrm{R}_+)$, (A.28) immediately yields

$$\lim_{n_k\to\infty}\int_\epsilon^t\int_{\mathrm{R}^d} p_{t-s}(x-y)v_s^{(n_k)}(y)^2\kappa^{(n_k)}(s)\,dy\,ds = \int_\epsilon^t\int_{\mathrm{R}^d} p_{t-s}(x-y)v_s(y)^2\kappa(s)\,dy\,ds.$$

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By choosing ϵ sufficiently small,

$$\int_0^\epsilon \int_{\mathrm{R}^d} p_{t-s}(x-y) v_s^{(n_k)}(y)^2 \kappa^{(n_k)}(s)\, dy\, ds$$

and

$$\int_0^\epsilon \int_{\mathrm{R}^d} p_{t-s}(x-y) v_s(y)^2 \kappa(s)\, dy\, ds$$

may be made arbitrarily small (uniformly on n_k) by Corollary A.13 and Fatou's lemma respectively. This shows that

$$\lim_{n_k\to\infty} \int_0^t \int_{\mathrm{R}^d} p_{t-s}(x-y) v_s^{(n_k)}(y)^2 \kappa^{(n_k)}(s)\, dy\, ds = \int_0^t \int_{\mathrm{R}^d} p_{t-s}(x-y) v_s(y)^2 \kappa(s)\, dy\, ds,$$

and (A.20) follows. (A.21) follows immediately by passing to the limit in (A.19) and then from Fatou's lemma. To get (A.22) pass to the limit in (A.13) and use Fatou's lemma. Our conditions on ρ and $\hat{\rho}$ together with Assumptions (ii), (iii) yield convergence of the right hand side of (A.13) to the finite right side of (A.22).

(A.22) together with the fact that $V(f,\kappa)$ is in $C\big(\mathrm{R}^d \times (0,\infty)\big)$ (which is an immediate consequence of the convergence of $v^{(n_k)}$ to $V(f,\kappa)$ in $C\big(\mathrm{R}^d \times (0,\infty)\big)$) yields (A.23). (A.24), (A.25) follow from (A.21). We leave to the reader to check the inequality $\|V_t^1\|_1 \le \langle V_0^1, 1\rangle + \|\kappa\|_\infty \|V\|^2_{2,(0,t]}$. This inequality together with (A.22) yields (A.26).

We proved that for each subsequence of $v^{(n)}$ there exists a further subsequence $v^{(n_k)}$ which converges to $V(f,\kappa)$ uniformly on compact sets in $(0,\infty)\times \mathrm{R}^d$. However, by Lemma A.9, $\big(V_t(f,\kappa), t>0\big)$ is unique, therefore each convergent subsequence converges to $V(f,\kappa)$. This implies that, in fact, $v^{(n)}$ converges to $V(f,\kappa)$ and (A.27) follows.

Proof of Proposition A.2 For each $f \in \tilde{S'}^\rho$, $\kappa \in L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$, the existence and all the properties of $V_t(f,\kappa)$ are proved in Corollary A.16. The uniqueness of $V_t(f,\kappa)$ is given by Lemma A.9. For $\kappa \in \overline{C}_{\mathrm{R}}(\mathrm{R}_+)_+$ use the smoothness properties of S_t to check that $V_t(f,\kappa)$ is a strong solution of (A.6).

(b) Let $\{\mu^{(n)}\} \in \tilde{S'}^\rho$, $\mu^{(n)} \to \mu$ in $\tilde{S'}^\rho$ and $\kappa_n \to \kappa$ weakly* in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$. Let Γ be an arbitrary compact set in $(0,\infty)\times \mathrm{R}^d$. We need to prove that

$$\text{(A.29)} \qquad \lim_{n\to\infty} \sup_{(t,x)\in\Gamma} |V_t(\mu^{(n)},\kappa_n)(x) - V_t(\mu,\kappa)(x)| = 0.$$

In Corollary A.16 we proved the following. If $\{f^{(n)}\}$ in $S(R^d)$ and $\{\tilde{\kappa}^{(n)}\}$ in $\overline{C}_{\mathrm{R}}(\mathrm{R}_+)_+$ are such that $f^{(n)} \to f$ in $\tilde{S'}^\rho$ and $\tilde{\kappa}^{(n)} \to \kappa$ *weakly** in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$, then

$$\text{(A.30)} \qquad \lim_{n\to\infty} \sup_{(t,x)\in\Gamma} |V_t(f^{(n)},\tilde{\kappa}^{(n)})(x) - V_t(f,\kappa)(x)| = 0.$$

For each $\epsilon > 0$ one can choose $\{f^{(n)}\}$ in $S(R^d)$ and $\{\tilde{\kappa}^{(n)}\}$ in $\overline{C}_{\mathrm{R}}(\mathrm{R}_+)_+$ such that $f^{(n)} \to \mu$ in $\tilde{S'}^\rho$, $\tilde{\kappa}^{(n)} \to \kappa$ *weakly** in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$ and

$$\text{(A.31)} \qquad \sup_{(t,x)\in\Gamma} |V_t(f^{(n)},\tilde{\kappa}^{(n)}) - V_{r,t}(\mu^{(n)},\kappa^{(n)})(x)| \le \epsilon$$

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for all n. This yields

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{(t,x)\in\Gamma} |V_t(\mu^{(n)},\kappa^{(n)})(x) - V_t(\mu,\kappa)(x)| \\
&\quad \le \lim_{n\to\infty} \sup_{(t,x)\in\Gamma} |V_t(\mu^{(n)},\kappa^{(n)}) - V_t(f^{(n)},\tilde\kappa^{(n)})| \\
&\qquad + \lim_{n\to\infty} \sup_{(t,x)\in\Gamma} |V_t(f^{(n)},\tilde\kappa^{(n)})(x) - V_t(\mu,\kappa)(x)| \\
&\quad \le \lim_{n\to\infty} \sup_{(t,x)\in\Gamma} |V_t(\mu^{(n)},\kappa^{(n)})(x) - V_t(f^{(n)},\tilde\kappa^{(n)})(x)| \le \epsilon,
\end{aligned}$$

where the last inequality follows by (A.30), (A.31). Since ϵ was arbitrary, we get (A.29). From (A.22) and Lemma A.1 we obtain

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \|V(\mu^{(n)},\kappa^{(n)})\|_{2,(0,\epsilon],\hat\rho} = 0. \tag{A.32}$$

From (A.21) we get

$$\limsup_{n\to\infty} \sup_{\epsilon\le s\le T} \|V_s(\mu^{(n)},\kappa^{(n)})\|_\infty < \infty, \quad \forall\, 0<\epsilon<T. \tag{A.33}$$

(A.29), (A.32), (A.33) imply that $V(\mu^{(n)},\kappa^{(n)})$ converges to $V(\mu,\kappa)$ in $L^{2,\hat\rho}_{c,\mathrm{loc},+}$.

