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"The equal internal bisectors theorem, 1840-1940. . . . Many solutions or none?" A centenary account

By JAMES A. M'BRIDE, B.Sc., B.A., F.R.S.E.

This paper contains (i) a short history of the geometrical theorem proposed in 1840 by Prof. Lehmus of Berlin to Jacob Steiner—" If BJY, CJZ are equal bisectors of the base angles of a triangle ABC, then AB equals AC," (ii) a selection of some half-dozen solutions from the 50 or 60 that have been given, (iii) some discussion of the logical points raised, and (iv) a list of references to the extensive literature of the subject.

Incidentally two widely current legends will be cleared up, (i) that J. J. Sylvester in 1852 proved that a solution was impossible, (ii) that nevertheless a valid proof, the first, was given in 1874 by a Girl of the Golden West, a contemporary of Bret Harte and Mark Twain. Both these stories are " much exaggerated."

Steiner, like most mathematicians, found the theorem " very difficult," and Sylvester remarks, referring to J. C. Adams—" If report may be believed, intellects capable of extending the bounds of the planetary system, and lighting up new regions of the universe with the torch of analysis, have been baffled by the difficulties of the elementary problem under consideration." (*Phil. Mag.*, 1853.)

Steiner gave a fine solution, both for external and internal bisection, and found an external case where the theorem is not true. This occurs when BY and CZ meet AC produced and BA produced respectively.

JACOB STEINER. (*Crelle's Journal,* 1844.) Also, later (1882), in his *Gesammelte Werke.*

His proof is given (see fig. 1) for the case of two equal bisectors of the angles made by BC with AB and AC produced (through B and C). Call the halves of these angles x and y. Let x be greater than y, and therefore CE than BE, where E is an Ex-Centre. Make $EH = EB$, $EK = EZ$; then $HC = KY$. The triangles BEZ, HEK are congruent; $ZHK = x$. Since x is greater than y, HK produced meets CY produced in L, at an angle equal to $x - y$ or twice HBC, *i.e.*, $2z$. Let $KYD = d$.

In the triangle YBA, $A + d = x$ (an ext. angle); therefore x is greater than d; therefore CY is greater than BC. Make $CD = BC$. The triangles CHB, CHD are congruent, and $CDH = z$. Thus, in the triangle HDL, the exterior angle z is greater than the interior angle $2z$, an absurdity. That is, x is not unequal to y. The angle B is equal to C, and (*Euc.* I. 6) $AB = AC$.

Meanwhile in 1842 the *Nouvelles Annales de Mathematiques* of Paris proposed it for solution. Two proofs were given immediately, one by Rougevin, Collège Louis le Grand, one by Grout de St Paer, Collège de Versailles. Rougevin's is given here, as the first actually printed, and as having suggested Sylvester's *Test Theorem.*

ROUGEVIN (*Nouvelles Annales de Math.,* 1842, p. 48). (See fig. 3).

Triangles BAY, CAZ have equal circum-circles. Place them in one circle on one side of a chord RS, equal to BY or CZ. The point A will take up two positions P, Q on the arc, AJ will take the two positions PU, QV. Produced, these met in M, mid point of the lower arc. The diameter $MTOH$ bisects RS at right angles. Rougevin says (angles) HMP, HMQ are equal. For, if HMQ is less than HMP, MP is less than MQ and MU is greater than MV. By difference, PU is less than QV, its equal (given). Similarly if HMP is less than HMQ, PU is greater than QV. Thus HMP, since $PU = QV$, equals HMQ, the figure is symmetrical, and $PS = QR$, or $AC = AB$.

From 1844 to 1852, about a dozen proofs appeared in *Grunert's Archiv der Mathematik*; in 1850-1 the theorem reached England. It was set in a Cambridge Examination Paper, with a new and disturbing element introduced. The proof was to be *direct, i.e.,* without *reductio ad absurdum.*

This came to the notice of J. J. Sylvester, then writing on Equations and their Roots, and in the *Philosophical Magazine* for Oct., 1852, he published two indirect proofs, one by B. L. Smith.

He gave a trigonometrical discussion of the general case, where the ratio of ABC to YBC (angles) is any number n, positive or negative. (See fig. 2.)

