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On the Uniqueness of Jordan Canonical Form Decompositions of Operators by K-theoretical Data

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Abstract. In this paper, we develop a generalized Jordan canonical form theorem for a certain class of operators in $\mathscr{L}(\mathscr{H})$. A complete criterion for similarity for this class of operators in terms of K-theory for Banach algebras is given.

1 Notations and Introduction

In this paper the authors continue the study in [10, 15] on generalizing the Jordan canonical form theorem for bounded linear operators on separable Hilbert spaces.

Denote by $\mathscr{L}(\mathscr{H})$ the set of bounded linear operators on a complex and separable Hilbert space $\mathscr{H}$. An *idempotent* P on $\mathscr{H}$ is an operator in $\mathscr{L}(\mathscr{H})$ such that $P^2 = P$. A *projection* Q in $\mathscr{L}(\mathscr{H})$ is an idempotent such that $Q = Q^*$. Following P. Halmos [9], an operator A in $\mathscr{L}(\mathscr{H})$ is said to be *irreducible* if its commutant $\{A\}' \triangleq \{B \in \mathscr{L}(\mathscr{H}) : AB = BA\}$ contains no projections other than 0 and the identity operator I on $\mathscr{H}$. (The separability assumption is necessary, because on a nonseparable Hilbert space every operator is reducible.) Following F. Gilfeather [8], an operator A in $\mathscr{L}(\mathscr{H})$ is said to be *strongly irreducible* if XAX^{-1} is irreducible for every invertible operator X in $\mathscr{L}(\mathscr{H})$. Equivalently, the commutant of a strongly irreducible operator contains no idempotents other than 0 and I. A Jordan matrix can be viewed as the prototype of a strongly irreducible operator. For an operator A in $\mathscr{L}(\mathscr{H})$, a nonzero idempotent P in $\{A\}'$ is said to be *minimal* if every idempotent Q in $\{A\}' \cap \{P\}'$ satisfies $QP = P$ or $QP = 0$. For a minimal idempotent P in $\{A\}'$, the restriction $A|_{\operatorname{ran} P}$ is strongly irreducible on $\operatorname{ran} P$.

The Jordan canonical form theorem states that each operator A in $M_n(\mathbb{C})$ is similar to a direct sum of Jordan matrices. The direct sum is unique up to permutations. An equivalent statement is that for any two (bounded) maximal Boolean algebras of idempotents $\mathscr{P}$ and $\mathscr{Q}$ in $\{A\}' \cap M_n(\mathbb{C})$, there exists an invertible operator X in

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$\{A\}' \cap M_n(\mathbb{C})$ such that $X\mathscr{P}X^{-1} = \mathscr{Q}$. (For Boolean algebras of idempotents, the reader is referred to [7].)

We say that an operator $A \in \mathscr{L}(\mathscr{H})$ has Property J if for any two bounded maximal Boolean algebras of idempotents $\mathscr{P}$ and $\mathscr{Q}$ in $\{A\}'$, there exists an invertible operator X in $\{A\}'$ such that $X\mathscr{P}X^{-1} = \mathscr{Q}$.

Inspired by [2], we gave a necessary and sufficient condition in [10] to represent an operator in a generalized Jordan canonical form. Precisely, an operator A in $\mathscr{L}(\mathscr{H})$ is similar to a direct integral of strongly irreducible operators if and only if its commutant $\{A\}'$ contains a bounded maximal Boolean algebra of idempotents. Furthermore, it is worth pointing out that there exist operators whose commutants contain no bounded maximal Boolean algebras of idempotents. The reader is referred to [10] for the pertinent examples and to [11,16,17] for more results about strongly irreducible operators. For related concepts and results concerning "direct integrals" in von Neumann's reduction theory, the reader is referred to [2, 4–6, 12, 14].

To generalize the Jordan canonical form theorem, a natural question following [10] is whether a generalized Jordan canonical form for an operator is unique up to similarity. In other words, does the above Property J holds for an operator A in $\mathscr{L}(\mathscr{H})$?

Let μ be (the completion of) a finite regular Borel measure supported on a compact subset Λ of $\mathbb{C}$. For the sake of simplicity, in what follows, we use elements in $L^\infty(\mu)$ as multiplication operators on $L^2(\mu)$ and matrices in $M_n(L^\infty(\mu))$ as bounded linear operators acting on $(L^2(\mu))^{(n)}$, the direct sum of n copies of $L^2(\mu)$. An operator A in $\mathscr{L}(\mathscr{H})$ is said to be *n-normal* if A is unitarily equivalent to an operator in $M_n(L^\infty(\mu))$ for some positive integer n. Every n-normal operator is unitarily equivalent to an upper triangular operator in $M_n(L^\infty(\mu))$, by [1, Corollary 2].

In [10], we proved that a direct integral of strongly irreducible operators can be written as a direct sum of upper triangular n-normal operators, where each summand has the same main diagonal entries and the symbols of 1-diagonal entries are nonzero a.e. on their supports. To answer the above question, we considered whether Property J holds for the summands. The result in [15] is developed from this. In this paper, we develop a more general result such that the result in [15] can be viewed as a special case.

In [15], we constructed an operator C to show that Property J does not hold for some non self-adjoint operators in general. The reason behind this is that the multiplicity function for S, a single generator of a maximal abelian self-adjoint subalgebra in $\{C\}'$, is not bounded.

For $A \in \mathscr{L}(L^2(\mu))$, denote by $A^{(m)}$ the direct sum of m copies of A acting on $(L^2(\mu))^{(m)}$. In [15], we mainly proved that an operator $A^{(m)}$ in the following upper triangular form has Property J,

$$A = \begin{pmatrix} M_{f_{11}} & \cdots & M_{f_{1n}} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & M_{f_{nn}} \end{pmatrix}_{n\times n}, \tag{1.1}$$

where m, n are positive integers, f_{ij} is in $L^\infty(\mu)$ for $1 \le i, j \le n$ such that the following hold:

(a) $f_{ii} = f_{11}$ for $1 \le i \le n$ and $M_{f_{11}}$ is star-cyclic;

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(b) $f_{i,i+1}(\lambda) \neq 0$ a.e. on $\operatorname{spt}(\mu)$, the support of μ, for $1 \leq i \leq n-1$,

where $M_{f_{ij}}$ is the multiplication operator on $L^2(\mu)$ with symbol f_{ij}. An operator N in $\mathscr{L}(L^2(\mu))$ is said to be *star-cyclic* if there exists a vector ξ in $L^2(\mu)$ such that $L^2(\mu)$ is the smallest reducing subspace for N containing ξ.