(c) Let us take arbitrary $\mu^{(n)} \to \mu$ in $\tilde{S}'^{\rho}$, $\kappa^{(n)} \to \kappa$ weakly* in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$ and $\epsilon_n \to 0$. We have to check that

$$\lim_{n\to\infty} \sup_{s<\epsilon_n} |\langle\psi, V_s(\mu^{(n)},\kappa^{(n)})\rangle - \langle\psi,\mu^{(n)}\rangle| = 0.$$

By definition

$$\begin{aligned}
|\langle\psi, V_{\epsilon_n}(\mu^{(n)},\kappa^{(n)})\rangle - \langle\psi,\mu^{(n)}\rangle| &\le |\langle\psi, S_{\epsilon_n}(\mu^{(n)}) - \mu^{(n)}\rangle| \\
&\quad + \left|\left\langle \psi, \int_0^{\epsilon_n} S_{\epsilon_n-u}\big(V_u(\mu^{(n)},\kappa^{(n)})^2\big)\right\rangle \kappa^{(n)}(u)\,du\right|.
\end{aligned}$$

By Lemma A.6, the first term converges to 0 as $n\to\infty$. The second term is bounded by

$$\begin{aligned}
\|\psi\|_\infty \|\kappa^{(n)}\|_\infty \int_0^{\epsilon_n} \|V_u(\mu^{(n)},\kappa^{(n)})\|_2^2\,du &= \|\psi\|_\infty \|\kappa^{(n)}\|_\infty \|V(\mu^{(n)},\kappa^{(n)})\|^2_{2,(0,\epsilon_n]} \\
&\le \|\psi\|_\infty \|\kappa^{(n)}\|_\infty \|V(\mu^{(n)},\kappa^{(n)})\|^2_{2,(0,\epsilon_n],\hat\rho} \\
&\to 0 \quad \text{as } n\to\infty
\end{aligned}$$

by (A.32). ■

Proof of Theorem 5.8 (a) For $r\ge 0$, $\mu\in\tilde{S}'^{\rho}$, $\kappa\in L^\infty_{\mathrm{R}}\big([r,\infty)\big)_+$, define

$$\kappa^{(r)}(t) \equiv \kappa(r+t), \quad \forall\, t\ge 0.$$

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Then $V_{r,t}(\mu,\kappa) = V_{t-r}(\mu,\kappa^{(r)})$ for all $t > r$, and (a) follows from Proposition A.2.

(b), (d) Fix arbitrary $\mu \in \tilde{S'}^{\rho}$, $t \in [0,T)$, $\kappa \in L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$, $x \in \mathrm{R}^d$ and let $\{\mu^{(n)}\}$, $\{t_n\}$, $\{\kappa^{(n)}\}$, $\{x^{(n)}\}$ be arbitrary sequences in $\tilde{S'}^{\rho}$, $[0,T)$, $L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$, R^d respectively such that $\mu^{(n)} \to \mu$ in $\tilde{S'}^{\rho}$, $t_n \to t$, $\kappa^{(n)} \to \kappa$ weakly* in $L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$ and $x^{(n)} \to x$ in R^d, as $n \to \infty$. For (b), (d) it is sufficient to show that

(A.34) $$\lim_{n\to\infty} |V_{t_n,T}(\mu^{(n)},\kappa^{(n)})(x^{(n)}) - V_{t,T}(\mu,\kappa)(x)| = 0.$$

and

(A.35) $$\sup_{y\in\mathrm{R}^d} |V_{t_n,T}(\mu^{(n)},\kappa^{(n)})(y)| < \infty.$$

But

(A.36) $$V_{t_n,T}(\mu^{(n)},\kappa^{(n)})(x^{(n)}) = V_{T-t_n}(\mu^{(n)},\kappa^{(n,t_n)})(x^{(n)}),$$

where $\kappa^{(n,t_n)}(s) \equiv \kappa^{(n)}(t_n + s)$ for all $s \geq 0$. Letting $n \to \infty$, it is easy to check that $\kappa^{(n,t_n)} \to \kappa^{(t)}$ weakly* in $L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$. Then Proposition A.2 (b) implies that $V(\mu^{(n)},\kappa^{(n,t_n)}) \to V(\mu,\kappa^{(t)})$ in $L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$. (A.36) and the definition of convergence in $L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$ immediately gives (A.34), (A.35).

(c) Let $\{\mu^{(n)}\}$, $\{\epsilon_n\}$ be arbitrary sequences in $\tilde{S'}^{\rho}$ and R_+ respectively such that $\mu^{(n)} \to \mu$ in $\tilde{S'}^{\rho}$ and $\epsilon_n \to 0$. Then it is sufficient to show that

$$\lim_{n\to\infty} \sup_{|t-s|<\epsilon_n, s\leq T} |\langle \psi, V_{s,t}(\mu^{(n)},\kappa^{(n)})\rangle - \langle \psi, \mu^{(n)}\rangle| = 0$$

uniformly in n. As in (b) it suffices to show that

$$\lim_{n\to\infty} \sup_{u<\epsilon_n, s\leq T} |\langle \psi, V_u(\mu^{(n)},\kappa^{(n,s)})\rangle - \langle \psi, \mu^{(n)}\rangle| = 0.$$

But $\{\kappa^{(n,s)}, n \geq 1, s \in [0,T]\}$ is weakly* compact in $L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$, therefore we are done by Proposition A.2 (c). ■

A.2 Proof of Theorem 5.9

We will prove a slightly more general result than Theorem 5.9; Theorem 5.9 will be an easy consequence of our more general setting.

Recall basic definitions from Section 5 and introduce some new notation.

$$\hat{\nu}(ds,dy) \equiv s^{\hat{\rho}} p_1(dy) 1(0 \leq s \leq 1)\, ds\, dy$$

$$\mathcal{G} f(x) \equiv \int_0^1 \int_{\mathrm{R}^d} f(s,y) p_s(x-y)\, \hat{\nu}(ds,dy).$$

For any function $f \in L^2\big(R^d \times \mathrm{R}_+, \hat{\nu}(ds,dy)\big)$ we define $\|f\|_{2,\hat{\nu}}$ by setting

$$\|f\|^2_{2,\hat{\nu}} = \int_0^1 \int_{\mathrm{R}^d} |f(s,y)|^2\, \hat{\nu}(ds,dy).$$

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The capacity of a set is given by

$$\mathcal{C}(B) = \inf\{\|f\|_{2,\hat{\nu}} : \mathcal{G}f(x) \geq 1, \forall x \in B\}, \quad \forall B \in \mathcal{B}(\mathrm{R}^d).$$

Let E be a metric space. Then $C_q(E \times \mathrm{R}^d)$ (resp. $\overline{C}_q(E \times \mathrm{R}^d)$) is the space of functions on $E \times \mathrm{R}^d$ such that

(a) for q.e. $x \in \mathrm{R}^d$, $f(\cdot, x) \in C(E)$ (resp. $\overline{C}(E)$),
(b) for all $y \in E$, $f(y, \cdot)$ is quasicontinuous.

