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A NOTE ON STRONGLY CLOSED 2-SUBGROUPS

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The purpose of this note is to present a proof of the following:

PROPOSITION. *Let T be a Sylow 2-subgroup of the finite group G and let $1 < S \le T$ with S strongly closed in T with respect to G. Let $\Gamma = \langle C_G(s), N_G(\Omega_1(S)) \mid s \in \mathscr{I}(S)\rangle$. Suppose that $\Gamma \le H < G$ for some subgroup H of G. Then $H \cap \langle S^G\rangle$ is strongly embedded in $\langle S^G\rangle$ and $S = T \cap \langle S^G\rangle$ or $S = \Omega_1(T \cap \langle S^G\rangle)$. Also if $m(S) = 1$, then $S = T \cap \langle S^G\rangle$.*

This result has several applications. As an illustration, we utilize the above Proposition to give an alternate proof of [1, Theorem 2(2)] (one of the many fundamental results of [1]).

The proof of our Proposition utilizes some of the basic results on strongly closed 2-subgroups in [2] and [3].

All groups in this article are finite and our notation is standard.

Section 1. A proof of the Proposition.

Throughout this section we assume that S, T, Γ, H and G satisfy the hypotheses of the Proposition. Set $X = \langle S^G\rangle$. Clearly the following two Lemmas imply the Proposition.

LEMMA 1.1. *Suppose that $m(S) = 1$. Then*

(1) $S \in Syl_2(X)$; *and*
(2) $H \cap X$ *is strongly embedded in* X.

Proof. For (1), we proceed by induction on $|G|$. Thus we may assume that $0(G) = 1$. Clearly [3, Corollary B3] implies that we may assume that S is quaternion, of order 8. Also $\Omega_1(S) \trianglelefteq G$ by Glauberman's Z^*-Theorem. Set $\bar{G} = G/\Omega_1(S)$. Then $\bar{S}$ is strongly closed in $\bar{T}$ with respect to $\bar{G}$ by [3, Lemma 2.2(a)], $\bar{S} \cong E_4$, $\bar{T} \in \mathrm{Syl}_2(\bar{G})$, $0(\bar{G}) = 1$ and $\bar{S} \ne \langle \bar{S}^{\bar{G}}\rangle = \bar{X}$. Also [2, Theorem A] implies that $\bar{X} \cong \mathrm{PSU}(3, 4)$ or $\bar{X} \cong \mathrm{PSL}(2, q)$ for some odd prime power $q > 3$ with $q \equiv \pm 3 \pmod 8$. Since the Schur multiplier of PSU(3, 4) is of odd order, it follows that $|X|_2 = |S|$ and (1) holds. Next observe that $\Gamma = C_G(\Omega_1(S)) \le H < G$. Hence $G = X\Gamma = XH$ and $C_X(\Omega_1(S)) \le H \cap X < X$. Thus (2) holds and the proof of the lemma is complete.

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LEMMA 1.2. *Suppose that* $m(S)>1$. *Then*
(1) $H\cap X$ *is strongly embedded in* X; *and*
(2) $S=T\cap X$ *or* $S=\Omega_1(T\cap X)$.