In this paper, we develop a technique to prove that an operator A in the form

$$A = \bigoplus_{i=1}^{k} A_{n_i}^{(m_i)} \tag{1.2}$$

has Property J, where A_{n_i} is in the upper triangular form

$$A_{n_i} = \begin{pmatrix} f_{11,i} & f_{12,i} & \cdots & f_{1n_i,i} \\ 0 & f_{11,i} & \cdots & f_{2n_i,i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{11,i} \end{pmatrix}_{n_i \times n_i}, \tag{1.3}$$

such that the following hold:

(a) $f_{st,i}$ is in $L^\infty(\mu)$ for $1 \leq s, t \leq n_i$;
(b) $f_{11,i} = f_{11,1}$ for $1 \leq i \leq k$ and $M_{f_{11,1}}$ is star-cyclic;
(c) $f_{j,j+1,i}(\lambda) \neq 0$ a.e. on $\operatorname{spt}(\mu)$ for $1 \leq j \leq n_i - 1$ and $1 \leq i \leq k$;
(d) $n_1 > n_2 > \cdots > n_k$ and all m_i are positive integers for $1 \leq i \leq k$.

The above condition (d) leads to difficulties in computing.

Secondly, we prove a complete similarity criterion for operators as in (1.3), expressed in terms of Banach algebra K-theory.

In K-theory for Banach algebras, we denote by $\mathscr{P}_n(\{A\}')$ the set of idempotents in $M_n(\{A\}')$ and by "$\sim$" the similarity relation in the corresponding algebra. The semigroup $\bigcup_{n=1}^{\infty} \mathscr{P}_n(\{A\}')/\sim$ is denoted by $V(\{A\}')$. By $K_0(\{A\}')$ we denote the Grothendieck group generated by $V(\{A\}')$, which is well known as the K_0-group of $\{A\}'$. The reader is referred to [3, 13] for details. For a compact subset Γ of $\mathbb{C}$, we define an additive group in the following form

$$BB(\Gamma, \mathbb{Z}^{(n)}) \triangleq \left\{ f(\lambda) \in \mathbb{Z}^{(n)} : f \text{ is bounded and Borel on } \Gamma \right\},$$

where n is a positive integer and $\mathbb{Z}^{(n)}$ is the n-fold direct sum of $\mathbb{Z}$ with itself. Precisely, we will prove the following theorems.

Theorem 1.1 *Let $A \in \mathscr{L}(\mathscr{H})$ be assumed as in (1.2). Then the following statements hold:*

(i) *A has Property J;*
(ii) *$K_0(\{A\}') \cong BB(\sigma(A), \mathbb{Z}^{(k)})$ (isomorphism of ordered groups).*

For operators as in (1.2), we characterize the similarity using K-theory for Banach algebras as follows.

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Theorem 1.2 *Let*

$$A = \sum_{i=1}^{s} \oplus A_{n_i}^{(m_i)} \quad \textit{and} \quad B = \sum_{j=1}^{t} \oplus B_{l_j}^{(k_j)}$$

be as in (1.2), *with every entry of* A_{n_i} *and* B_{l_j} *in* $L^\infty(\mu)$ *as in* (1.3), *for* $1 \le i \le s < \infty$ *and* $1 \le j \le t < \infty$, *where* $n_i \ne n_j$ *for* $i \ne j$. *Then* A *and* B *are similar if and only if there exists an isomorphism* θ *of ordered groups from* $K_0(\{A \oplus B\}')$ *to* $BB(\sigma(A), \mathbb{Z}^{(s)})$ *such that*

$$\theta\big([I_{\{A\oplus B\}'}]\big) = 2m_1e_1 + 2m_2e_2 + \cdots + 2m_se_s,$$

where $e_i(\lambda)$ *is an* s*-tuple vector with the* i*-th entry* 1 *and others* 0, $\{e_i(\lambda)\}_{i=1}^{s}$ *are the generators of the semigroup* $\mathbb{N}^{(s)}$ *of* $\mathbb{Z}^{(s)}$ *for every* λ *in* $\sigma(A)$, *and* $I_{\{A\oplus B\}'}$ *is the unit of* $\{A \oplus B\}'$.

When $\operatorname{spt}(\mu)$ is a single point, the above two theorems are identified with the Jordan canonical form theorem.

2 Proofs

Throughout this section, it is sufficient to prove the main theorems for $k = 3$. For an n-normal operator A in the form as in (1.1), an application of [15, Lemma 3.1] shows that for a fixed λ in the support of μ, the operator $A(\lambda)$ is strongly irreducible if and only if $f_{ii}(\lambda) = f_{nn}(\lambda)$ and $f_{i,i+1}(\lambda) \ne 0$ hold for $i = 1, \dots, n-1$. Therefore, for an n-normal operator A in the form as in (1.3), $A(\lambda)$ is strongly irreducible for almost every λ in the support of μ. We need to mention that the multiplication operators $M_{f_{j,j+1,i}}$ is not invertible in general. This makes the computation become more complicated. However, the commutant $\{A_n\}'$ is a subalgebra of $\{N_\mu^{(n)}\}'$ by [15, Lemma 3.2] for an operator A_n in the upper triangular form

$$A_n = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ 0 & f_{11} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{11} \end{pmatrix}_{n\times n},$$

as in (1.3) such that the following hold:

(a) f_{ij} is in $L^\infty(\mu)$ for μ (the completion of) a finite positive regular Borel measure supported on a compact subset Λ of $\mathbb{C}$ and $1 \le i, j \le n$;
(b) $M_{f_{11}}$ is star-cyclic;
(c) $f_{j,j+1}(\lambda) \ne 0$ a.e. on $\operatorname{spt}(\mu)$ for $1 \le j \le n-1$;
(d) N_μ is defined by $(N_\mu\xi)(z) = z \cdot \xi(z)$ for every ξ in $L^2(\mu)$.