We say that a sequence of functions $\{f^{(n)}(\cdot,\cdot)\}$ in $\mathcal{B}(E \times \mathrm{R}^d)$ converges to $f(\cdot,\cdot)$ almost uniformly q.e. if

$$f^{(n)}(y,x) \to f(y,x), \quad \forall y \in E, \quad \text{for q.e. } x,$$

and, moreover, for each $\delta > 0$, there exists an open set $B_\delta \subset \mathrm{R}^d$ with $\mathcal{C}(B_\delta) \leq \delta$ such that

$$\sup_{x \in B_\delta^c} |f^{(n)}(\cdot,x) - f(\cdot,x)| \to 0$$

uniformly on compact subsets of E. The notation $f^{(n)}(\cdot,\cdot) \xrightarrow{\text{q.e.}} f(\cdot,\cdot)$ stands for this convergence. We say that $\{f^{(n)}(\cdot,\cdot)\}$ in $\mathcal{B}(E \times \mathrm{R}^d)$ converges to $f(\cdot,\cdot)$ in $\mathcal{C}$ if, for each $\epsilon > 0$,

$$\mathcal{C}\big(x : |f^{(n)}(\cdot,x) - f(\cdot,x)| > \epsilon\big) \to 0,$$

uniformly on compact subsets of E. $\xrightarrow{\mathcal{C}}$ stands for this type of convergence.

Remark A.17 There is some possible confusion over the above definitions since functions are defined on the product space $E \times \mathrm{R}^d$ and most of their properties are verified "for every" y in E and q.e. x in R^d. In order to help the reader to distinguish between E and R^d (since in many cases E will be R^d as well!), we reserve the letter x for an element in the space R^d over which functions and their properties are defined q.e.

Let $v_\cdot(\cdot) \in L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$ and $\kappa \in L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$. Then $W_t(v,\kappa,x)(\cdot)$ denotes a solution (if it exists) of the following evolution equation:

$$u(t,y) = p_t(x-y) - \int_0^t 2\kappa(s) S_{t-s}\big(v(s)u(s)\big)(y)\,ds, \quad t > 0, \quad y \in \mathrm{R}^d. \tag{A.37}$$

For $\mu \in M_F$, $W_t(v,\kappa,\mu)(\cdot)$ denotes a solution (if it exists) of the following equation

$$u(t,y) = S_t\mu(y) - \int_0^t 2\kappa(s) S_{t-s}\big(v(s)u(s)\big)(y)\,ds, \quad t > 0, \quad y \in \mathrm{R}^d. \tag{A.38}$$

As we will see later Theorem 5.9 is an easy consequence of the following proposition.

Proposition A.18

(a) *For each* $v \in L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$, $\kappa \in L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$ *and q.e. x, there exists unique solution* $W_t(v,\kappa,x)(\cdot)$ *for (A.37) such that*

$$W_\cdot(v,\kappa,x)(\cdot) \in C\big((0,\infty)\times \mathrm{R}^d\big),$$
$$W_t(v,\kappa,x)(\cdot) \in \overline{C}(\mathrm{R}^d), \quad \forall\, t>0.$$

For each $t>0$ *and* $y \in \mathrm{R}^d$, *the function* $W_t(v,\kappa,\cdot)(y)$ *is quasicontinuous and there exists* $\mathcal{N} \subset \mathrm{R}^d$ *with* $\mathcal{C}(\mathcal{N})=0$ *such that*

$$|W_t(v,\kappa,x)(y)| \le p_t(x-y), \quad \forall\,(t,y,x) \in (0,\infty)\times \mathrm{R}^d \times (\mathrm{R}^d \setminus \mathcal{N}).$$

(b) *For each* $\mu \in M_F^{\rho}(\mathrm{R}^d)$,

$$W_\cdot(v,\kappa,\mu)(\cdot) = \int_{\mathrm{R}^d} W_\cdot(v,\kappa,x)(\cdot)\,\mu(dx),$$

that is, the solution for (A.38) is given by the integration of the fundamental solution with respect to initial condition.

(c) *Let* $\{\kappa^{(n)}\}$ *in* $L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$ *and* $\{v^{(n)}\}$ *in* $L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$ *be such that* $\kappa^{(n)} \to \kappa$ *weakly* in* $L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$ *and* $v^{(n)} \to v$ *in* $L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$. *Then*

$$\mathcal{C}\big(x : |W_\cdot(v^{(n)},\kappa^{(n)},x)(\cdot) - W_\cdot(v,\kappa,x)(\cdot)| > \epsilon\big) \to 0,$$

uniformly on compact subsets of $(0,\infty)\times\mathrm{R}^d$, *that is, according to our notation* $W_\cdot(v^{(n)},\kappa^{(n)},\cdot)(\cdot) \stackrel{\mathcal{C}}{\to} W_\cdot(v,\kappa,\cdot)(\cdot)$.

We will prove this proposition via a series of lemmas. We start with the lemma that gives the "uniqueness" part of the proposition.

Lemma A.19 *(A.38) (and therefore also (A.37)) has at most one solution such that, for each* $s>0$, $u(s,\cdot) \in \overline{C}(\mathrm{R}^d)_+$.

Proof Immediately from Lemma A.8 (see also the proof of Lemma A.9). ■

In the following lemmas we investigate some properties of convergence q.e. and in $\mathcal{C}$.

Lemma A.20 *Let* $\{f^{(n)}\}$ *be a sequence in* $C_q(E\times\mathrm{R}^d)$ *such that* $f^{(n)} \xrightarrow{q.e.} f$ *as* $n\to\infty$. *Then* $f \in C_q(E\times\mathrm{R}^d)$.

Proof For each $\delta>0$, there exists an open set $B_\delta \subset \mathrm{R}^d$ with $\mathcal{C}(B_\delta) \le \delta$ such that, for each compact set $\Gamma \subset E$, $\lim_{n\to\infty} f^{(n)} = f$ uniformly on $\Gamma\times B_\delta^c$. The convergence is uniform, hence $f \restriction \Gamma\times B^c_\delta \in \overline{C}(\Gamma\times B^c_\delta)$. Since Γ and B_δ were arbitrary, we are done. ■

Lemma A.21 *Let* $\{f^{(n)}\}$ *be a sequence in* $\mathcal{B}(E\times\mathrm{R}^d)$ *such that* $f^{(n)} \xrightarrow{q.e.} f$ *as* $n\to\infty$. *Then* $f^{(n)} \stackrel{\mathcal{C}}{\to} f$.

