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NEST ALGEBRAS OF OPERATORS AND THE DUNFORD-PETTIS PROPERTY

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ABSTRACT. A Banach space X is said to have the Dunford-Pettis Property if every weakly compact linear operator $T: X \longrightarrow Y$, where Y is any Banach space, is completely continuous (that is, T maps weakly convergent sequences to strongly convergent ones). In this paper, we prove that if $\mathcal{A}$ is a nest algebra of operators on a separable, infinite dimensional Hilbert space, then $\mathcal{A}$ fails to have the Dunford-Pettis Property. We also investigate a certain algebra associated to $\mathcal{A}$, analogous to a construction used by Bourgain and others in connection with the Dunford-Pettis Property for function algebras. We show that this algebra must lie between $\mathcal{A}$ and the quasi-triangular algebra $\mathcal{A} + \mathcal{K}$ and we give examples to show that either extreme or something in between is possible. Finally, we consider the algebra of analytic Toeplitz operators and give a result for the corresponding associated algebra which is analogous to a result of Cima, Jansen, and Yale for H^∞.

A Banach space X is said to have the Dunford-Pettis Property if every weakly compact linear operator $T: X \longrightarrow Y$, where Y is any Banach space, is completely continuous (that is, T maps weakly convergent sequence to strongly convergent ones). This general definition was made in the 1950's by Grothendieck following the earlier work of Dunford and Pettis ([10]) showing that $L^1(\mu)$ spaces have this property. There are various equivalent formulations of the Dunford-Pettis Property (cf. [9]) of which the following will be used in this article.

(DPP) The Banach space X has the Dunford-Pettis Property provided that, whenever the sequence (x_n) in X converges weakly to 0 and the sequence (f_n) in the dual space X^* converges weakly to 0, then the sequence $(f_n(x_n))$ converges to 0.

We remark that it is clear from this formulation and the canonical imbedding of a Banach space X into its second dual X^{**} that if X^* has the Dunford-Pettis Property then so does X.

In studying the Dunford-Pettis Property for nest algebras of operators on Hilbert space, we were motivated by the parallels which exist between these algebras and the space H^∞ of functions bounded and analytic in the open unit disc of the complex plane and by the fact that H^∞ satisfies (DPP), proven by Bourgain ([4]) using the theory of ultrafilters. In this note, we provide an elementary proof that no nest algebra on a separable Hilbert space satisfies (DPP), thus exhibiting an example of the limitations of the analogy with H^∞.

Bourgain has also shown ([3]) that if X is a subspace of a $C(K)$ space such that a certain algebra associated with X coincides with $C(K)$, then X has the Dunford-Pettis Property.

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Cima et al ([5], [6]) labeled this associated algebra the 'Bourgain algebra' of X and fell short of a purely function-theoretic proof that H^∞ has the Dunford-Pettis Property when they showed that the Bourgain algebra of H^∞ is $H^\infty + C$ rather than L^∞. In this note, we will study the 'Bourgain algebra' of a nest algebra and get results somewhat analogous to those for H^∞.

1. **Preliminaries & Notation.** Throughout, let $\mathcal{H}$ denote a separable, infinite-dimensional Hilbert space, $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathcal{K}$ the ideal of compact operators in $\mathcal{L}(\mathcal{H})$. The trace-class ideal in $\mathcal{K}$ will be denoted by (τc) and we remind the reader that, for $X \in (\tau c)$, the trace of X is given by $\mathrm{tr}(X) = \sum_j (Xe_j, e_j)$ where $\{e_j\}$ is any orthonormal basis for $\mathcal{H}$. For any vectors x and y in $\mathcal{H}$, the symbol $x \otimes y^*$ denotes the operator on $\mathcal{H}$ defined by $(x \otimes y^*)z = (z, y)x$. Note that $\|x \otimes y^*\| = \|x\|\ \|y\|$ while $\mathrm{tr}\,(x \otimes y^*) = (x, y)$.