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Precisely, by [15, Lemma 3.2], every operator X in $\{A_n\}'$ is in the form

$$X=\begin{pmatrix}\psi & \psi_{12} & \psi_{13} & \cdots & \psi_{1n}\\ 0 & \psi & \psi_{23} & \cdots & \psi_{2n}\\ 0 & 0 & \psi & \cdots & \psi_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \psi\end{pmatrix}_{n\times n},$$

and in particular, every idempotent E in $\{A_n\}'$ is in the form $E=M^{(n)}_{\chi_\Delta}$ for some characteristic function χ_Δ in $L^\infty(\mu)$, where Δ is a Borel subset of $\operatorname{spt}(\mu)$. Let $\mathscr{E}_n$ denote the set of idempotents in $\{A_n\}'$. Then $\mathscr{E}_n$ is the only maximal Boolean algebra of idempotents in $\{A_n\}'$ and obviously, $\mathscr{E}_n$ is bounded. We observe that the bounded Boolean algebra of idempotents

$$\mathscr{E}\triangleq(\overbrace{\mathscr{E}_{n_1}\oplus\cdots\oplus\mathscr{E}_{n_1}}^{m_1})\oplus(\overbrace{\mathscr{E}_{n_2}\oplus\cdots\oplus\mathscr{E}_{n_2}}^{m_2})\oplus(\overbrace{\mathscr{E}_{n_3}\oplus\cdots\oplus\mathscr{E}_{n_3}}^{m_3}) \tag{2.1}$$

is maximal in the commutant of $A=A^{(m_1)}_{n_1}\oplus A^{(m_2)}_{n_2}\oplus A^{(m_3)}_{n_3}$ as mentioned in (1.2) and (1.3). In this article, we define $\mathscr{E}$ to be the *standard* bounded maximal Boolean algebra of idempotents in $\{A\}'$, where A is defined as in (1.2). The following two preliminary lemmas are needed to prove Theorem 1.1.

Lemma 2.1 *Let A_{n_1} and A_{n_2} $(n_1>n_2)$ be as in (1.3). Then*

(i) *the equality $A_{n_1}X=XA_{n_2}$ yields that $X=(X_1^{\mathrm T},\mathbf{0}_{n_2\times(n_1-n_2)})^{\mathrm T}$, where X_1 is an upper triangular $n_2\times n_2$ operator-valued matrix with every entry of X_1 in $\{N_\mu\}'$, and the transpose of X_1 is denoted by $X_1^{\mathrm T}$;*

(ii) *the equality $A_{n_2}Y=YA_{n_1}$ yields that $Y=(\mathbf{0}_{n_2\times(n_1-n_2)},Y_1)$, where Y_1 is an upper triangular $n_2\times n_2$ operator-valued matrix with every entry of Y_1 in $\{N_\mu\}'$.*

Proof If $A_{n_1}=A_{n_2}$, then this lemma is identified with [15, Lemma 3.2]. For the sake of simplicity, let operators A_{n_1} and A_{n_2} be in the form

$$A_{n_2}=\begin{pmatrix}f & f_{12} & f_{13} & \cdots & f_{1n_2}\\ 0 & f & f_{23} & \cdots & f_{2n_2}\\ 0 & 0 & f & \cdots & f_{3n_2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & f\end{pmatrix}_{n_2\times n_2}\begin{matrix}L^2(\mu)\\ L^2(\mu)\\ L^2(\mu)\\ \vdots\\ L^2(\mu),\end{matrix}$$

and

$$A_{n_1}=\begin{pmatrix}f & g_{12} & g_{13} & \cdots & g_{1n_1}\\ 0 & f & g_{23} & \cdots & g_{2n_1}\\ 0 & 0 & f & \cdots & g_{3n_1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & f\end{pmatrix}_{n_1\times n_1}\begin{matrix}L^2(\mu)\\ L^2(\mu)\\ L^2(\mu)\\ \vdots\\ L^2(\mu).\end{matrix}$$

Let $E_f(\cdot)$ denote the spectral measure for M_f. For a Borel subset Δ of $\sigma(M_f)$ such that $E_f(\Delta)$ is a nontrivial projection in $\{M_f\}'$, we write $P_1=E_f(\Delta)$ and

$P_2 = E_f(\sigma(M_f)\backslash\Delta)$; we also write μ_1 for $\mu|_{f^{-1}(\Delta)}$ and μ_2 for $\mu|_{f^{-1}(\sigma(M_f)\backslash\Delta)}$. Denote $f_1 = f|_{\mathrm{spt}(\mu_1)}$ and $f_2 = f|_{\mathrm{spt}(\mu_2)}$. Hence, the operators A_{n_1}, A_{n_2} and X can be expressed in the form

$$A_{n_i} = \begin{array}{c} \begin{pmatrix} A_{n_i,1} & \mathbf{0} \\ \mathbf{0} & A_{n_i,2} \end{pmatrix} \end{array} \begin{array}{l} \operatorname{ran} P_1^{(n_i)} \\ \operatorname{ran} P_2^{(n_i)} \end{array}, \quad \text{for } i = 1,2, \quad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where

$$A_{n_1,i} = \begin{pmatrix} f_i & g_{12,i} & \cdots & g_{1n_1,i} \\ 0 & f_i & \cdots & g_{2n_1,i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_i \end{pmatrix}_{n_1\times n_1} \begin{array}{l} \operatorname{ran} P_i \\ \operatorname{ran} P_i \\ \vdots \\ \operatorname{ran} P_i \end{array}, \quad i = 1,2,$$

and

$$A_{n_2,i} = \begin{pmatrix} f_i & f_{12,i} & \cdots & f_{1n_2,i} \\ 0 & f_i & \cdots & f_{2n_2,i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_i \end{pmatrix}_{n_2\times n_2} \begin{array}{l} \operatorname{ran} P_i \\ \operatorname{ran} P_i \\ \vdots \\ \operatorname{ran} P_i \end{array}, \quad i = 1,2.$$