Proof Fix any $\delta > 0$. Let $B_\delta \subset \mathrm{R}^d$ be a set such that $\mathcal{C}(B_\delta) < \delta$ and

$$\sup_{x \in B_\delta^c} |f^{(n)}(\cdot, x) - f(\cdot, x)| \to 0$$

uniformly on compact subsets of E. Let Γ be an arbitrary compact set in R^d. Then

$$\sup_{y\in\Gamma} \mathcal{C}\big(x : |f^{(n)}(y,x) - f(y,x)| > \epsilon\big)$$
$$\leq \sup_{y\in\Gamma} \mathcal{C}(B_\delta) + \sup_{y\in\Gamma} \mathcal{C}\big(x : |f^{(n)}(y,x) - f(y,x)| > \epsilon, x \in B_\delta^c\big).$$

Since $f^{(n)} \xrightarrow{\text{q.e.}} f$, there exists N such that, for any $n > N$, the second term equals to 0. The first term is less than δ. Since δ was arbitrary, we are done. ■

Lemma A.22 *Let $\{f^{(n)}\}$ be a sequence in $C_q(E \times \mathrm{R}^d)$. Suppose that, for each subsequence $\{f^{(n_k)}\}$, there exists a further subsequence $\{f^{(n'_k)}\}$ such that $f^{(n'_k)} \xrightarrow{\text{q.e.}} f$ as $n'_k \to \infty$. Then $f^{(n)} \stackrel{\mathcal{C}}{\to} f$ as $n \to \infty$.*

Proof Let $f^{(n)} \stackrel{\mathcal{C}}{\not\to} f$. This means that, for any $\epsilon > 0$, there exists $\{f^{(n_k)}\}$ and $0 < a \leq \infty$ such that

$$\sup_{y\in\Gamma} \mathcal{C}\big(x : |f^{(n_k)}(x,y) - f(x,y)| > \epsilon\big) \to a > 0. \tag{A.39}$$

$\{f^{(n_k)}\}$ contains a further subsequence $\{f^{(n'_k)}\}$ such that $f^{(n'_k)} \xrightarrow{\text{q.e.}} f$. Then the previous lemma implies that $f^{(n'_k)} \stackrel{\mathcal{C}}{\to} f$ and this contradicts (A.39). ■

Lemma A.23 *Let $\{f^{(n)}\}, \{g^{(n)}\}, f, g$ be in $\mathcal{B}(E\times\mathrm{R}^d)$ and suppose that $f^{(n)} \stackrel{\mathcal{C}}{\to} f$, $g^{(n)} \stackrel{\mathcal{C}}{\to} g$ as $n \to \infty$. Define*

$$h^{(n)}(y,z,x) \equiv f^{(n)}(y,x)g^{(n)}(z,x) \in \mathcal{B}(E \times E \times \mathrm{R}^d),$$
$$h(y,z,x) \equiv f(y,x)g(z,x) \in \mathcal{B}(E \times E \times \mathrm{R}^d).$$

Then $h^{(n)} \stackrel{\mathcal{C}}{\to} h$.

Proof Trivial.

Lemma A.24 *Let $\{f^{(n)}\}$ be a sequence in $\mathcal{B}(E \times \mathrm{R}^d)$ such that $f^{(n)} \stackrel{\mathcal{C}}{\to} f$ as $n \to \infty$. Suppose that there exist sets $\mathcal{N}^{(n)}, \mathcal{N} \subset \mathrm{R}^d$ of nil capacity with the following properies. For any compact set $\Gamma \in E$ there exists a function $g^\Gamma \in S(R^d)_+$ such that*

$$\sup_{y\in\Gamma} |f^{(n)}(y,x)| \leq g^\Gamma(x), \quad \forall x \in \mathrm{R}^d \setminus \mathcal{N}^{(n)}$$
$$\sup_{y\in\Gamma} |f(y,x)| \leq g^\Gamma(x), \quad \forall x \in \mathrm{R}^d \setminus \mathcal{N}.$$

Let $\{\mu^{(n)}\}$ *be a sequence in* $M_F^\rho(\mathrm{R}^d)$ *such that* $\mu^{(n)} \Rightarrow \mu$ *in* $M_F^\rho(\mathrm{R}^d)$. *Then*

$$\int_{\mathrm{R}^d} |f^{(n)}(\cdot,x) - f(\cdot,x)|\mu^{(n)}(dx) \to 0 \quad \textit{as} \quad n \to \infty$$

uniformly on compact subsets of E.

Proof Let $\epsilon > 0$ be arbitrary. For each $n \geq 1$, $y \in E$, set

$$B_{y,n,\epsilon} \equiv \{x : |f^{(n)}(y,x) - f(y,x)| > \epsilon\}.$$

Take an arbitrary compact set $\Gamma \subset \mathrm{R}^d$. Since $g^\Gamma \in S(R^d)_+$ and $\mu^{(n)}$ converges in $M_F^\rho(\mathrm{R}^d)$, therefore for each $\epsilon' > 0$ we can choose another compact set $\Gamma_{\epsilon'} \subset \mathrm{R}^d$ such that

$$\int_{\Gamma^c_{\epsilon'}} g^\Gamma(x)\mu^{(n)}(dx) \leq \epsilon', \quad \forall n.$$

Then

$$\begin{aligned}
&\sup_{y\in\Gamma}\int_{\mathrm{R}^d} |f^{(n)}(y,x) - f(y,x)|\,\mu^{(n)}(dx) \\
&\leq \sup_{y\in\Gamma}\int_{B_{y,n,\epsilon}} |f^{(n)}(y,x) - f(y,x)|\,\mu^{(n)}(dx) \\
&\qquad + \sup_{y\in\Gamma}\int_{B^c_{y,n,\epsilon}\cap\Gamma_{\epsilon'}} |f^{(n)}(y,x) - f(y,x)|\,\mu^{(n)}(dx) \\
&\qquad + \sup_{y\in\Gamma}\int_{B^c_{y,n,\epsilon}\cap\Gamma^c_{\epsilon'}} |f^{(n)}(y,x) - f(y,x)|\,\mu^{(n)}(dx) \qquad \text{(A.40)} \\
&\leq \sup_{y\in\Gamma}\int_{B_{y,n,\epsilon}} |f^{(n)}(y,x) - f(y,x)|\,\mu^{(n)}(dx) \\
&\qquad + \epsilon\sup_{y\in\Gamma}\mu^{(n)}(B^c_{y,n,\epsilon}\cap\Gamma_{\epsilon'}) + 2\mu^{(n)}(g^\Gamma\cdot\cap\Gamma^c_{\epsilon'}) \\
&\leq \sup_{y\in\Gamma}\int_{B_{y,n,\epsilon}} |f^{(n)}(y,x) - f(y,x)|\,\mu^{(n)}(dx) + \epsilon\mu^{(n)}(\Gamma_{\epsilon'}) + 2\epsilon'.
\end{aligned}$$