For any Banach space X, the triple dual space X^{***} can be written as a direct sum $X^{***} = X^\perp \oplus X^*$ where $X^\perp$ denotes the annihilator of X. In light of the duality relations $\mathcal{K}^* \cong (\tau c)$ and $(\tau c)^* \cong \mathcal{L}(\mathcal{H})$ (cf. [13]), this yields $\mathcal{L}(\mathcal{H})^* \cong \mathcal{K}^\perp \oplus (\tau c)$. For $\phi \in \mathcal{L}(\mathcal{H})^*$, we will customarily write $\phi = \phi_0 + \phi_X$ where $\phi_0 \in \mathcal{K}^\perp$ and ϕ_X is induced by the trace-class operator X, namely, $\phi_X(A) = \mathrm{tr}\,(XA)$ for A in $\mathcal{L}(\mathcal{H})$.

A nest is a set of (self-adjoint) projections in $\mathcal{L}(\mathcal{H})$ which is linearly ordered by range inclusion, contains 0 and 1, and is closed in the stong operator topology. If $\mathcal{P}$ is a nest, then, for each P in $\mathcal{P}$, the projection P_- is defined by $P_- = \sup\{Q \in \mathcal{P} : Q < P\}$. The nest $\mathcal{P}$ is said to be continuous if $P = P_-$ for all P in $\mathcal{P}$ and purely atomic in case $\sum_{P\in\mathcal{P}}(P - P_-) = 1$. If $\mathcal{P}$ is any nest, then the associated nest algebra is given by $\mathrm{Alg}\mathcal{P} = \{T \in \mathcal{L}(\mathcal{H}) : PTP = TP, \text{ for all } P \in \mathcal{P}\}$. For a projection P, the projection $1 - P$ will be denoted by $P^\perp$. We will use Arveson's distance formula for nest algebras ([1]):

$$\text{(1.1)} \qquad \mathrm{dist}\,(T,\ \mathrm{Alg}\mathcal{P}) = \sup\{\|P^\perp TP\| : P \in \mathcal{P}\}, \text{ for } T \in \mathcal{L}(\mathcal{H}).$$

2. **Main Results.**

THEOREM 1. *Let $\mathcal{A} = Alg\mathcal{P}$ be a nest algebra on a separable infinite-dimensional Hilbert space $\mathcal{H}$. Then $\mathcal{A}$ fails to have the Dunford-Pettis Property.*

PROOF. Since $\mathcal{H}$ is infinite-dimensional, there is an orthonormal sequence $\{e_n : n \geqq 1\}$ in $\mathcal{H}$ such that either $e_1 \otimes e_n^*$ belongs to $\mathcal{A}$ for all $n \geqq 1$ or $e_n \otimes e_1^*$ is in $\mathcal{A}$ for all $n \geqq 1$. The arguments for the two cases are similar so we assume the former and set $A_n = e_1 \otimes e_n^*$.

First, $A_n \longrightarrow 0$ weakly in $\mathcal{A}$ since, if $\phi \in \mathcal{L}(\mathcal{H})^*$, then we may write $\phi = \phi_0 + \phi_X$ where ϕ_0 annihilates $\mathcal{K}$ and X is a trace-class operator. Thus, $\phi(A_n) = \phi_0(A_n) + \mathrm{tr}\,(XA_n) = \mathrm{tr}\,(Xe_1 \otimes e_n^*) = (Xe_1, e_n)$. This tends to 0 as n tends to ∞ since $\{e_n\}$ is an orthonormal set.

Next, let $X_n = A_n^* = e_n \otimes e_1^* \in (\tau c)$. For $T \in \mathcal{L}(\mathcal{H})$, we have $\mathrm{tr}\,(X_nT) = (e_n, T^*e_1) \longrightarrow 0$ as n tends to ∞. If follows that $\{\phi_{X_n} + \mathcal{A}^\perp\}$ converges to 0 weakly in the dual space $\mathcal{A}^* \cong \mathcal{L}(\mathcal{H})^*/\mathcal{A}^\perp$.