The equality $A_{n_1}X = XA_{n_2}$ yields $A_{n_1,1}X_{12} = X_{12}A_{n_2,2}$, and this equality can be expressed in the form

$$\begin{pmatrix} f_1 & g_{12,1} & \cdots & g_{1n_1,1} \\ 0 & f_1 & \cdots & g_{2n_1,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_1 \end{pmatrix} \begin{pmatrix} X_{12,11} & X_{12,12} & \cdots & X_{12,1n_2} \\ X_{12,21} & X_{12,22} & \cdots & X_{12,2n_2} \\ \vdots & \vdots & \ddots & \vdots \\ X_{12,n_11} & X_{12,n_12} & \cdots & X_{12,n_1n_2} \end{pmatrix} =$$
$$\begin{pmatrix} X_{12,11} & X_{12,12} & \cdots & X_{12,1n_2} \\ X_{12,21} & X_{12,22} & \cdots & X_{12,2n_2} \\ \vdots & \vdots & \ddots & \vdots \\ X_{12,n_11} & X_{12,n_12} & \cdots & X_{12,n_1n_2} \end{pmatrix} \begin{pmatrix} f_2 & f_{12,2} & \cdots & f_{1n_2,2} \\ 0 & f_2 & \cdots & f_{2n_2,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_2 \end{pmatrix}.$$

Since μ_1 and μ_2 are mutually singular, $M_{f_1}X_{12,n_11} = X_{12,n_11}M_{f_2}$ yields that $X_{12,n_11} = 0$. Thus, the equality $M_{f_1}X_{12,n_12} = X_{12,n_12}M_{f_2}$ yields that $X_{12,n_12} = 0$. In this way, we obtain that every entry in the n_1-th row of X_{12} is zero. The same result holds for the (n_1-1)-th row of X_{12}. By induction, we obtain that $X_{12} = \mathbf{0}$. By a similar computation, we have that $X_{21} = \mathbf{0}$. This means that the equality $P_i^{(n_1)}X = XP_i^{(n_2)}$ holds for every Borel subset Δ of $\sigma(M_f)$. Therefore, the operator X can be expressed in the form

$$X = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n_2} \\ h_{21} & h_{22} & \cdots & h_{2n_2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n_11} & h_{n_12} & \cdots & h_{n_1n_2} \end{pmatrix}_{n_1\times n_2},$$

where h_{ij} is in $L^\infty(\mu)$, $1 \le i \le n_1$ and $1 \le j \le n_2$. By the assumption, we have that $f_{i,i+1}(\lambda) \neq 0$ and $g_{j,j+1}(\lambda) \neq 0$ for $1 \le i \le n_2 - 1$, $1 \le j \le n_1 - 1$, and almost every λ in

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$\sigma(M_f)$. The equality $A_{n_1}X = XA_{n_2}$ yields that $fh_{n_1-1,1} + g_{n_1-1,n_1}h_{n_1 1} = h_{n_1-1,1}f$. This equality yields that $h_{n_1 1} = 0$. Thus, the equality

$$fh_{n_1-2,1} + g_{n_1-2,n_1-1}h_{n_1-1,1} = h_{n_1-2,1}f$$

yields that $h_{n_1-1,1} = 0$. By computation, $h_{j,1} = 0$ for $j = 2, \dots, n_1$.

By the equality $A_{n_1}X = XA_{n_2}$, we have $fh_{n_1-1,2} + g_{n_1-1,n_1}h_{n_1 2} = h_{n_1-1,2}f$. This yields that $h_{n_1 2} = 0$. Thus, the equality

$$fh_{n_1-2,2} + g_{n_1-2,n_1-1}h_{n_1-1,2} = h_{n_1-2,2}f$$

yields that $h_{n_1-1,2} = 0$. By computation, $h_{j,2} = 0$ for $j = 3, \dots, n_1$. By induction, we have $h_{j,i} = 0$ for $i < j$. The proof of the first assertion is finished.

In the proof of the second assertion, by a similar computation, Y is obtained as required. ∎

A fact we need to mention is that if $n_1 = n_2$, then X is an $n_1 \times n_1$ upper triangular operator-valued matrix with every entry of X in $\{M_f\}'$, whose entries have further relations between themselves.

Lemma 2.2 *For an operator A defined as in* (1.2) *and* (1.3)*, given an idempotent P in* $\{A\}'$*, there exists an invertible operator X in* $\{A\}'$ *such that* XPX^{-1} *is in* $\mathscr{E}$ *(defined as in* (2.1)*).*

Proof As defined in (1.2) and (1.3), we have $A = A_{n_1}^{(m_1)} \oplus A_{n_2}^{(m_2)} \oplus A_{n_3}^{(m_3)}$ for positive integers $n_1 > n_2 > n_3$.

Let B be an operator in $\{A\}'$. Then B can be expressed in the form

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}, \tag{2.2}$$

where

$$B_{ij} = \begin{pmatrix} B_{ij;11} & \cdots & B_{ij;1m_j} \\ \vdots & \ddots & \vdots \\ B_{ij;m_i 1} & \cdots & B_{ij;m_i m_j} \end{pmatrix}_{m_i \times m_j}, \tag{2.3}$$

and $B_{ij;st}$ is in the set $\{X \text{ is bounded linear} : A_{n_i}X = XA_{n_j}\}$, for $1 \le i, j \le 3$. For B in $\{A\}'$, there exists a unitary operator U that is a composition of finitely many row-switching transformations such that $C = UBU^*$ is in the form

$$C = \begin{pmatrix} C_{11} & \cdots & C_{1n_1} \\ \vdots & \ddots & \vdots \\ C_{n_1 1} & \cdots & C_{n_1 n_1} \end{pmatrix}, \tag{2.4}$$

where C_{lk} consists of the (l,k) entries of all the $B_{ij;st}$'s, and the relative positions of these entries stay invariant in C_{lk}. Note that C_{lk} is not square for $l \neq k$, and C_{11}, C_{n_3+1,n_3+1}, and C_{n_2+1,n_2+1} are not of the same size. By Lemma 2.1, we have that $C_{ij} = 0$ for $i > j$.