By fixing ϵ' and using the fact that $\mu^{(n)}(\Gamma_{\epsilon'})$ is bounded uniformly in n we can make the second term less than ϵ' by choosing ϵ sufficiently small. By Lemma 5.7, $\mu^{(n)} \in M_F^\rho(\mathrm{R}^d)$ does not charge sets of nil capacity, hence we obtain

$$\begin{aligned}
&\sup_{y\in\Gamma}\int_{B_{y,n,\epsilon}} |f^{(n)}(y,x) - f(y,x)|\,\mu^{(n)}(dx) \\
&= \sup_{y\in\Gamma}\int_{B_{y,n,\epsilon}\setminus(\mathcal{N}^{(n)}\cup\mathcal{N})} |f^{(n)}(y,x) - f(y,x)|\,\mu^{(n)}(dx) \\
&\leq \sup_{y\in\Gamma,x\in\mathrm{R}^d\setminus(\mathcal{N}^{(n)}\cup\mathcal{N})} \big(|f^{(n)}(y,x)| + |f(y,x)|\big)\sup_{y\in\Gamma}\mu^{(n)}(B_{y,n,\epsilon}).
\end{aligned}$$

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By Theorem 2.5.1 [1] (see inequality (2.5.1) there)

$$\mu^{(n)}(B_{y,n,\epsilon}) \leq \|S.\mu^{(n)}\|_{2,\hat{\nu}} \sqrt{\mathcal{C}(B_{y,n,\epsilon})}.$$

$\mu^{(n)} \Rightarrow \mu$ in $M_F^{\rho}(\mathrm{R}^d)$, hence $\|S.\mu^{(n)}\|_{2,\hat{\nu}}$ is bounded uniformly in n. $f^{(n)} \overset{\mathcal{C}}{\to} f$, therefore

$$\sup_{y\in\Gamma} \sqrt{\mathcal{C}(B_{y,n,\epsilon})} \to 0 \quad \text{as} \quad n \to \infty$$

by definition of $B_{y,n,\epsilon}$. This together with our assumptions on $f^{(n)}$, f, $\mathcal{N}^{(n)}$, $\mathcal{N}$ yields

$$\begin{aligned}
\lim_{n\to\infty} \sup_{y\in\Gamma, x\in\mathrm{R}^d\setminus(\mathcal{N}^{(n)}\cup\mathcal{N})} &\big(|f^{(n)}(y,x)| + |f(y,x)|\big) \sup_{y\in\Gamma} \mu^{(n)}(B_{y,n,\epsilon}) \\
&\leq \lim_{n\to\infty} \|g^{\Gamma}\|_{\infty} \sup_{y\in\Gamma} \mu^{(n)}(B_{y,n,\epsilon}) \\
&= 0,
\end{aligned}$$

and it follows that the first term in (A.40) approaches 0 as $n \to \infty$. Since ϵ' was arbitrary we are done. ∎

Lemma A.25 *Let $\{g^{(n)}\}$ be in $L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$ and suppose that $g^{(n)} \to g$ in $L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$. Define $f^{(n)}(s,\cdot) \equiv g^{(n)}(s,\cdot)s^{-\hat{\rho}}$. Then there exists $\{f^{(n_k)}\}$ such that $\mathcal{G}f^{(n_k)} \overset{q.e.}{\longrightarrow} \mathcal{G}f$. (To be consistent with our definition of q.e. convergence, one can set $E = \varnothing$ in this case.)*

Proof $f^{(n)} \to f$ uniformly on compact subsets of $(0,\infty) \times \mathrm{R}^d$. $\hat{\nu}$ is a finite measure on $(0,1] \times \mathrm{R}^d$; the fact that $g^{(n)} \to g$ in $L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$ easily gives the uniform integrability condition for the sequence $\{f^{(n)}\}$ with respect to measure $\hat{\nu}$. This yields $f^{(n)} \to f$ in $L^2\big((0,\infty) \times \mathrm{R}^d, \hat{\nu}(ds,dx)\big)$. Now the result follows from [1, Proposition 2.3.8]. ∎

Throughout the rest of the proof of Proposition A.18 the following assumptions are used.

Assumptions

(i) Let $v \in L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$ and $\kappa \in L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$ be arbitrary.

(ii) Let $\{v^{(n)}\}$ be a sequence of functions in $\overline{C}\big([0,\infty) \times \mathrm{R}^d\big)_+$ such that $v^{(n)} \to v$ in $L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$ as $n \to \infty$. For each n define $\tilde{v}^{(n)}(s,\cdot) \equiv s^{-\hat{\rho}} v^{(n)}(s,\cdot)$.

(iii) Let $\{\kappa^{(n)}\} \in \overline{C}(\mathrm{R}_+)$, $\kappa^{(n)} \to \kappa$ *weakly** in $L^{\infty}_{\mathrm{R}}(\mathrm{R}_+)_+$ as $n \to \infty$.

Since $v^{(n)}$, $\kappa^{(n)}$ are bounded continuous functions, it follows from the theory of parabolic equations that the solution $W.(v^{(n)}, \kappa^{(n)}, x)(\cdot)$ to (A.37) exists for each $x \in \mathrm{R}^d$, and $W.(v^{(n)}, \kappa^{(n)}, \cdot)(\cdot) \in C\big((0,\infty) \times \mathrm{R}^d \times \mathrm{R}^d\big) \subset C_q\big((0,\infty) \times \mathrm{R}^d \times \mathrm{R}^d\big)$. The following simple lemma will be frequently used.

Lemma A.26 $|W_t(v^{(n)}, \kappa^{(n)}, x)(y)| \leq p_t(x-y), \forall (t,y,x) \in (0,\infty) \times \mathrm{R}^d \times \mathrm{R}^d, \forall n \geq 1.$

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Proof Fix $x \in \mathbb{R}^d$. Take a sequence of functions $\{\psi^{(k)}\}$ in $\overline{C}_{\mathbb{R}}(\mathbb{R}^d)_+$ such that $\psi^{(k)} \Rightarrow \delta_x$ as $k \to \infty$. Apply Feynman-Kac formula to see that

$$\begin{aligned} |W_t(v^{(n)}, \kappa^{(n)}, \psi^{(k)})(y)| &= |E_y[\psi^{(k)}(B_t)e^{-\int_0^t v(z,B(t-z))\,dz}]| \\ &\le S_t(\psi^{(k)})(y), \end{aligned}$$

where the inequality follows from our assumption $\mathrm{Re}(v) \ge 0$. Then passing to the limit as $k \to \infty$, one can easily complete the proof. We leave the details to the reader. ■

We wish to prove that $W_\cdot(v^{(n)}, \kappa^{(n)}, \cdot)(\cdot) \xrightarrow{\mathcal{C}} W_\cdot(v, \kappa, \cdot)(\cdot) \in C_q(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d)$, where

(a) for q.e. $x \in \mathbb{R}^d$, $W_\cdot(v, \kappa, x)(\cdot) \in C\big((0,\infty) \times \mathbb{R}^d\big)$,
(b) for all $(t, y) \in (0,\infty) \times \mathbb{R}^d$, $W_t(v, \kappa, \cdot)(y)$ is quasicontinuous.