Thus, let $ABC = n.YBC = 2ny$, and $ACB = n.ZCB = 2nx$. Then, BY being equal to CZ,
$BC \sin 2ny = CZ \sin (2ny + 2x)$; $BC \sin 2nx = BY \sin (2nx + 2y)$. Therefore

$$\sin 2nx. \sin (2ny + 2x) = \sin 2ny. \sin (2ny + 2x),$$

reducing to

$$\tan (n-1)(x-y) \tan n (x+y) = \tan (n+1)(x+y) \tan n (x-y).$$

If B is not less than C, then for any value of $x-y$,

$$\frac{\tan (n-1)(x-y)}{\tan n (x-y)} = \frac{\tan (n+1)(x+y)}{\tan n (x+y)}.$$

Then (i) Keeping to internal division, n positive and greater than 1, $2n(x+y)$ is less than two right angles, as is $2n(x-y)$.

The left hand ratio is a positive proper fraction, and (a) if $(n+1)(x+y)$ is less than a right angle, the right hand ratio is positive and improper, (b) if $(n+1)(x+y)$ is greater than a right angle, said ratio is negative.

Thus the equation cannot be true unless $x = y$.

(ii) By writing $n = -m$, he shows that $x = y$ if n is negative and greater than 1.

So except for n between +1 and −1, the theorem is true.

For n between +1 and −1 not necessarily true.

Thus for $n = \frac{1}{2}$, equation is $\tan \frac{3}{2}(x+y) + \tan \frac{1}{2}(x+y) = 0$, and given $x-y$, this equation is consistent with $x+y = 90°$.

Sylvester says that for $n = 2$, geometrical proof must be indirect.

He surmises (does not prove) that " when a theorem depends on the necessary non-existence of real roots (within prescribed limits) of the analytical equation expressing the conditions, no other form of proof then *reductio ad absurdum* is possible. If this is erroneous, it can be refuted in particular instances." But, he says, all proofs of the Bisectors Theorem have been hitherto

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indirect. He invited mathematicians to give a direct proof of the theorem (see fig. 3)—" If from M, mid point of an arc RS, two chords of the circle MUP, MVQ are drawn, crossing the chord RS in U and V, and if $UP = VQ$, then $MU = MV$."

T. K. ABBOTT. (*Phil. Mag.*, 1853.)

This mathematician took up the challenge of Sylvester and gave the following proof of the latter's Test Theorem. (See Rougevin's second fig. 3).

Bisect PU, QV in E, F. Then $QV.VM = SV.VR$. Add to each side the square on MV or square on VT + square on MT. Then $QM.MV$ = square on ST + square on MT = square on MS. Similarly, $PM.MU$ = square on RM. Therefore $QM.MV$ = $PM.MU$; add to each square on EU or on FV; then square on ME = square on MF, and $ME = MF$. Thus $MU = MV$, and $MP = MQ$.

I believe this proof is indirect. It is supported by Euc. III. 35, which uses the Theorem of Pythagoras, and by Euc. II. 6, involving the existence and construction of a square. This depends on Euc. I. 29, indirectly proved. J. J. S. said nothing. The principle had been laid down definitely by him, that " all lemmas and supporting propositions must be provable directly." He did not admit that Rougevin had a right, if he claimed his proof as direct (which he did not) to assume that two triangles with equal vertical angles standing on the same side of the same base, have the same circum-circle. Few solvers have paid any attention to these just principles.

Rev. Dr Adamson replied to Sylvester in three articles, admitting his Mathematics, but denying his logical deductions. He indicated what he thought might be a direct proof, but did not go fully into details.

He was followed by

AUGUSTUS DE MORGAN ON DIRECT AND INDIRECT PROOFS. (*Lond. Edin. and Dub. Phil. Mag.*—Dec. 1852.)

Proofs of the Proposition—Every A is B.

1. The Positive Proposition is :—Every A is B. The Direct Positive Proof takes any A, and shows that it is B.

2. The Contrapositive Proposition is :—Every not − B is not −A. The Direct Contrapositive Proof takes any not − B, and shows that it is not − A.

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3. The Indirect Positive Proof attacks the Positive Contradiction :—Some *A's* are not - *B's*, and taking an *A*, assumed to be not - *B*, shows that an absurdity is involved.
4. The Indirect Contrapositive Proof attacks the Contrapositive Contradiction :—Some not - *B's* are *A's*, and taking a not - *B*, assumed to be *A*, shows that an absurdity is involved.