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For $1 \le i \le n_3$, the block entry C_{ii} is in the form

$$C_{ii} = \begin{pmatrix} C_{ii;11} & C_{ii;12} & C_{ii;13} \\ \mathbf{0} & C_{ii;22} & C_{ii;23} \\ \mathbf{0} & \mathbf{0} & C_{ii;33} \end{pmatrix},$$

where

$$C_{ii;kl} = \begin{pmatrix} b^{ii}_{kl;11} & \cdots & b^{ii}_{kl;1m_l} \\ \vdots & \ddots & \vdots \\ b^{ii}_{kl;m_k1} & \cdots & b^{ii}_{kl;m_km_l} \end{pmatrix}_{m_k \times m_l},$$

and the operator $b^{ii}_{kl;st}$ is the (i,i) entry of the block $B_{kl;st}$, for $1 \le k,l \le 3$, and $1 \le s \le m_k$, and $1 \le t \le m_l$.

For $n_3 < i \le n_2$, the block entry C_{ii} is in the form

$$\left(\begin{array}{ccc|ccc} b^{ii}_{11;11} & \cdots & b^{ii}_{11;1m_1} & b^{ii}_{12;11} & \cdots & b^{ii}_{12;1m_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b^{ii}_{11;m_11} & \cdots & b^{ii}_{11;m_1m_1} & b^{ii}_{12;m_11} & \cdots & b^{ii}_{12;m_1m_2} \\ & & & b^{ii}_{22;11} & \cdots & b^{ii}_{22;1m_2} \\ & \mathbf{0}_{m_2\times m_1} & & \vdots & \ddots & \vdots \\ & & & b^{ii}_{22;m_21} & \cdots & b^{ii}_{22;m_2m_2} \end{array}\right),$$

and for $n_2 < j \le n_1$ the block entry C_{jj} is in the form

$$\begin{pmatrix} b^{jj}_{11;11} & \cdots & b^{jj}_{11;1m_1} \\ \vdots & \ddots & \vdots \\ b^{jj}_{11;m_11} & \cdots & b^{jj}_{11;m_1m_1} \end{pmatrix},$$

where the operator $b^{ii}_{kl;st}$ is the (i,i) entry of the block $B_{kl;st}$, for $1 \le k,l \le 2$, and $1 \le s \le m_k$, and $1 \le t \le m_l$, and the operator $b^{jj}_{11;st}$ is the (j,j) entry of the block $B_{11;st}$, for $1 \le s \le m_1$, and $1 \le t \le m_1$.

Let C'_{ii} be the block diagonal matrix, where the diagonal blocks are the same as in C_{ii}. For example, the operator C'_{11} is in the form

$$C'_{11} = \begin{pmatrix} C_{11;11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{11;22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{11;33} \end{pmatrix}. \tag{2.5}$$

We observe that an operator C' in the form

$$C' = \begin{pmatrix} C'_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & C'_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & C'_{n_1n_1} \end{pmatrix} \tag{2.6}$$

is in the commutant $\{UAU^*\}'$. Let $\sigma_{\{UAU^*\}'}(C - C')$ denote the spectrum of $C - C'$ in the unital Banach algebra $\{UAU^*\}'$. Then for every operator D in the commutant

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$\{UAU^*\}'$, we obtain the following equality:

$$\sigma_{\{UAU^*\}'}\big(D(C-C')\big) = \sigma_{\{UAU^*\}'}\big((C-C')D\big) = \{0\}.$$

Therefore, the operator $C - C'$ is in the Jacobson radical of $\{UAU^*\}'$ denoted by $\operatorname{Rad}(\{UAU^*\}')$.

Let C be an idempotent in $\{UAU^*\}'$. Then C' is also an idempotent in $\{UAU^*\}'$. Note that $2C' - I$ is invertible in $\{UAU^*\}'$. Then the equality

$$(2C'-I)(C+C'-I) = I + (2C'-I)(C-C')$$

yields that the operator $C + C' - I$ is invertible in $\{UAU^*\}'$, since $C - C'$ is in $\operatorname{Rad}(\{UAU^*\}')$. Therefore, we obtain the equality $(C+C'-I)C = C'(C+C'-I)$, which means that the operators C and C' are similar in $\{UAU^*\}'$.

Next, it suffices to show that the $(1,1)$ block of C'_{11} denoted by $C_{11;11}$ is similar to an element of the standard bounded maximal abelian set of idempotents in $M_{m_1}(L^\infty(\mu))$.

We assert that for every positive integer k, there exists a positive integer l_k such that for every idempotent P in $\mathscr{L}(\mathscr{H})$ with $\|P\| \le k$, there exists an invertible operator X in $\mathscr{L}(\mathscr{H})$ with $\|X\| \le l_k$, $\|X^{-1}\| \le l_k$, and XPX^{-1} the corresponding Jordan canonical form. The idea is from considering the the following equality

$$\begin{pmatrix} I & R \\ 0 & I \end{pmatrix}\begin{pmatrix} I & R \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & -R \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for} \quad P = \begin{pmatrix} I & R \\ 0 & 0 \end{pmatrix}.$$