Finally, we observe that $\phi_{X_n}(A_n) = \text{tr}\,(X_n A_n) = \text{tr}\,(e_n \otimes e_n^*) = (e_n, e_n) = 1$ for all $n \geqq 1$. Thus, the condition (DPP) is not satisfied and therefore $\mathcal{A}$ fails to have the Dunford-Pettis Property. ∎

We remark that $\mathcal{A} \cong ((\tau c)/{}^{\perp}\mathcal{A})^*$ so that, by reversing the roles of A_n and X_n, the above proof shows that $(\tau c)/{}^{\perp}\mathcal{A}$ and, consequently, its dual $\mathcal{A}$ fail to have the Dunford-Pettis Property.

DEFINITION. For any subset $\mathcal{B}$ of $\mathcal{L}(\mathcal{H})$ the *Bourgain algebra* of $\mathcal{B}$ is the set

$$\mathcal{B}_b = \{T \in \mathcal{L}(\mathcal{H}) : \text{dist}\,(TA_n,\ \mathcal{B}) \longrightarrow 0 \text{ whenever } A_n \longrightarrow 0 \text{ weak}^* \text{ in } \mathcal{B}\}.$$

It is straightforward to see that $\mathcal{B}_b$ is a norm-closed subalgebra of $\mathcal{L}(\mathcal{H})$ and that, if $\mathcal{B}$ is itself a subalgebra of $\mathcal{L}(\mathcal{H})$, then $\mathcal{B}_b \supseteq \mathcal{B}$. In particular, if $\mathcal{A} = Alg\mathcal{P}$ is a nest algebra, then $\mathcal{A} \subseteq \mathcal{A}_b$.

LEMMA 2. *If $\mathcal{A} = Alg\mathcal{P}$ is a nest algebra, then $\mathcal{A} \subseteq \mathcal{A}_b \subseteq \mathcal{A} + \mathcal{K}$.*

PROOF. We have already discussed the first containment. The second follows from the theorem of J. A. Erdos ([11]) that the algebra generated by the rank-one elements of a nest algebra is dense in the nest algebra in the weak* topology. Thus, there is a sequence $\{R_n\}$ of compact operators in $\mathcal{A}$ such that $(1 - R_n) \longrightarrow 0$ in the weak* topology. If $T \notin \mathcal{A} + \mathcal{K}$, then we have $0 < \text{dist}\,(T, \mathcal{A} + \mathcal{K}) \leqq \text{dist}\,(T - TR_n, \mathcal{A})$ for all n so that the sequence $\{\text{dist}\,(T(1 - R_n), \mathcal{A})\}$ does not tend to 0. This implies that $T \notin \mathcal{A}_b$. Hence, $\mathcal{A}_b \subseteq \mathcal{A} + \mathcal{K}$ as desired. ∎

We remark that Erdos' result has been extended by Laurie and Longstaff ([12]) who have shown that, if $\mathcal{P}$ is any commutative, completely distributive lattice of projections, then the algebra generated by the finite-rank elements of Alg$\mathcal{P}$ is dense in Alg$\mathcal{P}$ in the weak* topology. The argument above then applies to this case as well and shows that $(\text{Alg}\mathcal{P})_b \subseteq \text{Alg}\mathcal{P} + \mathcal{K}$ for such lattices.

COROLLARY 3. *If $\mathcal{A}$ and $\mathcal{B}$ are two nest algebras for which $\mathcal{A}_b = \mathcal{B}_b$, then $\mathcal{A} + \mathcal{K} = \mathcal{B} + \mathcal{K}$.*

PROOF. Lemma 2 and the hypothesis imply that $\mathcal{A} \subseteq \mathcal{B} + \mathcal{K}$ and $\mathcal{B} \subseteq \mathcal{A} + \mathcal{K}$. Hence, $\mathcal{A} + \mathcal{K} \subseteq \mathcal{B} + \mathcal{K}$ and $\mathcal{B} + \mathcal{K} \subseteq \mathcal{A} + \mathcal{K}$ from which the conclusion follows. ∎

We remark that necessary and sufficient conditions for $\mathcal{A} + \mathcal{K} = \mathcal{B} + \mathcal{K}$ are known. (cf. [8]) Also, by the remark following Lemma 2, the Corollary holds as well when $\mathcal{A}$ and $\mathcal{B}$ are algebras having commutative, completely distributive lattices of projections.