Any proof of the Contrapositive of a Proposition is a proof of the Positive, for their content is the same. Euclid, not writing for expert logicians, but for persons who through Geometry desired to become logicians, used *reductio ad absurdum* to pass from Contrapositive to Positive—*quite unnecessarily.*

De Morgan means that the Contrapositive and the Positive being identical logically, no further discussion is necessary if you have proved the Contrapositive. It has been lately pointed out to me by Mr J. A. Fullarton, ex-Headmaster, Ballymena Academy, that in Nixon's *Euclid Revised* and elsewhere it is proved that the bisector of the smaller of the two base angles of a triangle is longer than that of the other. This is the Contrapositive of our Theorem. But of course it has to be proved, and in doing so Nixon and others use supporting propositions only provable indirectly.

Here may be noted that
PROFESSOR L. SUSAN STEBBING) *A Modern Introduction to Logic*, Chap. V.) observes that Contraposition is a form of Immediate Inference. " All *S* is *P* " is identical with " All non-*P* is non-*S*." " All organic substances contain Carbon " means exactly the same thing as " All substances not containing Carbon are inorganic," except in the mode of expression. Proof of one statement is Proof of the other.

On this question of the identical content of the positive and the contrapositive form of a proposition Bertrand Russell (*Principles of Mathematics,* Vol. I., p. 17) may be quoted—

" *p* implies *q* " IMPLIES " not-*q* implies not-*p*."

This Russell regards as one of the ten fundamental principles of Mathematics and Formal Logic. It is unprovable, that is, cannot be reduced to anything simpler. But it is recognised by the mind as true.

" The Contrapositive is different *in form,*" says De Morgan. " The Positive and the Contrapositive are identical, *except in the mode of expression,*" says Miss Stebbing.

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Notwithstanding all this, Sylvestrian purists could not accept proof of the Contrapositive Form as a *direct* proof of the Theorem.

As so many solvers (*e.g.* Casey) chose this method, it is desirable to give an example—Euc. I. 19 (fig. 11).

"In a triangle ABC if B is greater than C, b is greater than c." For if not, b is (i) equal to, or (ii) less than c. If (i) is true $B = C$ (I. 5); if (ii) is true, B is less than C (I. 18). Either is a contradiction of the hypothesis, therefore both are false, that is b is greater than c. This (Euclid's) proof is indirect.

Now take a proof by contraposition.

(i) If $b = c$, $B = C$ (I. 5). By contraposition, if B is not equal to C, b is not equal to c.

(ii) If x is the greater of b and c, the greater of B and C is opposite to it (Euc. I. 18), that is B is opposite to x, or x is b. The contents of the second proof is the same as that of the first; the supporting propositions are the same; the first is indirect; so also is the other.

I shall hold that proofs in the Contrapositive form (open or hidden) are indirect. It is worth while bringing in

J. P. HENNESSY (later Sir J. P. H.).

He asked (*Phil Mag.* 1852), "What changes in Euclid, Book I., would make all the proofs direct?"

He finally decided that Props. XIV., XXVII., XXIX. would never be proved directly, and that XXXIX. was doubtful.

But Hennessy placed five Props. after XXXII., instead of before it, relying probably on either the Playfair-Hamilton rotation proof of XXXII., or Legendre's. Both of these claimed to prove that "In any triangle ABC, the sum of the angles is equal to two right angles," independent of Euclid's Theory of Parallels. (See Casey's *Euclid*, pp. 299-302.)

JOHN CASEY (Adaptation of Hamilton's Quaternion Proof), fig. 7.

Let AB be (i) rotated round B till it lies along BE, then translated till B comes to C.

(ii) rotated round C till it lies along CF, then translated till B comes to A.

(iii) rotated round A till it lies along AD, then translated till B comes to B. AB is now back in its original position.

It has described 4 right angles therefore $A+B+C = (6-4)$ right angles = 2 right angles.

In the above rotation proof are introduced postulates (like the independence of translation and rotation) which, I believe, ultimately depend on the Theory of Parallels. Further, the proof would apply, word for word, to a spherical triangle, where the theorem is false. But there is a more definite and cogent reason why we must reject both of these proofs.