Let $J_m(\mathbb{C})$ be the subset of $M_m(\mathbb{C})$ consisting of matrices in Jordan normal form. Then, for a subset of $M_m(\mathbb{C}) \times J_m(\mathbb{C}) \times M_m(\mathbb{C})$ defined in the form

$$\mathscr{S}_{l_k} = \{(S, J, Y) : \|Y\| \le l_k,\ \|Y^{-1}\| \le l_k, YSY^{-1} = J\},$$

as in [1, Corollary 3], the set $\pi_1(\mathscr{S}_{l_k})$ contains every idempotent with norm less than k. Using [1, Theorem 1], the Borel map $\phi_{l_k} : \pi_1(\mathscr{S}_{l_k}) \to \pi_3(\mathscr{S}_{l_k})$ is bounded. Therefore, the equivalent class of

$$\phi_{l_{\lceil \|C_{11;11}\| \rceil}} \circ C_{11;11}(\cdot)$$

is the invertible operator $X_{11;11}$ we need in $M_{m_1}(L^\infty(\mu))$. In the same way, we obtain the invertible operators $X_{11;22}$ and $X_{11;33}$ for $C_{11;22}$ and $C_{11;33}$, respectively. Note that the diagonal entries of $B_{ii;st}$ are the same for $1 \le i \le 3$ and $1 \le s, t \le m_i$. Construct an invertible operator X in the commutant $\{UAU^*\}'$ with $X_{11;ii}$ for $1 \le i \le 3$ such that $XC'X^{-1}$ is in the standard bounded maximal abelian set of idempotents of $\{UAU^*\}'$. ∎

Lemma 2.3 *Let $\mathscr{P}$ be a bounded maximal Boolean algebra of idempotents in the commutant $\{A\}'$, where A is defined as in (1.2) and (1.3). Then there exists a finite subset $\mathscr{P}_0$ of $\mathscr{P}$ such that the equality $\mathscr{P}_0(\lambda) = \mathscr{P}(\lambda)$ holds almost everywhere on* $\operatorname{spt}(\mu)$.

Proof By Lemma 2.2, for an idempotent P in $\mathscr{P}$, there exists a unitary operator U such that the operator $C = UPU^*$ is in the form of (2.4), and C is similar to C' in $\{UAU^*\}'$, where C' is in the form of (2.6).

Let E_i be a projection in $\{UAU^*\}'$ such that

$$E_i = \begin{pmatrix} E_{i;1} & 0 & \cdots & 0 \\ 0 & E_{i;2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{i;n_1} \end{pmatrix} \text{ for } i = 1, 2, 3,$$

which is as in (2.6), where, as in the form of (2.5), we write $E_{i;1}$ as a 3×3 block matrix; the (i, i) block of $E_{i;1}$ is the identity of $M_{m_i}(L^\infty(\mu))$ and other blocks are 0, compared with C'_{11} in (2.5). Thus, the projections $E_{i;2}, \ldots, E_{i;n_1}$ can be fixed corresponding to $E_{i;1}$. Therefore, we have the equality $E_i C E_i = E_i C' E_i$.

Define a μ-measurable function r_i in the form

$$\mathrm{r}_i(P)(\lambda) \triangleq \frac{1}{n_i} \mathrm{Tr}_{n_i m_i}\big(E_i UPU^* E_i(\lambda)\big), \text{ for almost every } \lambda \text{ in } \mathrm{spt}(\mu),$$

where $\mathrm{Tr}_{n_i m_i}$ denotes the standard trace on $M_{n_i m_i}(\mathbb{C})$.

We assert that there exists an idempotent P in $\mathscr{P}$ such that the inequality $0 < \mathrm{r}_1(P)(\lambda) < m_1$ holds almost everywhere on $\mathrm{spt}(\mu)$.

If $\mathrm{r}_1(P)(\lambda) = 0$ or $\mathrm{r}_1(P)(\lambda) = m_1$ holds almost everywhere on $\mathrm{spt}(\mu)$ for every P in $\mathscr{P}$, then $\mathscr{P}$ is not bounded maximal. Therefore, there exists a subset Γ_1 of $\mathrm{spt}(\mu)$ with $\mu(\Gamma_1) > 0$ and an idempotent P_1 in $\mathscr{P}$ such that $0 < \mathrm{r}_1(P_1)(\lambda) < m_1$ holds almost everywhere on Γ_1. In the same way, we have a subset Γ_2 of $\mathrm{spt}(\mu)\backslash\Gamma_1$ with $\mu(\Gamma_2) > 0$ and an idempotent P_2 in $\mathscr{P}$ such that $0 < \mathrm{r}_1(P_2)(\lambda) < m_1$ holds almost everywhere on Γ_2. By Zorn lemma, there are sequences $\{P_i\}_{i=1}^\infty$ in $\mathscr{P}$ and $\{\Gamma_i\}_{i=1}^\infty$ with $\mu(\Gamma_i) > 0$ for every i and $\bigcup_{i=1}^\infty(\Gamma_i) = \mathrm{spt}(\mu)$ such that $0 < \mathrm{r}_1(P_i)(\lambda) < m_1$ holds almost everywhere on Γ_i. Denote by P the sum of the restrictions of P_i on Γ_i. Therefore, we obtain the above assertion.

Next, we assert that there exists an idempotent P in $\mathscr{P}$ such that the equality $\mathrm{r}_1(P)(\lambda) = 1$ everywhere on $\mathrm{spt}(\mu)$.

If P is described as in the fist assertion, then $\mathrm{spt}(\mu)$ can be divided into at most $m_1 - 1$ pairwise disjoint Borel subsets $\{\Gamma_i\}_{i=1}^{m_1-1}$ corresponding to $\mathrm{r}_1(P)$ such that the equality $\mathrm{r}_1(P)(\lambda) = i$ holds almost everywhere on Γ_i. Assume that $\mu(\Gamma_{m_1-1}) > 0$. By a similar proof of the first assertion, there exists an idempotent P_1 in $\mathscr{P}$ such that the inequality $0 < \mathrm{r}_1(P_1)(\lambda) < m_1 - 1$ holds almost everywhere on Γ_{m_1-1}. Let Q_1 denote the sum of the restriction of P_1 on Γ_{m_1-1} and the restriction of P on $\mathrm{spt}(\mu)\backslash\Gamma_{m_1-1}$. Redivide $\mathrm{spt}(\mu)$ into at most $m_1 - 2$ pairwise disjoint Borel subsets $\{\Gamma_i\}_{i=1}^{m_1-2}$ corresponding to $\mathrm{r}(Q_1)$ as above. Assume that $\mu(\Gamma_{m_1-2}) > 0$. There exists an idempotent P_2 in $\mathscr{P}$ such that the inequality $0 < \mathrm{r}_1(P_2)(\lambda) < m_1 - 2$ holds almost everywhere on Γ_{m_1-2}. Construct Q_2 with P_2 and Q_1, as above. After at most $m_1 - 2$ steps, we obtain an idempotent in $\mathscr{P}$ as required in the second assertion.

