



Auslander’s theorem for dihedral actions on preprojective algebras of type A

Jacob Barahona Kamsvaag and Jason Gaddis

Abstract. Given an algebra R and G a finite group of automorphisms of R , there is a natural map $\eta_{R,G} : R\#G \rightarrow \text{End}_{R^G} R$, called the Auslander map. A theorem of Auslander shows that $\eta_{R,G}$ is an isomorphism when $R = \mathbb{C}[V]$ and G is a finite group acting linearly and without reflections on the finite-dimensional vector space V . The work of Mori–Ueyama and Bao–He–Zhang has encouraged the study of this theorem in the context of Artin–Schelter regular algebras. We initiate a study of Auslander’s result in the setting of nonconnected graded Calabi–Yau algebras. When R is a preprojective algebra of type A and G is a finite subgroup of D_n acting on R by automorphism, our main result shows that $\eta_{R,G}$ is an isomorphism if and only if G does not contain all of the reflections through a vertex.

1 Introduction

In [18], Qin, Wang, and Zhang initiated a study of the McKay correspondence for nonconnected \mathbb{N} -graded algebras in (global) dimension 2. An important component to this study is Auslander’s theorem. This project is an attempt to study this result in the context of preprojective algebras of type A .

Let V be a finite-dimensional vector space, and let G be a finite group acting linearly on $R = \mathbb{C}[V]$. The *Auslander map* $\eta_{R,G} : R\#G \rightarrow \text{End}_{R^G} R$ is defined as

$$(1) \quad a\#g \mapsto \begin{pmatrix} R & \rightarrow & R \\ b & \mapsto & ag(b) \end{pmatrix}.$$

Auslander’s theorem, then, states that $\eta_{R,G}$ is an isomorphism if and only if G acts without reflections. That is, G is a small group [1].

The Auslander map may be defined for any algebra R , commutative or noncommutative, and any subgroup G of $\text{Aut } R$. However, in general, the map need not be injective or surjective. Bao, He, and Zhang introduced the pertinency invariant as a tool to study Auslander’s theorem in the noncommutative setting [3, 4]. If δ is any dimension function on an algebra R , G is a finite group acting on R , and $f_G = \sum_{g \in G} 1\#g \in R\#G$, the *pertinency* of the G -action is defined as

$$p(R, G) = \delta(R) - \delta(R\#G/(f_G)).$$

Received by the editors November 1, 2021; revised June 3, 2022, accepted June 4, 2022.

Published online on Cambridge Core June 9, 2022.

AMS subject classification: 16W22, 16S35.

Keywords: Auslander’s theorem, preprojective algebras, Calabi–Yau algebras.



Throughout this work, we take δ to be the Gelfand–Kirillov dimension. Under suitable hypotheses, the Auslander map is an isomorphism for the pair (R, G) if $\text{p}(R, G) \geq 2$.

Kirkman, Moore, Won, and the second author proved that the Auslander map is an isomorphism for $(\mathbb{C}_{-1}[x_1, \dots, x_n], G)$, where G is any subgroup of \mathcal{S}_n acting linearly as permutations of the generators (i.e., $\sigma(x_i) = x_{\sigma(i)}$) [11]. Chan, Young, and Zhang computed the explicit pertinency value for many cyclic subgroups of \mathcal{S}_n in search of noncommutative cyclic isolated singularities [8]. Crawford proved that the Auslander map is an isomorphism for any pair (R, G) where R is a two-dimensional Artin–Schelter regular algebra and G is a small group, in which “smallness” is generalized to the noncommutative setting using the homological determinant [10]. Further study of Auslander's theorem and applications of the pertinency invariant can be found in [7, 9, 12, 14, 17].

A natural generalization of the above is then to study the Auslander map in the context of nonconnected algebras. That is, to replace the condition of Artin–Schelter regularity with the twisted Calabi–Yau condition (see [13]). By a result of Bocklandt, if R is Calabi–Yau of dimension 2, then R is the preprojective algebra of a non-Dynkin quiver [6, Theorem 3.2]. More generally, Reyes and Rogalski have classified graded twisted Calabi–Yau algebras of global dimension 2 that are generated in degree 1 [19].