First, by Lemma A.25, we can choose a subsequence $\{v^{(n_k)}\}$ such that $\mathcal{G}\tilde{v}^{(n_k)} \xrightarrow{\text{q.e.}} \mathcal{G}\tilde{v}$. Fix $\delta > 0$ and choose an open set $B_\delta \subset \mathbb{R}^d$ such that $\mathcal{C}(B_\delta) \le \delta$ and $\mathcal{G}\tilde{v}^{(n_k)} \to \mathcal{G}\tilde{v}$ uniformly on B_δ^c. Take any $x_{n_k} \to x$ in B_δ^c. Then the definition of $\mathcal{G}$ and the fact that $\mathcal{G}\tilde{v}^{(n_k)}(x_{n_k}) \to \mathcal{G}\tilde{v}(x)$ yield

$$\lim_{\epsilon \downarrow 0} \limsup_{n_k \to \infty} \int_0^\epsilon \int_{\mathbb{R}^d} \tilde{v}^{(n_k)}(s, y) p_s(x_{n_k} - y) s^{\hat{\rho}} p_1(y)\,dy\,ds = 0.$$

Using the definition of $\{\tilde{v}^{(n)}\}$ one can easily check that, in fact, for each compact $\Gamma \in \mathbb{R}^d$ and for each $\epsilon' > 0$,

$$\lim_{\epsilon \downarrow 0} \limsup_{n_k \to \infty} \sup_{z \in \Gamma, t \ge \epsilon'} \int_0^\epsilon \int_{\mathbb{R}^d} v^{(n_k)}(s, y) p_s(x_{n_k} - y) p_{t-s}(y - z)\,ds\,dy = 0. \tag{A.41}$$

In what follows set

$$u^{(n)}(\cdot,\cdot) \equiv W_\cdot(v^{(n)}, \kappa^{(n)}, x^{(n)})(\cdot).$$

Lemma A.27 *Set $\psi^{(n_k)} \equiv v^{(n_k)} u^{(n_k)} \kappa^{(n_k)}$ and $G_\epsilon(\psi^{(n_k)})(t, z) = \int_\epsilon^t S_{t-s}\big(\psi^{(n_k)}(s)\big)(z)\,ds$. Then $G_0(\psi^{(n_k)})(\cdot,\cdot)$ is relatively compact in $C\big((0,\infty) \times \mathbb{R}^d\big)$.*

Proof One can represent $G_0(\psi^{(n_k)})(t, z)$ as

$$G_0(\psi^{(n_k)})(t, z) = \int_0^\epsilon S_{t-s}\big(\psi^{(n_k)}(s)\big)(z)\,ds + G_\epsilon(\psi^{(n_k)})(t, z). \tag{A.42}$$

By Lemma A.26 $|u^{(n)}(s, z)| \le p_s(x^{(n)} - z)$. Therefore, using (A.41) it is easy to make $\int_0^\epsilon S_{t-s}\big(\psi^{(n_k)}(s)\big)(z)\,ds$ arbitrarily small on compact subsets of $\mathbb{R}^d$ by taking ϵ sufficiently small. To show relative compactness of the second term at the right side of (A.42), change the variables to get that $G_\epsilon\big(\psi^{(n_k)}(\cdot)\big)(t, z) = G_0\big(\psi^{(n_k)}(\epsilon + \cdot)\big)(t - \epsilon, z)$. Then Lemma A.4 gives the relative compactness of $G_0\big(\psi^{(n_k)}(\epsilon + \cdot)\big)(t - \epsilon, z)$ in $C\big((\epsilon,\infty) \times \mathbb{R}^d\big)$. ■

The proof of the following corollary is completely analogous to the proof of Corollary A.16 and hence omitted.

Corollary A.28 *$u^{(n_k)}$ is relatively compact in $C\big((0,\infty)\times \mathrm{R}^d\big)$ and if u is any limit point of $u^{(n_k)}$ then $u(t,\cdot) = W_t(v,\kappa,x)(\cdot)$ for any $t>0$.*

Lemma A.29 *$W_\cdot(v^{(n_k)},\kappa^{(n_k)},\cdot)(\cdot) \xrightarrow{q.e.} W_\cdot(v,\kappa,\cdot)(\cdot)$ as $n_k\to\infty$.*

Proof Recall that $\{x^{(n_k)}\}$ was an arbitrary sequence in B^c_δ such that $\lim_{n_k\to\infty} x_{n_k} = x$. Therefore, for any compact set $\Gamma \in \mathrm{R}^d$,

$$\sup_{x\in\Gamma\cap B^c_\delta} |W_\cdot(v^{(n_k)},\kappa^{(n_k)},x)(\cdot) - W_\cdot(v,\kappa,x)(\cdot)| \to 0 \quad \text{as} \quad n_k\to\infty$$

uniformly on compact subsets of $(0,\infty)\times\mathrm{R}^d$. Using $|W_\cdot(v^{(n_k)},\kappa^{(n_k)},x)(\cdot)| \le p_\cdot(x-\cdot)$, it is easy to verify that, in fact,

$$\sup_{x\in B^c_\delta} |W_\cdot(v^{(n_k)},\kappa^{(n_k)},x)(\cdot) - W_\cdot(v,\kappa,x)(\cdot)| \to 0 \quad \text{as} \quad n_k\to\infty$$

uniformly on compact subsets of $(0,\infty)\times\mathrm{R}^d$. ■

Lemma A.30 *There exists $\mathcal{N}\subset\mathrm{R}^d$ with $\mathcal{C}(\mathcal{N})=0$ such that*

$$|W_t(v,\kappa,x)(y)| \le p_t(x-y), \quad \forall\,(t,y,x)\in(0,\infty)\times\mathrm{R}^d\times(\mathrm{R}^d\setminus\mathcal{N}).$$