Finally, we assert that there are m_1 idempotents $\{P_i\}_{i=1}^{m_1}$ in $\mathscr{P}$ such that the equality $\mathrm{r}_1(P_i)(\lambda) = 1$ holds almost everywhere on $\mathrm{spt}(\mu)$, and $P_i P_j = 0$ for $i \neq j$.

By the second assertion, we obtain P_1 in $\mathscr{P}$ such that $\mathrm{r}_1(P_1)(\lambda) = 1$ holds almost everywhere on $\mathrm{spt}(\mu)$. Then we obtain P_2 in $(I - P_1)\mathscr{P}$ such that $\mathrm{r}_1(P_2)(\lambda) = 1$ holds almost everywhere on $\mathrm{spt}(\mu)$ by applying the first two assertions. Take these idempotents one by one, and we prove the third assertion.

By the above three assertions, we obtain $m_1 + m_2 + m_3$ idempotents $\{P_{j;i}\}_{i=1;j=1}^{3;m_i}$ in $\mathscr{P}$ such that the equality $\mathrm{r}_i(P_{j;i})(\lambda) = 1$ holds almost everywhere on $\mathrm{spt}(\mu)$, and $(P_{j;i})(P_{l;k}) = 0$ for $i \neq k$ or $j \neq l$. Construct $\mathscr{P}_0$ in the form

$$\mathscr{P}_0 \triangleq \Big\{ \sum_{i=1}^{3} \sum_{j=1}^{m_i} \alpha_{ij}(P_{j;i}) : \alpha_{ij} \in \{0,1\} \Big\}.$$

Then the equality $\mathscr{P}_0(\lambda) = \mathscr{P}(\lambda)$ holds almost everywhere on $\sigma(N_\mu)$. ■

Proof of Theorem 1.1 Let $\mathscr{P}$ be a bounded maximal Boolean algebra of idempotents in $\{A\}'$. By Lemma 2.3, there exist $m_1 + m_2 + m_3$ idempotents $\{P_{j;i}\}_{i=1;j=1}^{3;m_i}$ in $\mathscr{P}$ such that the equality $\mathrm{r}_i(P_{j;i})(\lambda) = 1$ holds almost everywhere on $\sigma(N_\mu)$, and $P_{j;i}P_{l;k} = 0$ for $i \neq k$ or $j \neq l$. By Lemma 2.2, there exists an invertible operator $X_{1;1}$ in $\{A\}'$ such that $X_{1;1}P_{1;1}X_{1;1}^{-1}$ is in the standard bounded maximal Boolean algebra of idempotents $\mathscr{E}$ in $\{A\}'$. Precisely, the idempotent $X_{1;1}P_{1;1}X_{1;1}^{-1}$ is in the form

$$X_{1;1}P_{1;1}X_{1;1}^{-1} = (I \oplus 0^{(m_1-1)}) \oplus (0^{(m_2)}) \oplus (0^{(m_3)}),$$

where I is the identity operator in $M_{n_1}(L^\infty(\mu))$. In a similar way, there exists an invertible operator $X_{2;1}$ in $\{A\}'$ such that

$$(X_{2;1}X_{1;1})P_{1;1}(X_{2;1}X_{1;1})^{-1} \quad \text{and} \quad (X_{2;1}X_{1;1})P_{2;1}(X_{2;1}X_{1;1})^{-1}$$

are both in the standard bounded maximal abelian set of idempotents in $\{A\}'$. The invertible operator $X_{2;1}$ is in the form

$$X_{2;1} = \begin{pmatrix} I & 0 \\ 0 & * \end{pmatrix},$$

where I is the identity operator in $M_{n_1}(L^\infty(\mu))$. Furthermore, there exist $m_1 + m_2 + m_3 - 3$ invertible operators $\{X_{j;i}\}_{i=1;j=1}^{3;m_i-1}$ in $\{A\}'$ such that $X(P_{j;i})X^{-1}$ is in the standard bounded maximal abelian set of idempotents in $\{A\}'$ for every i and j, where X denotes the product

$$X = X_{m_3-1;3} \cdots X_{1;3}X_{m_2-1;2} \cdots X_{1;2}X_{m_1-1;1} \cdots X_{1;1}.$$

Therefore, the set $X\mathscr{P}X^{-1}$ is the standard bounded maximal abelian set of idempotents in the commutant $\{A\}'$. Equivalently, the operator A has Property J.

Next, we compute the K_0 group of $\{A\}'$. We denote by $\mathscr{J}$ the subset of $\{A\}'$ consisting of all the operators B in $\{A\}'$ with every main diagonal entry of $B_{ii;st}$ being 0 for $1 \leq i \leq 3$ and $1 \leq s, t \leq m_i$, where B and $B_{ii;st}$ are as in (2.2) and (2.3). We claim that $\mathscr{J}$ is a closed two-sided ideal of $\{A\}'$. This can be proved by computation. By $\mathscr{B}$ we denote the subalgebra of $\{A\}'$ consisting of all the operators B in $\{A\}'$ such that every entry of $B_{ij;st}$ is 0 except ones in the main diagonal of $B_{ii;st}$, for $1 \le i, j \le 3$ and $1 \le s, t \le m_i$. By observation, we obtain the following split short exact sequence:

$$0 \longrightarrow \mathscr{J} \xrightarrow{\iota} \{A\}' \underset{\alpha}{\overset{\pi}{\rightleftarrows}} \mathscr{B} \longrightarrow 0 ,$$

where we denote by ι and α the inclusion maps and by π, from $\{A\}'$ to $\{A\}'$, the map such that for every operator B in $\{A\}'$, every entry of $\pi(B)_{ij;st}$ is 0 except ones in the main diagonal being the same as their counterparts of $B_{ii;st}$, for $1 \le i, j \le 3$ and $1 \le s, t \le m_i$. Essentially, π is the quotient map. Furthermore, we obtain

$$\mathscr{B} \cong M_{m_1}\big(L^\infty(\mu)\big) \oplus M_{m_2}\big(L^\infty(\mu)\big) \oplus M_{m_3}\big(L^\infty(\mu)\big).$$