3. **Examples.** We will now present a few examples which show that by appropriate choice of the nest algebra $\mathcal{A}$ we can have $\mathcal{A}_b = \mathcal{A}$, $\mathcal{A}_b = \mathcal{A} + \mathcal{K}$, or the proper containment $\mathcal{A} \subset \mathcal{A}_b \subset \mathcal{A} + \mathcal{K}$.

EXAMPLE 1. Let $\{e_n : n \in \mathcal{Z}\}$ be an orthonormal basis for $\mathcal{H}$ and, for each n, let E_n be the projection onto the subspace spanned by $\{e_j : j \leqq n\}$. Let $\mathcal{P} = \{E_n : n \in$

$\mathcal{Z}\} \cup \{0, 1\}$ and $\mathcal{A} = \mathrm{Alg}\mathcal{P}$. We will show that, in this case, $\mathcal{A}_b = \mathcal{A} + \mathcal{K}$. This case is analogous to the result of Cima et al ([6]) for H^∞.

Since $\mathcal{A} \subseteq \mathcal{A}_b \subseteq \mathcal{A} + \mathcal{K}$ and since $\mathcal{A}_b$ is a linear space, it is enough to show that $\mathcal{K} \subseteq \mathcal{A}_b$. Since $\mathcal{A}_b$ is also norm-closed, it will suffice to show that $\mathcal{A}_b$ contains every rank-one operator of the form $e_m \otimes e_n^*$ for $m, n \in \mathcal{Z}$. For this, let m and n be integers and set $R = e_m \otimes e_n^*$. If $n \geqq m$ then $R \in \mathcal{A} \subseteq \mathcal{A}_b$ and we are done. On the other hand, if $m > n$ and if $T \in \mathcal{L}(\mathcal{H})$ then

$$\mathrm{dist}\,(RT,\ \mathcal{A}) = \sup_{k \in \mathcal{Z}} \|E_k^\perp RT E_k\| = \left(\sum_{j=-\infty}^{m-1} |(Te_j,\ e_n)|^2 \right)^{1/2}.$$

If, in addition, $T \in \mathcal{A}$, then $(Te_j, e_n) = 0$ whenever $j < n$ and hence

$$\mathrm{dist}\,(RT,\ \mathcal{A}) = \left(\sum_{j=n}^{m-1} |(Te_j,\ e_n)|^2 \right)^{1/2}, \text{ for } T \in \mathcal{A}.$$

Suppose now that the operators $\{T_k\}$ in $\mathcal{A}$ converge to 0 in the weak* topology. Then, for each j with $n \leqq j \leqq m-1$, we have $\lim_{k\to\infty} |(T_k e_j, e_n)| = 0$. As there are only finitely many such j to consider, it follows that $\lim_{k\to\infty} \mathrm{dist}\,(RT_k, \mathcal{A}) = 0$ and, hence, R belongs to $\mathcal{A}_b$ as desired. Thus, $\mathcal{A}_b = \mathcal{A} + \mathcal{K}$.

Note that a similar argument can be applied to show that $(\mathrm{Alg}\mathcal{P})_b = \mathrm{Alg}\mathcal{P} + \mathcal{K}$ whenever the nest $\mathcal{P} = \{P_k : k \in \mathcal{Z}\} \cup \{0, 1\}$ where $(P_{k+1} - P_k)$ is finite-dimensional for all k.