Proof Lemma A.29 implies that there exists set $\mathcal{N}\subset\mathrm{R}^d$ with $\mathcal{C}(\mathcal{N})=0$ such that

$$W_t(v,\kappa,x)(y) = \lim_{n_k\to\infty} W_t(v^{(n_k)},\kappa^{(n_k)},x)(y), \quad \forall\,(t,y,x)\in(0,\infty)\times\mathrm{R}^d\times(\mathrm{R}^d\setminus\mathcal{N}).$$

By Lemma A.26 $|W_t(v^{(n_k)},\kappa^{(n_k)},x)(y)| \le p_t(x-y)$ for all $n\ge 1$, and the result follows. ■

Lemma A.31 *$u^{(n)} \xrightarrow{\mathcal{C}} W_t(v,\kappa,\cdot)(\cdot)$ as $n\to\infty$.*

Proof In Lemma A.29 we proved that there is a subsequence $\{u^{(n_k)}\}$ such that $u^{(n_k)} \xrightarrow{\text{q.e.}} W_t(v,\kappa,\cdot)(\cdot)$ as $n_k\to\infty$. But the same arguments say that for any subsequence $\{u^{(n')}\}$ of $\{u^{(n)}\}$ there exists a further subsequence $\{u^{(n'_k)}\}$ such that $u^{(n'_k)} \xrightarrow{\text{q.e.}} W_t(v,\kappa,\cdot)(\cdot)$ as $n'_k\to\infty$. Hence the desired result follows from Lemma A.22. ■

Proof of Proposition A.18 (a) Follows from Lemma A.19, Lemma A.20, Lemma A.29, Lemma A.30.

(b) For q.e. x $W_t(v^{(n)},\kappa^{(n)},x)(\cdot)$ solves (A.37), which means that

$$W_t(v^{(n)},\kappa^{(n)},x)(y) = p_t(x-y) - \int_0^t\int_{\mathrm{R}^d} 2\kappa(s)p_{t-s}(y-z)v(s,z)W_s(v^{(n)},\kappa^{(n)},x)(z)\,dz\,ds,$$

$$\forall\,(t,y)\in(0,\infty)\times\mathrm{R}^d, \quad \text{for q.e. } x.$$

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Since any $\mu \in M_F^\rho$ does not charge sets of capacity nil, we have

$$\int_{\mathrm{R}^d} W_t(v^{(n)}, \kappa^{(n)}, x)(y)\,\mu(dx)$$
$$= S_t\mu(y) - \int_{\mathrm{R}^d}\int_0^t \int_{\mathrm{R}^d} 2\kappa(s) p_{t-s}(y-z) v(s,z) W_s(v^{(n)}, \kappa^{(n)}, x)(z)\,dz\,ds\,\mu(dx),$$
$$\forall\,(t,y) \in (0,\infty)\times \mathrm{R}^d.$$

Using Fubini's theorem and putting

$$W_t(v^{(n)}, \kappa^{(n)}, \mu)(y) \equiv \int_{\mathrm{R}^d} W_t(v^{(n)}, \kappa^{(n)}, x)(y)\,\mu(dx), \quad \forall\, t > 0,$$

we are done.

(c) Let $\{v^{(n)}\}$ in $L^{2,\hat\rho}_{c,\mathrm{loc},+}$ and $\{\kappa^{(n)}\}$ in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$ be arbitrary sequences such that $v^{(n)} \to v$ in $L^{2,\hat\rho}_{c,\mathrm{loc},+}$, $\kappa^{(n)} \to \kappa$ *weakly** in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$. It is sufficient to show that, for any $\epsilon > 0$ and any compact set $\Gamma \subset (0,\infty)\times\mathrm{R}^d$, we have

$$\text{(A.43)} \qquad \lim_{n\to\infty} \sup_{(t,y)\in\Gamma} \mathcal{C}\big(x : |W_t(v^{(n)}, \kappa^{(n)}, x)(y) - W_t(v,\kappa,x)(y)| > \epsilon\big) = 0.$$

Lemma A.31 shows that, for any $\{\tilde v^{(n)}\}$ in $\overline{C}\big((0,\infty)\times\mathrm{R}^d\big)_+$ and $\{\tilde\kappa^{(n)}\}$ in $\overline{C}_{\mathrm{R}}(\mathrm{R}_+)_+$ such that $\tilde v^{(n)} \to v$ in $L^{2,\hat\rho}_{c,\mathrm{loc},+}$ and $\kappa^{(n)} \to \kappa$ *weakly** in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$, we have

$$\text{(A.44)} \qquad \lim_{n\to\infty} \sup_{(t,y)\in\Gamma} \mathcal{C}\big(x : |W_t(\tilde v^{(n)}, \tilde\kappa^{(n)}, x)(y) - W_t(v,\kappa,x)(y)| > \epsilon\big) = 0.$$

For each compact $\Gamma \subset (0,\infty)\times\mathrm{R}^d$ and $\epsilon, \epsilon' > 0$ choose $\{\tilde v^{(n)}\}$ in $\overline{C}\big((0,\infty)\times\mathrm{R}^d\big)_+$ and $\{\tilde\kappa^{(n)}\}$ in $\overline{C}_{\mathrm{R}}(\mathrm{R}_+)_+$ such that $\tilde v^{(n)} \to v$ in $L^{2,\hat\rho}_{c,\mathrm{loc},+}$, $\kappa^{(n)} \to \kappa$ *weakly** in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$ and

$$\text{(A.45)} \qquad \sup_{(t,y)\in\Gamma} \mathcal{C}\big(x : |W_t(\tilde v^{(n)}, \tilde\kappa^{(n)}, x)(y) - W_t(v^{(n)},\kappa^{(n)},x)(y)| > \epsilon\big) \le \epsilon',$$

for all n. This yields

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{(t,y)\in\Gamma} \mathcal{C}\big(x : |W_t(v^{(n)}, \kappa^{(n)}, x)(y) - W_t(v,\kappa,x)(y)| > \epsilon\big)\\
&\quad \le \lim_{n\to\infty} \sup_{(t,y)\in\Gamma} \mathcal{C}\big(x : |W_t(\tilde v^{(n)}, \tilde\kappa^{(n)}, x)(y) - W_t(v^{(n)},\kappa^{(n)},x)(y)| > \epsilon\big)\\
&\qquad + \lim_{n\to\infty} \sup_{(t,y)\in\Gamma} \mathcal{C}\big(x : |W_t(\tilde v^{(n)}, \tilde\kappa^{(n)}, x)(y) - W_t(v,\kappa,x)(y)| > \epsilon\big)\\
&\quad \le \lim_{n\to\infty} \sup_{(t,y)\in\Gamma} \mathcal{C}\big(x : |W_t(\tilde v^{(n)}, \tilde\kappa^{(n)}, x)(y) - W_t(v^{(n)},\kappa^{(n)},x)(y)| > \epsilon\big) \le \epsilon',
\end{aligned}$$