If we assume (i) that there is a valid proof independent of the Theory of Parallels that in every triangle ABC, $A+B+C$ = two right angles, we can prove the *Parallel Postulate.*

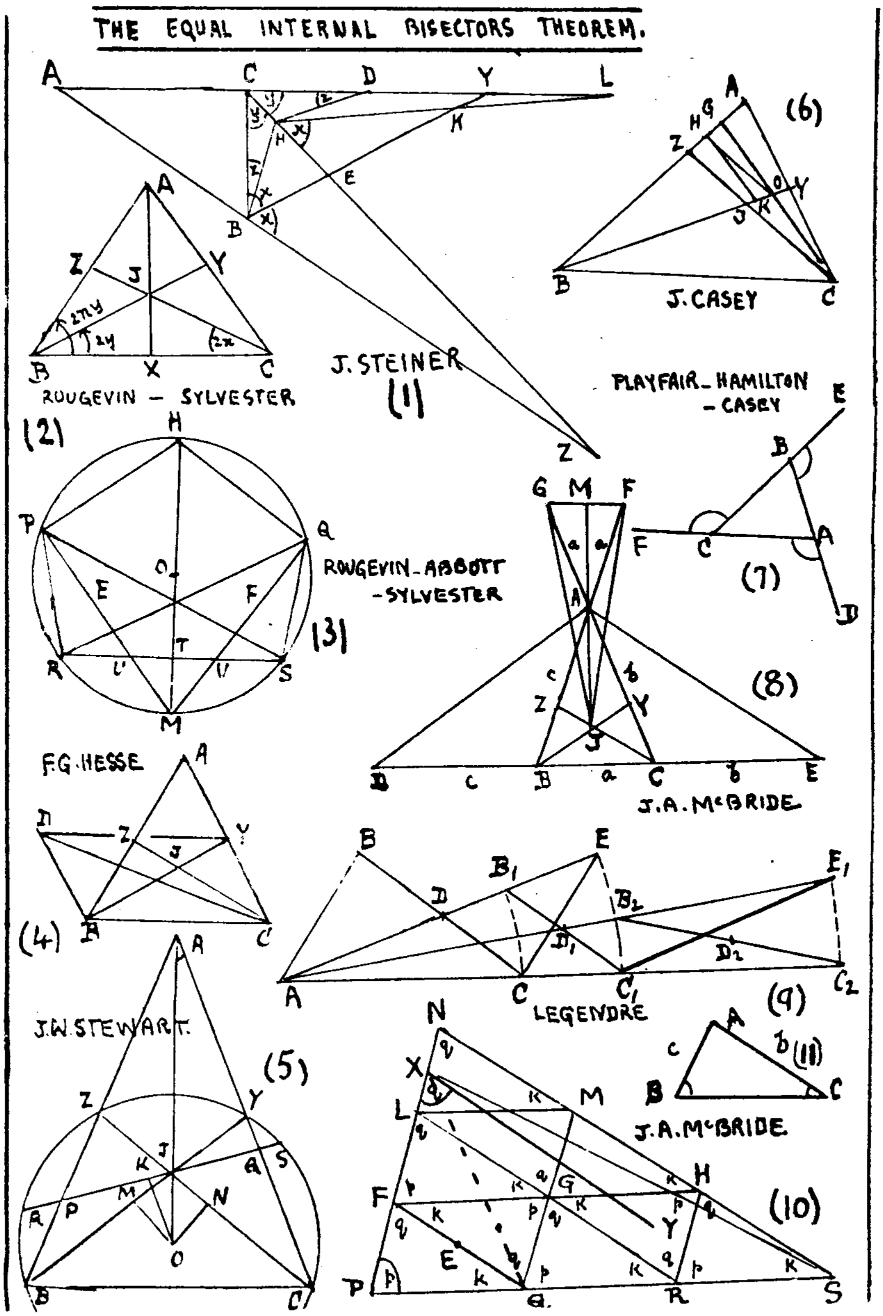
PROOF. (See fig. 10.)

Given that XP is a transversal meeting two straight lines XY, PQ, so that the sum $YXP + SPX$ is less than two right angles, we are to prove that XY produced meets PQ produced.

Let $YXP = q$, $SPX = p$, $180° - (p+q) = k$, a finite angle. Take any point Q on PS, make $PQE = k$, and join QX. Then if XY does not meet the segment PQ, q is greater than PXQ, and therefore k is less than PQX, since $PXQ + PQX = q + k$. Thus QE meets PX, say in F. Set off distances each equal to PF along PX, let LN be that which contains X. It is the nth, as PF is the first. Set off also equal distances along PS, beginning with PQ, ending with the nth, RS. On QR, etc., erect triangles congruent with PFQ, viz. QGR . . . RHS. Join FG, GH. Then since $k+p+q = 180°$, $FQG = GRH = q$ and the triangles FQG, GRH are congruent with QFP.