By [3, Theorem 5.6.1] and Lemma 2.2, we obtain that $K_0(\pi)$ is an isomorphism of ordered groups. We may also achieve this by an analogue of [13, Proposition 4.3.3]. Therefore, $K_0(\{A\}') \cong K_0(\mathscr{B})$ and by computation, we obtain

$$K_0(\{A\}') \cong BB\big(\sigma(A), \mathbb{Z}^{(3)}\big). \qquad \blacksquare$$

By Theorem 1.1, we can compute the K_0 group of $\{A\}'$ if A has Property J. Next, we investigate the Property J of A by the K_0 group of $\{A\}'$. Let operators A and B be as in the form of (1.2) and (1.3):

$$A = \begin{pmatrix} f & f_{12} & \cdots & f_{1n} \\ 0 & f & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}_{n\times n} \quad \text{and} \quad B = \begin{pmatrix} f & g_{12} & \cdots & g_{1n} \\ 0 & f & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}_{n\times n},$$

where f, f_{ij}, and g_{ij} are in $L^\infty(\mu)$, for $1 \le i < j \le n$, M_f is star-cyclic. Then we have the following lemma.

Lemma 2.4 *The operators $A^{(m_1)}$ and $B^{(m_2)}$ are similar in $M_{nm_1}(L^\infty(\mu))$ $(m_1 \ge m_2)$ if and only if there exists an isomorphism θ of ordered groups from $K_0(\{T\}')$ to $BB(\sigma(T), \mathbb{Z})$ such that*

$$\theta\big([I_{\{T\}'}]\big) = 2m_1 e, \tag{2.7}$$

where $T = A^{(m_1)} \oplus B^{(m_2)}$ and $e(\lambda)$ is the generator of the semigroup $\mathbb{N}$ of $\mathbb{Z}$ for almost every λ in $\sigma(T)$.

Proof If the operators $A^{(m_1)}$ and $B^{(m_2)}$ are similar in $M_{nm_1}(L^\infty(\mu))$, then we obtain $K_0(\{T\}')$ as required by the proof of Theorem 1.1.

On the other hand, we suppose that the relations in (2.7) hold. Let P and Q be idempotents in $\{A^{(m_1)}\}'$ and $\{B^{(m_2)}\}'$ respectively such that the equalities

$$\mathrm{r}_{\{A^{(m_1)}\}'}(P)(\lambda) = 1 \quad \text{and} \quad \mathrm{r}_{\{B^{(m_2)}\}'}(Q)(\lambda) = 1$$

hold for almost every λ in $\sigma(N_\mu)$. If $P \oplus 0$ and $0 \oplus Q$ are not similar in $\{T\}'$, then θ is not an isomorphism that contradicts the assumption in (2.7), since $\theta([P \oplus 0]) = e = \theta([0 \oplus Q])$. Therefore $P \oplus 0$ and $0 \oplus Q$ are similar in $\{T\}'$. We can choose projections

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E and F similar to $P \oplus 0$ in $\{T\}'$ such that $T|_{\mathrm{ran}E} = A$ and $T|_{\mathrm{ran}\,F} = B$. Thus, $A \oplus 0$ is similar to $0 \oplus B$ in $\{T\}'$. The equality $\theta([I_{\{T\}'}]) = 2m_1 e_1$ yields that $m_1 + m_2 = 2m_1$. Hence, $m_1 = m_2$ and $A^{(m_1)}$ is similar to $B^{(m_2)}$. ∎

Proof of Theorem 1.2 If the operators

$$A = \oplus_{i=1}^{3} A_{n_i}^{(m_i)} \quad \text{and} \quad B = \oplus_{j=1}^{3} B_{l_j}^{(k_j)}$$

are similar, then we can obtain an isomorphism θ of ordered groups and the group $K_0(\{T\}')$ as required in the theorem by a routine computation.

On the other hand, suppose that there exists an isomorphism θ of ordered groups from $K_0(\{T\}')$ to $\{f : \sigma(T) \to \mathbb{Z}^{(3)}, f \text{ is bounded Borel}\}$ such that

$$\theta([I_{\{T\}'}]) = 2m_1 e_1 + 2m_2 e_2 + 2m_3 e_3.$$

In the commutant $\{T\}'$, there exist 3 projections $\{E_i\}_{i=1}^3$ and 3 projections $\{F_j\}_{j=1}^3$ such that

(a) $T|_{\mathrm{ran}E_i} = A_{n_i}$ and $T|_{\mathrm{ran}F_j} = B_{l_j}$;
(b) $E_i E_j = F_i F_j = 0$ and $E_i F_j = 0$ for $i \neq j$;
(c) the equalities $\mathrm{r}_i(E_i)(\lambda) = 1$ and $\mathrm{r}_j(F_j)(\lambda) = 1$ hold for almost every λ in $\sigma(N_\mu)$ and $1 \le i, j \le 3$.

For the equivalence classes $\{[E_i]\}_{i=1}^3$, if F_i is not similar to E_i in $\{T\}'$ for some i, then for $K_0(\{T\}')$, there exists a λ in the $\sigma(N_\mu)$ such that the set $\{E_j(\lambda)\}_{j=1}^3 \cup \{F_i(\lambda)\}$ generates $\mathbb{Z}^{(4)}$, which is a contradiction, since λ cannot be removed from $\sigma(N_\mu)$. Therefore, F_i is similar to E_i in $\{T\}'$ for $1 \le i \le 3$. The coefficient of e_i in $\theta([I_{\{T\}'}])$ is $m_i + k_i = 2m_i$ for $1 \le i \le 3$. Therefore, the equality $m_i = k_i$ holds for $1 \le i \le 3$. Thus, we obtain that the operators A and B are similar to each other. ∎

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