We review the definition of preprojective algebras, as well as relevant ring-theoretic and homological definitions necessary to our study, in Section 2. Our main focus, however, will be the preprojective algebra $\Pi_{\widetilde{A}_{n-1}}$, where \widetilde{A}_{n-1} is the extended Dynkin diagram of type A . In Theorem 2.6, we establish some cases where the Auslander map is an isomorphism for cyclic subgroups of scalar automorphisms on $\Pi_{\widetilde{A}_{n-1}}$.

In Section 3, we study dihedral group actions on $R = \Pi_{\widetilde{A}_{n-1}}$. Each graph automorphism of the underlying graph Γ_n of \widetilde{A}_{n-1} extends to a graded algebra automorphism of R . From this, we obtain a subgroup of $\text{Aut}_{\text{gr}}(R)$ which is isomorphic to the graph automorphism group of Γ_n , namely the dihedral group on n vertices D_n . We identify this group with D_n itself. Our main result classifies the subgroups G of D_n for which $\eta_{R,G}$ is an isomorphism.

Theorem 1.1 *Let $R = \Pi_{\widetilde{A}_{n-1}}$ and G a subgroup of D_n . The Auslander map $\eta_{R,G}$ is an isomorphism if and only if there exists a reflection $\tau \in D_n$ that fixes a vertex and $\tau \notin G$.*

Proof Sufficiency is proved in Theorem 3.2. Necessity follows from Theorems 3.7 and 3.13. ■

A natural next question would be to study actions on preprojective algebras associated with other extended Dynkin types. However, as these exhibit significantly fewer symmetries, we do not consider them here.

2 Preprojective algebras

Throughout, we assume that \mathbb{k} is an algebraically closed field of characteristic zero. All algebras are assumed to be \mathbb{k} -algebras unless otherwise noted.

An algebra R is called $(\mathbb{N}\text{-})$ graded if there exists a collection of \mathbb{k} -vector subspaces $\{R_n\}_{n=0}^\infty$ of R such that $R = \bigoplus_{n \in \mathbb{N}} R_n$ and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{N}$. We say that R is *locally finite* if $\dim_{\mathbb{k}}(R_n) < \infty$ for all $n \in \mathbb{N}$. If $R_0 = \mathbb{k}$, we say that R is *connected*. A \mathbb{k} -algebra automorphism ϕ of R is called *graded* if $\phi(R_n) = R_n$ for all $n \in \mathbb{N}$. We denote the group of \mathbb{N} -graded automorphisms of R by $\text{Aut}_{\text{gr}}(R)$.

Let R be an algebra, and let G be a finite subgroup of $\text{Aut}(R)$. Let $R\#G$ denote the set of formal sums

$$R\#G = \left\{ \sum a_g \# g : a_g \in R, g \in G \right\}.$$

Define a multiplication on $R\#G$ by

$$(r_1 \# g_1)(r_2 \# g_2) = r_1 g_1(r_2) \# g_1 g_2, \quad r_i \# g_i \in R\#G,$$

extended linearly. We call $R\#G$ the *skew group ring* $R\#G$. Note that, under this definition, the group ring $\mathbb{k}G$ is $\mathbb{k}\#G$ with trivial G action.

A *quiver* Q is a tuple (Q_0, Q_1, s, t) consisting of a set of *vertices* $Q_0 = \{e_0, \dots, e_n\}$, a set of *arrows* Q_1 , and *source* and *target* functions $s, t : Q_1 \rightarrow Q_0$. For any arrow α , we call the vertex $s(\alpha)$ the *source* of α and the vertex $t(\alpha)$ the *target* of α . We say Q is *finite* if Q_0 and Q_1 are finite, and *Schurian* if, given any two vertices i and j , there is at most one arrow with source i and target j . The *adjacency matrix* of a quiver Q is the matrix M_Q in which $(M_Q)_{ij}$ denotes the number of arrows $i \rightarrow j$.