Continue this process as far as the final triangle LNM. All the angles in the lattice are p, q, or k. It is seen by Euc. I. 14 that $SHMN$, FGH, etc., are straight lines. Join SX. Then PXS is greater than PNS, therefore than PXY. That is XY is within PXS and must meet the line segment PS. This is the *Parallel Postulate,* universally held to be unprovable. Thus the assumption that Euc. I. 32 (angle sum of a triangle) is provable apart from the Theory of Parallels is false. In particular, the proof by rotation is fallacious.

Consequently, we reject as indirect all proofs of the Bisectors theorem that depend on Euc. I. 32, which is proved indirectly by using Euc. I. 29. We must reject also proofs using the Theory of Proportion, or depending on the Theorem of Pythagoras, which



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ultimately depends on Euc. I. 29. (Transversal across two parallel lines makes interior angle-sum on one side equal to two right angles.) We have already rejected, as indirect, proofs of the Contrapositive, like Casey's, Steiner's, etc.

From 1852 to 1874 one finds in England alone about a dozen proofs—in the *Lady's and Gentleman's Diary* (devoted to Poetry and Mathematics), and in other journals. They are (1) frankly indirect, or (2) proofs of the Contrapositive, or (3) " direct " proofs depending on indirectly proved lemmas.

In 1874 arrived what is perhaps the best known solution. It was sent by Miss Christine Chart, Oakland, California, to Rev. Dr N. M. Ferrers, Master of Gonville and Caius, who forwarded it with a covering letter to the *Phil. Mag.*

It had been made out in 1842, not by Miss Chart, but by her friend, Mr F. G. Hesse. Sylvester is said to have accepted it. I can hardly think he did, for it depends on Euc. I. 32, which depends, as shown above, on Euc. I. 29.

F. G. HESSE. (*Phil. Mag.* 1874.)

On BY erect a triangle with $YD = BC$, and $BD = BZ$. Then

(i) (Angle) $DYB = BCZ = \frac{1}{2}C$ (Euc. I. 8); $DBY = BZC = A + \frac{1}{2}C$; and $YDB = CBZ$.

(ii) (Angle) $BJC = JZB + JBZ = DBY + \frac{1}{2}B = DBC$ (Euc. I. 32).

(iii) (Angle) $BJC = 90 + \frac{1}{2}A$, therefore obtuse, and $DYC = DBC$ (obtuse).

(iv) The triangles DYC, CBD have one pair of angles equal, the sides about a second pair equal severally, and the third pair of angles both acute (DCY, CDB), therefore they are congruent. Thus $YC = BD = BZ$.

(v) ZCB and YBC are now congruent triangles, $ABC = ACB$, and $AB = AC$.

The best Contrapositive proof is attributed to Casey by a writer in the *Mathematical Gazette.* It avoids Euc. I. 29 and 47, the Blue Symplegades on which many a solver's ship foundered.

H. G. Forder noted that it is independent of the Parallel Postulate. I have joined two points in fig 6, and added some words.

JOHN CASEY. (Proof re-printed in the *Math. Gazette,* 1933). (Fig. 6.)

Take AB greater than AC, or C greater than B (Euc. I. 18). Make $ZCG = \frac{B}{2}$. Then BCG is greater than GBC, and BG than GC (Euc. I. 19). Make $BH = GC$ and $BHK = BGC$, K being on BY. Then BHK, CGZ are congruent triangles (I. 4), therefore $CZ = BK$, less than BY. Proof as follows—(Join HO. Then BHO is greater than BGO, therefore than BHK; therefore K lies between B and O, therefore between B and Y). Thus, if AB is greater than AC, BY is greater than CZ. This is the contrapositive of "If $BY = CZ$, $AB = AC$."

I give my own proof because (i) it is on different lines from the others, (ii) has not hitherto been printed (see fig. 8). I do not claim it as direct since it uses Proportion, depending ultimately on I. 29.