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where the last inequality follows from (A.44), (A.45). Since ϵ' was arbitrary, (A.43) follows and this finishes the proof of Proposition A.18. ∎

With Proposition A.18 it is easy to accomplish the

Proof of Theorem 5.9 (a), (b) Let $\mu \in \tilde{S}'^{\rho}(\mathbb{R}^d)$, $r > 0$, $\kappa \in L^{\infty}_{\mathbb{R}}(\mathbb{R}_+)_+$. For arbitrary $r > 0$ define

$$\kappa^{(r)}(t) \equiv \kappa(r+t), \quad \forall t \geq 0,$$
$$v_t \equiv V_{r,r+t}(\mu,\kappa), \quad \forall t > 0.$$

Now we have the following representation for v and $U_{r,t}\big(V_{r,\cdot}(\mu,\kappa),x\big)$:

$$v_t = V_t(\mu,\kappa^{(r)}), \quad \forall t > 0,$$
$$U_{r,t}\big(V_{r,\cdot}(\mu,\kappa),x\big) = W_{t-r}\big(v,\kappa(r+\cdot),x\big), \quad \forall t > r.$$

By Proposition A.2 $v \in L^{2,\hat{\rho}}_{c,\mathrm{loc},+}$, therefore parts (a) and (b) of the theorem follow trivially from Proposition A.18 (a), (b).

(c) Let $T > 0$ and $\psi_1, \psi_2 \in C^{\infty}_{c,\mathbb{R}}(\mathbb{R}^d)_+$, $\mu^{(n)} \to \mu$ in $\tilde{S}'^{\rho}(\mathbb{R}^d)$, $r_n \to r$ in $[0,T)$ and $\kappa^{(n)} \to \kappa$ *weakly** in $L^{\infty}_{\mathbb{R}}(\mathbb{R}_+)_+$. $\{\mu^{(n)}\}$, μ, $\{r_n\}$, r are arbitrary, hence the proof of part (c) of the theorem will be finished if we show that

$$\text{(A.46)}\qquad \lim_{n\to\infty}\Big|\int_{\mathbb{R}^d} U_{r_n,T}\big(V_{r_n,\cdot}(\mu^{(n)},\kappa^{(n)}),x\big)(\psi_1)U_{r_n,T}\big(V_{r_n,\cdot}(\overline{\mu}^{(n)},\kappa^{(n)}),x\big)(\psi_2)\,\mu_1^{(n)}(dx)$$
$$-\int_{\mathbb{R}^d} U_{r,T}\big(V_{r,\cdot}(\mu,\kappa),x\big)(\psi_1)U_{r,T}\big(V_{r,\cdot}(\overline{\mu},\kappa),x\big)(\psi_2)\,\mu_1(dx)\Big| = 0.$$

As in (a) and (b) we have

$$U_{r_n,T}\big(V_{r_n,\cdot}(\mu^{(n)},\kappa^{(n)}),x\big) = W_{T-r_n}(v^{(n)},\kappa^{(n,r_n)},x), \quad \forall T \geq r_n, \quad \text{q.e. } x,$$
$$U_{r,T}\big(V_{r,\cdot}(\mu,\kappa),x\big) = W_{T-r}(v,\kappa^{(r)},x), \quad \forall T \geq r, \quad \text{q.e. } x,$$

where

$$\kappa^{(r)}(t) \equiv \kappa(r+t), \qquad \kappa^{(n,r_n)}(t) \equiv \kappa^{(n)}(r_n+t), \quad \forall t \geq 0,$$
$$v_t \equiv V_t(\mu,\kappa^{(r)}), \qquad v_t^{(n)} \equiv V_t(\mu^{(n)},\kappa^{(n,r_n)}), \quad \forall t > 0.$$

Recall that $U_{r_n,T}\big(V_{r_n,\cdot}(\overline{\mu}^{(n)},\kappa),x\big) = \overline{U}_{r_n,T}\big(V_{r_n,\cdot}(\mu^{(n)},\kappa),x\big)$ for all $\kappa \in L^{\infty}_{\mathbb{R}}(\mathbb{R}_+)_+$. Hence (A.46) is equivalent to

$$\text{(A.47)}\qquad \lim_{n\to\infty}\Big|\int_{\mathbb{R}^d} W_{T-r_n}(v^{(n)},\kappa^{(n,r_n)},x)(\psi_1)\overline{W}_{T-r_n}(v^{(n)},\kappa^{(n,r_n)},x)(\psi_2)\,\mu_1^{(n)}(dx)$$
$$-\int_{\mathbb{R}^d} W_{T-r}(v,\kappa^{(r)},x)(\psi_1)\overline{W}_{T-r}(v,\kappa^{(r)},x)(\psi_2)\,\mu(dx)\Big| = 0.$$

Since $r_n \to r$, $\kappa^{(n)} \to \kappa$ we obtain that $\kappa^{(r_n)} \to \kappa^{(r)}$, $\kappa^{(n,r_n)} \to \kappa^{(r)}$ *weakly** in $L^\infty_{\mathrm{R}}(\mathrm{R}_+)_+$. From Proposition A.2 it follows that

$$v^{(n)} \to v = V(\mu, \kappa^{(r)}), \quad \text{in } L^{2,\hat{\rho}}_{c,\mathrm{loc},+} \quad \text{as} \quad n \to \infty.$$

Apply Proposition A.18 (c) to see that

$$W_\cdot(v^{(n)}, \kappa^{(n,r_n)}, \cdot)(\cdot) \overset{\mathcal{C}}{\to} W_\cdot(v, \kappa^{(r)}, \cdot)(\cdot).$$

Apply Lemma A.23, Lemma A.30, Lemma A.24 and triangle inequality to verify that

$$\text{(A.48)} \qquad \lim_{n\to\infty} \tilde{h}^{(n)} \equiv \lim_{n\to\infty} \Big| \int_{\mathrm{R}^d} W_\cdot(v^{(n)}, \kappa^{(n,r_n)}, x)(\cdot)\overline{W}_\cdot(v^{(n)}, \kappa^{(n,r_n)}, x)(\cdot)\, \mu_1^{(n)}(dx) - \int_{\mathrm{R}^d} W_\cdot(v, \kappa^{(r)}, x)(\cdot)\overline{W}_\cdot(v, \kappa^{(r)}, x)(\cdot)\, \mu_1(dx) \Big| = 0,$$

uniformly on compact subsets of $(0, \infty) \times \mathrm{R}^d \times \mathrm{R}^d$. Since $\psi_1, \psi_2 \in C^\infty_{c,\mathrm{R}}(\mathrm{R}^d)_+$, (A.47) follows immediately.

(d) The proof of (d) is analogous to the proof of (c) and therefore is omitted. ■

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