A *path of length ℓ with source e_i and target e_j* is a word of the form $\alpha_1 \alpha_2 \dots \alpha_\ell$ where $\alpha_k \in Q_1$ for each $k = 1, \dots, \ell$, and $s(\alpha_1) = e_i, t(\alpha_\ell) = e_j$, and $t(\alpha_k) = s(\alpha_{k+1})$ for each $k = 1, \dots, \ell - 1$. If $p = \alpha_1 \dots \alpha_\ell$, we extend the source and target functions to the set of all paths by the rule $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_\ell)$. We denote the set of paths of length ℓ by Q_ℓ , and treat the vertices e_i as *trivial paths* of length 0 with $s(e_i) = t(e_i) = e_i$.

Let Q be a quiver. The *path algebra* of a quiver Q over the field \mathbb{k} , denoted $\mathbb{k}Q$, is the algebra with \mathbb{k} -basis the set of paths $\bigcup_{\ell=0}^\infty Q_\ell$ and multiplication defined by concatenation. That is, given paths $p = \alpha_1 \dots \alpha_\ell$ and $q = \beta_1 \dots \beta_k, pq = \alpha_1 \dots \alpha_\ell \beta_1 \dots \beta_k$ if $t(p) = s(q)$ and $pq = 0$ otherwise. For any $e \in Q_0$ and any path p , we also have the following:

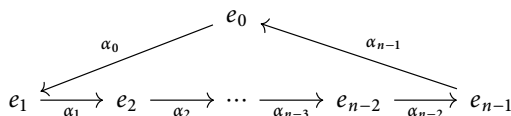
$$ep = \begin{cases} p, & s(p) = e, \\ 0, & s(p) \neq e, \end{cases} \quad pe = \begin{cases} p, & t(p) = e, \\ 0, & t(p) \neq e. \end{cases}$$

Note that a path algebra $\mathbb{k}Q$ on a finite quiver is graded by $\{(\mathbb{k}Q)_\ell\}_{\ell=0}^\infty$ where $(\mathbb{k}Q)_\ell = \text{Span}_{\mathbb{k}}(Q_\ell)$, and there are only finitely many paths of any given length, so $\mathbb{k}Q$ is locally finite.

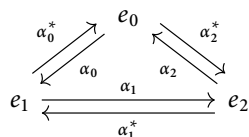
The *double* of a quiver Q , denoted \overline{Q} , is defined by setting $\overline{Q}_0 = Q_0$, and for every arrow $\alpha \in Q_1$ with $s(\alpha) = e_i$ and $t(\alpha) = e_j$, we add an arrow α^* with $s(\alpha^*) = e_j$ and $t(\alpha^*) = e_i$. We call Q_1 the set of *nonstar arrows* and $Q_1^* := \overline{Q}_1 \setminus Q_1$ the set of *star arrows*.

Definition 2.1 Let Q be a finite quiver, and set $\Omega = \sum_{\alpha \in Q_1} \alpha \alpha^* - \alpha^* \alpha \in \overline{Q}$. The *preprojective algebra* associated with Q , denoted Π_Q , is the quotient $\mathbb{k}\overline{Q}/(\Omega)$. We call Ω the *preprojective relation*.

The type A extended Dynkin quivers \widetilde{A}_{n-1} , $n \geq 3$, are given by



We focus on preprojective algebras corresponding to this type. If $Q = \widetilde{A}_{n-1}$, then \overline{Q} can be characterized as follows. The vertex set \overline{Q}_0 of \overline{Q} is $\{e_0, \dots, e_{n-1}\}$, and there is exactly one nonstar arrow, α_i , from e_i to e_{i+1} and one star arrow, α_i^* , from e_{i+1} to e_i for each $i = 0, \dots, n - 1$ where the index is taken mod n . That is, \overline{Q} is Schurian. For example, the double of \widetilde{A}_2 is presented below:



We note that \widetilde{A}_1 is also defined. However, as its double is not Schurian, it does not fit into the theory we have developed.