J. A. M'BRIDE (1939).

Produce (i) BC both ways, making $BD = AB$, $CE = AC$, (ii) produce BA, CA through A, making $AF = AG = a$. Join AD, AE, JF, JG and produce JA to M. Then JBC, ADE are equiangular triangles ($\frac{1}{2}B$, $\frac{1}{2}C$, $90 + \frac{1}{2}A$) and

$$\frac{BY}{a} = \frac{AD}{a+c}\,;\ \frac{CZ}{a} = \frac{AE}{a+b}\,;\ BY = CZ\,;\ \frac{BF}{CG} = \frac{a+c}{a+b} = \frac{AD}{AE} = \frac{JB}{JC}$$

JAM bisects FG at right angles; $JF = JG$; (*angle*) $JGA = JFA$. The triangles BJF, CJG have (i) angles at F and G equal, (ii) $\frac{JB}{FB} = \frac{JC}{CG}$, (iii) BJF, CJG both obtuse; therefore (Euc. VI. 7) they are similar. Also $JF = JG$, therefore they are congruent, and $a + b = a + c$. Therefore $b = c$.

Many proofs have appeared in the last 60 years (see list at end). I am greatly indebted for valuable details to Mr J. W. Stewart, of Sunderland, formerly of Dumfries and Ayr. Also to an article by the late Dr J. S. Mackay in the *Proc. Edin. Math. Society.*

I give Mr Stewart's fine proof, which is a concealed contrapositive. That is now no failing.

J. W. STEWART. (*Phil. Mag.* 1913.)

Lemma. If through a point J on the bisector of an angle BAC a transversal BJY is drawn to the lines containing it, then (i) one and only one other transversal can be drawn through J equal to BY, (ii) the segments of BY, CZ made by J are severally equal, (iii) the figure is symmetrical (axis AJ).

(i) Make (angle) $CJA = BJA$. Then $BJ = CJ$, $JY = JZ$, $BY = CZ$.

(ii) Draw the circle passing through B, Z, Y, C, and let a transversal PQ, within the angle BJZ, meet the circle in R and S. Draw perpendiculars OM, ON, OK to the three transversals. Then OK is greater than OM; RS is less than BY, still less is PQ than BY. Similarly, any transversal within YJZ is greater than BY.

(iii) It is easily seen that, omitting RS, the figure is symmetrical.

The Theorem. If BY bisects ABC, CZ is the bisector of ACB, and is uniquely equal to BY. The rest follows.

When this account was being written, in 1940, I wrote to Mr Stewart telling him that I was going to claim his as the only direct proof, provided he could give me a direct proof of Euc. I. 47. He at once sent the fine demonstration of the Theorem of Pythagoras attributed to Leonardo da Vinci. But, alas! a simple case of Euc. I. 14 is required in this, and the last hope was gone!

FINAL CONCLUSIONS.

(i) More than 60 distinct proofs of the Theorem have been given, many frankly indirect.

(ii) Some of the best are proofs of the Contrapositive, i.e., indirect.

(iii) If it is held, as I hold, that Euc. I. 14, Euc. I. 29, Euc. I. 32, and the Theorem of Pythagoras have no direct proof, then the Bisectors Theorem has not been proved directly, nor is it likely to be.

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THE PSEUDO-ISOSCELES TRIANGLE.

(Two external Bisectors equal, triangle not isosceles).

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The inextensible string

By A. G. Walker.

An object to which we were all introduced at an early stage in mechanics is the inextensible string. This appears frequently without causing much trouble, but there is one type of problem which, in my opinion, stands apart from the rest, and which certainly caused me a lot of trouble. Such a problem is when impulses are given to a system which includes an inextensible string, as, for example, a system consisting of two rigid parts joined by a string. If an impulse is applied to one of these parts, an impulsive tension (T) may be set up in the string, which, in turn, gives an impulse to the other part. One new quantity, T, has appeared, and one equation in addition to the ordinary dynamical equations is thus required before the problem of finding the change in motion of the system can be solved. It is at this stage that opinions can differ, for this extra equation depends essentially upon what concept of an inextensible string is being adopted, and there is more than one. The usual procedure is to employ a " geometrical equation " based upon the argument that the two ends must have equal component velocities in the line of the string as long as the string is taut. This seems almost obvious when described in such general terms, and is followed by such eminent writers as Routh[1] and Loney,[2] amongst others. I suggest,

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[1] See for example the worked exercise (170) on p. 149 of his *Elementary Rigid Dynamics* (1882).

[2] Loney devotes two sections to methods involving the geometrical equation in *Dynamics of a Particle and of Rigid Bodies* (1919), p. 180.