EXAMPLE 2. Suppose that the nest $\mathcal{P}$ contains projections F, P, and Q such that $F < P < Q$ and $Q - F$ is finite-rank. Let R be any non-zero operator satisfying $R = (Q - P)R(P - F)$. For $A \in \mathrm{Alg}\mathcal{P}$ we then have $RA = R(P - F)A = R(P - F)AF^\perp = RAF^\perp$ so that $E^\perp RAE = 0$ whenever $E \in \mathcal{P}$ satisfies $E \leqq F$ or $E \geqq Q$. For $F < E < Q$, we have $\|E^\perp RAE\| \leqq \|RAE\| = \|RAF^\perp E\| \leqq \|RA(Q - F)\|$. Thus, for all $A \in \mathrm{Alg}\mathcal{P}$,

$$\mathrm{dist}\,(RA,\ \mathrm{Alg}\mathcal{P}) = \sup_{F<E<Q} \|E^\perp RAE\| \leqq \|RA(Q - F)\|.$$

If the sequence $\{A_k\}$ in $\mathrm{Alg}\mathcal{P}$ converges to 0 in the weak* topology, then, since R and $(Q - F)$ are compact, it follows that RA_k converges to 0 in the strong operator topology and that $RA_k(Q - F)$ tends to 0 in norm. Hence, dist $(RA_k, \mathrm{Alg}\mathcal{P})$ converges to 0 as k tends to ∞ which implies that $R \in (\mathrm{Alg}\mathcal{P})_b$. But clearly R does not belong to $\mathrm{Alg}\mathcal{P}$ so we conclude that, in this case, $(\mathrm{Alg}\mathcal{P})_b$ properly contains $\mathrm{Alg}\mathcal{P}$.

We remark that if the nest contains a projection P for which $P_+ = \inf\{E \in \mathcal{P} : E > P\} > P$ then, taking $Q = P_+$ in the above argument, we would have dist $(RA, \mathrm{Alg}\mathcal{P}) = \sup\{\|E^\perp RAE\| : F < E \leqq P\} \leqq \|RA(P - F)\|$ for all A in $\mathrm{Alg}\mathcal{P}$ so we would only require the assumption that $(P - F)$ is finite-rank in order to conclude that $R \in (\mathrm{Alg}\mathcal{P})_b \setminus \mathrm{Alg}\mathcal{P}$.

EXAMPLE 3. Let $\mathcal{A} = \mathrm{Alg}\mathcal{P}$ where the nest $\mathcal{P}$ contains projections P and Q such that $0 < P < Q < 1$ and such that $Q - P$ has infinite rank. Then there is an orthonormal set $\{e_j : j \geqq 1\}$ in $\mathcal{H}$ such that $Pe_1 = e_1$ and $(Q - P)e_j = e_j$ for all $j \geqq 2$.

Clearly, the operator $A_j = e_1 \otimes e_j^*$ is in $\mathcal{A}$ for each $j \geqq 2$. For each $X \in (\tau c)$, we have $\lim_{j\to\infty} \operatorname{tr}(XA_j) = \lim_{j\to\infty}(Xe_1, e_j) = 0$ since $\{e_j\}$ is an orthonormal set. Thus, the sequence $\{A_j\}$ converges to 0 in the weak* topology.

Next, let R be any compact operator satisfying $R = Q^\perp R$ and such that $\|Re_1\| = 1$. For each $j \geqq 2$, we then have

$$\begin{aligned}\operatorname{dist}(RA_j, \mathcal{A}) &= \sup_{E\in\mathcal{P}} \|E^\perp RA_j E\| \geqq \|Q^\perp RA_j Q\|\\ &= \|RA_j\| = \|Re_1 \otimes e_j^*\| = \|Re_1\|\, \|e_j\| = 1.\end{aligned}$$

It follows that $R \notin \mathcal{A}_b$. Since R is compact, we conclude that, in this case, $\mathcal{A}_b$ is properly contained in $\mathcal{A} + \mathcal{K}$.