Proof. We proceed by induction on $|G|$. Since $0(G)\le\Gamma\le H$, we may assume that $0(G)=1$. Also $G=XN_G(S)=XH$, $X=\langle S^G\rangle=\langle S^X\rangle$ and $\langle C_X(s), N_X(\Omega_1(S)) \mid s\in\mathscr{I}(S)\rangle\le H\cap X<X$. Thus we may assume that $G=X$. Let $Y=\langle\Omega_1(S)^G\rangle$. Then $G=YN_G(\Omega_1(S))=YH$ and hence $Y\not\le H$. Thus $Y\not\le\langle C_G(s)\mid s\in\mathscr{I}(S)\rangle$ and Y has a strongly embedded subgroup by [3, Lemma 2.7]. Consequently Y is a simple Bender group and $\Omega_1(S)=\Omega_1(T\cap Y)\le S\cap Y\le T\cap Y\in \mathrm{Syl}_2(Y)$. If $G=Y$, then we are clearly done. Thus we may assume that $Y<G$, $S\not\le Y$ and $S\cap Y=\Omega_1(S)=\Omega_1(T\cap Y)$ or $S\cap Y=T\cap Y$. Set $L=YS$ and $C=C_L(Y)$. Thus $C=0_2(L)\le(T\cap Y)S$. Suppose that $C\ne 1$. Choose $c\in\mathscr{I}(C)$ and write $c=t^{-1}s$ where $t\in T\cap Y$ and $s\in S$. Since $Y\not\le H$, we have $c\notin S$ and $S\cap Y=\Omega_1(S)=\Omega_1(T\cap Y)$. Since $c\notin Y$, we have $s\notin Y$ and $|s|\ge 4$. As $s=ct=tc$, we also have $|t|\ge 4$. In particular, $T\cap Y=\langle t^{-1}t^n \mid n\in N_Y(T\cap Y)\rangle$. But if $n\in N_Y(T\cap Y)$, then $s^n=ct^n\in T$, $s^n\in S$ and $t^{-1}t^n=s^{-1}s^n\in S\cap Y=\Omega_1(T\cap Y)$. Since this is impossible, we have $C=1$. Let $s\in S-Y$ be such that $s^2\in Y$ and set $L_1=Y\langle s\rangle$. Then $T_1=T\cap L_1=(T\cap Y)\langle s\rangle\in\mathrm{Syl}_2(L_1)$ and T_1 contains an involution τ such that $T_1=(T\cap Y)\langle\tau\rangle$ and τ acts like a "field automorphism" on Y. Thus $\mathscr{I}(T_1)>\mathscr{I}(S)=\Omega_1(S)^{\#}=\Omega_1(T\cap Y)^{\#}$, $S\cap Y=\Omega_1(S)=\Omega_1(T\cap Y)<T\cap Y$, $Y\cong \mathrm{PSU}(3,2^n)$ for some integer $n\ge 2$ and $\bar T_1=T_1/\Omega_1(T\cap Y)$ is not abelian. However $\overline{S\cap T_1}=\overline{(S\cap Y)\langle s\rangle=\Omega_1(S)\langle s\rangle=\Omega_1(T\cap Y)\langle s\rangle}\trianglelefteq T_1$ and hence $\overline{S\cap T_1}\le Z(\bar T_1)$. But $\bar T_1=\overline{(T\cap Y)}\,\overline{(S\cap T_1)}$ and $\overline{T\cap Y}$ is abelian, so that $\bar T_1$ is abelian. This contradiction establishes Lemma 2.2.

Section 2. An Application of the Proposition.

Recall that a subgroup Q of a finite group G is said to be tightly embedded in G if $|Q|$ is even and if $|Q\cap Q^g|$ is odd for every $g\in G-N_G(Q)$. The following corollary is [1, Theorem 2(2)].

COROLLARY. *Let* Q *be a tightly embedded subgroup of the finite group* G *and let* $H=N_G(Q)$. *Suppose That* $|Q^g\cap H|$ *is odd for all* $g\in G-H$. *Then either* $G=H$ *or* $H<G$ *and* $H\cap\langle Q^G\rangle$ *is strongly embedded in* $\langle Q^G\rangle$.

Proof. Let $S\in\mathrm{Syl}_2(Q)$ and let $T\in\mathrm{Syl}_2(H)$ with $S\le T$. The hypotheses imply that $T\in\mathrm{Syl}_2(G)$ and that S is strongly closed in T with respect to G. Suppose that $H<G$. Clearly $\Gamma_1(S,G)=\langle N_G(U)\mid 1\ne U\le S\rangle\le H$. Set $X=\langle S^G\rangle$ and $Y=\langle Q^G\rangle$. Then $H\cap X$ is strongly embedded in X and $S=T\cap X$ or $S=\Omega_1(T\cap X)$ by the Proposition. *Also* $G=XN_G(S)=XH=YH$, $Y=\langle Q^G\rangle=\langle Q^X\rangle=XQ$, $|Y/X|$ *is odd and* $H\cap Y<Y$. *But* $T\cap X=T\cap Y\in\mathrm{Syl}_2(Y)$ *and* $\Gamma_1(T\cap X, Y)\le$

$H \cap Y < Y$ since $\Omega_1(T \cap X) = \Omega_1(S)$. Consequently $H \cap Y$ is strongly embedded in Y and the proof is complete.

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