Note that a similar argument applies in case there is a projection Q for which $Q - Q_-$ is infinite dimensional and either $0 \leqq Q_- < Q < 1$ or $0 < Q_- < Q \leqq 1$.

We remark also that it is clear from Examples 2 and 3 that we can construct a nest $\mathcal{P}$ so that $(\operatorname{Alg}\mathcal{P})_b$ does not coincide with either $\operatorname{Alg}\mathcal{P}$ or $\operatorname{Alg}\mathcal{P} + \mathcal{K}$.

EXAMPLE 4. Let the Hilbert space $\mathcal{H}$ have the concrete form $\mathcal{H} = L^2([0,1])$. For $0 \leqq t \leqq 1$, set $\mathcal{N}_t = \{f \in \mathcal{H} : f = 0 \text{ a.e. on } [t,1]\}$. With N_t denoting the projection onto $\mathcal{N}_t$, the nest $\mathcal{N} = \{N_t : 0 \leqq t \leqq 1\}$ is the well-known Volterra nest. Let $\mathcal{A} = \operatorname{Alg}\mathcal{N}$. We will show that $\mathcal{A} = \mathcal{A}_b$ in this case. (Note that the previous example shows that $\mathcal{A}_b \neq \mathcal{A} + \mathcal{K}$.)

Suppose the operator T does not belong to $\mathcal{A}$. Then, for some $0 < t < 1$, we have $N_t^\perp T N_t \neq 0$. Since N_s converges to N_t in the strong operator topology as s increases to t, it follows that $N_t^\perp T N_s$ converges to $N_t^\perp T N_t$ and, hence, that $N_t^\perp T N_s \neq 0$ for some $0 < s < t$. Choose a unit vector f in $\mathcal{H}$ such that $f = N_s f$ and $N_t^\perp Tf \neq 0$ and select an orthonormal sequence $\{f_n : n \geqq 1\}$ of vectors satisfying $(N_t - N_s)f_n = f_n$ for all n. For each $n \geqq 1$, set $A_n = f \otimes f_n^*$. Then each A_n belongs to $\mathcal{A}$ and the sequence $\{A_n\}$ converges to 0 in the weak* topology since $\{f_n\}$ tends to 0 weakly in $\mathcal{H}$. For $n \geqq 1$, we have

$$\begin{aligned}\operatorname{dist}(TA_n, \mathcal{A}) &\geqq \|N_t^\perp TA_n N_t\| = \|N_t^\perp Tf \otimes (N_t f_n)^*\|\\ &= \|N_t^\perp Tf\|\, \|N_t f_n\| = \|N_t^\perp Tf\| > 0.\end{aligned}$$

Thus, $\operatorname{dist}(TA_n, \mathcal{A})$ does not tend to 0 which implies that $T \notin \mathcal{A}_b$. We have shown that $\mathcal{A}_b \subseteq \mathcal{A}$. Since one always has $\mathcal{A} \subseteq \mathcal{A}_b$ for a nest algebra, we conclude that $\mathcal{A}_b = \mathcal{A}$ in this case.

A simple modification of the above argument shows that $(\operatorname{Alg}\mathcal{P})_b = \operatorname{Alg}\mathcal{P}$ for any continuous nest $\mathcal{P}$.

EXAMPLE 5. As a last example, we consider the algebra of analytic Toeplitz operators on the Hilbert space $H^2 = H^2(T)$, where T denotes the unit circle. For each f in L^∞, the operator T_f is defined on H^2 by $T_f g = P_{H^2} fg$ where P_{H^2} is the projection of $L^2(T)$ onto H^2. The analytic Toeplitz algebra is then $\mathcal{T} = \{T_f : f \in H^\infty\}$ which coincides with the weak* closed subalgebra of $\mathcal{L}(H^2)$ generated by 1 and the unilateral shift T_z. It is well known that, for each $f \in L^\infty$, the operator norm $\|T_f\| = \|f\|_\infty$. Since the unilateral shift is a completely non-unitary contraction, the Sz.Nagy – Foias functional Calculus then implies that the mapping $\alpha : H^\infty \longrightarrow \mathcal{T}$ defined by $\alpha(f) = T_f$ is an isometric Banach algebra isomorphism which is a homeomorphism when H^∞ and $\mathcal{T}$ are equipped with their respective weak* topologies. (cf. [2])

Since H^∞ and $\mathcal{T}$ are isometrically isomorphic and since Bourgain has shown that H^∞ has the Dunford-Pettis Property, it follows that $\mathcal{T}$ has the Dunford-Pettis Property as well.

Cima, Janson, and Yale have shown that

$$\{f \in L^\infty : \text{dist}\,(fh_n,\ H^\infty) \longrightarrow 0 \text{ whenever } h_n \longrightarrow 0 \text{ weak* in } H^\infty\} = H^\infty + C.$$

Since the mapping α is isometric and a weak* homeomorphism, and since $T_f T_h = T_{fh}$ for all f in L^∞ and all h in H^∞, it follows that T_f belongs to $\mathcal{T}_b$ if and only if f is in $H^\infty + C$. In particular, $\mathcal{T}_b$ is not all of $\mathcal{L}(H^2)$.

Finally, we show that $\mathcal{T}_b$ contains all compact operators. For this, let $\{e_n : n \geqq 0\}$ denote the usual orthonormal basis for H^2. Given non-negative integers n and m, let $R_{nm} = e_n \otimes e_m^*$. For $f \in H^\infty$ and $k \geqq 0$, we then have

$$R_{nm}T_f e_k = (T_f e_k,\ e_m)e_n = \begin{cases} \hat{f}(m-k)e_n, & \text{if } m \geqq k; \\ 0, & \text{if } m < k. \end{cases}$$

Suppose now that the sequence $\{f_j\}$ in H^∞ converges weak* to 0. It is well known that this implies that $\lim_{j\to\infty} \hat{f}_j(k) = 0$ for each $k \geqq 0$. Thus, given $\epsilon > 0$, there exists J such that $|\hat{f}_j(k)| \leqq \epsilon/\sqrt{2^{k+1}}$ for all $k = 0, \ldots, m$ and all $j \geqq J$. For $j \geqq J$, we then have

$$\begin{aligned} \text{dist}\,(R_{nm}T_{f_j},\ \mathcal{T}) \leqq \|R_{nm}T_{f_j}\| &= \left(\sum_{k=0}^{m} |\hat{f}_j(k)|^2\right)^{1/2} \\ &\leqq \left(\sum_{k=0}^{m} \epsilon^2/2^{k+1}\right)^{1/2} < \epsilon. \end{aligned}$$

Hence, $\lim_{j\to\infty} \text{dist}\,(R_{nm}T_{f_j}, \mathcal{T}) = 0$ which implies that R_{nm} is in $\mathcal{T}_b$ for all non-negative integers n and m. Since $\mathcal{T}_b$ is a norm-closed subalgebra of $\mathcal{L}(H^2)$, it follows that every compact operator belongs to $\mathcal{T}_b$.

We conclude this note with two questions. First, if $\mathcal{A}$ is an operator subalgebra for which $\mathcal{A}_b = \mathcal{L}(\mathcal{H})$, does it follow that $\mathcal{A}$ has the Dunford-Pettis property? For instance, if $\mathcal{H}$ is finite-dimensional then it can be shown that $\mathcal{A}_b = \mathcal{L}(\mathcal{H})$ and that $\mathcal{A}$ satisfies (DPP) for every subalgebra $\mathcal{A}$. Example 5 shows that, on an infinite-dimensional space, the converse of this question does not hold. Second, if the subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ does have the Dunford-Pettis property, must $\mathcal{A}_b \supseteq \mathcal{K}$? Again, the converse question is not true as illustrated by Example 1 and Theorem 1.

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