The preprojective algebra $\Pi_{\widetilde{A}_{n-1}}$ has nice ring-theoretic properties, as discussed in the next proposition. Before this, we review some of the definitions. For others, such as *graded injectively smooth* and the *generalized Gorenstein* condition, we refer to [20, 22], respectively.

Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be a locally finite graded algebra. The *Gelfand-Kirillov (GK) dimension* of R is defined as

$$\text{GKdim}(R) = \overline{\lim}_{n \rightarrow \infty} \log_n \left(\sum_{i \leq n} \dim_{\mathbb{k}}(R_i) \right).$$

Now, let δ be a dimension function on R . For example, we may have $\delta = \text{GKdim}$ or $\delta = \text{Kdim}$, the Krull dimension. The ring R is δ -Cohen-Macaulay (δ -CM) if $\delta(R)$ is finite, and for every nontrivial finitely generated right R -module M , $j(M) + \delta(M) = \delta(R)$, where $j(M) = \min\{i : \text{Ext}_R^i(M, R) \neq 0\}$ denotes the *grade* of M .

The *(total) Hilbert series* of R is the formal power series

$$H_R^{tot} = \sum_{k=0}^{\infty} \dim_{\mathbb{k}}(R_k) t^k.$$

If $R_0 = \mathbb{k}^n$ with primitive idempotents $\{e_0, \dots, e_{n-1}\}$, then the *matrix-valued Hilbert series* is the matrix with entries

$$(H_R)_{ij} = \sum_{k=0}^{\infty} \dim_{\mathbb{k}}(e_{i+1} R_k e_{j+1}) t^k.$$

Proposition 2.1 *Let $R = \Pi_{\widetilde{A}_{n-1}}$. Then R is a locally finite graded Noetherian algebra of global and GK dimension 2. Moreover, R is GKdim-CM and $H_R = (1 - M_{\overline{Q}}t)^{-2}$.*

Proof That R is locally finite graded follows because Q (and hence \overline{Q}) is finite. By [2, Theorem 6.5], R is a Noetherian polynomial identity ring. Furthermore, R satisfies the generalized Gorenstein condition [21, Proposition 2.11], which implies that R is right graded injectively smooth. Then, by [20, Theorem 1.3], R is Kdim-CM , which implies that R is GKdim-CM by [20, Lemma 4.3]. The statement on the Hilbert series follows from [6]. ■

The next lemma gives a canonical form for paths in the preprojective algebra $\Pi_{\widetilde{A_{n-1}}}$.

Lemma 2.2 (Structure Lemma) *Let $Q = \widetilde{A_{n-1}}$, and let $R = \Pi_Q$. Let p be a (non-constant) monomial in R . Then there exist nonstar arrows $\beta_1, \dots, \beta_\ell$ and star arrows $\gamma_1, \dots, \gamma_m$ such that $p = \beta_1 \cdots \beta_\ell \gamma_1 \cdots \gamma_m$.*

Proof Given any idempotent $e_i \in Q_0$, $e_i \Omega e_i \in (\Omega)$ and so the following relations hold in $\Pi_{\widetilde{A_{n-1}}}$:

$$(2) \quad 0 = e_i \Omega e_i = \alpha_i \alpha_i^* - \alpha_{i-1}^* \alpha_{i-1},$$

where the indices are taken mod n .

Given any star arrow α_i^* , the only nonstar arrow β such that $\alpha_i^* \beta \neq 0$ is α_i . Consequently, whenever a star arrow is followed by a nonstar arrow, we can use (2) to obtain a nonstar arrow followed by a star arrow instead. By repeated use of (2), we have

$$\begin{aligned} \alpha_j^* \alpha_j \alpha_{j+1} \cdots \alpha_{j+k} &= \alpha_{j+1} \alpha_{j+1}^* \alpha_{j+1} \alpha_{j+2} \cdots \alpha_{j+k} \\ &= \alpha_{j+1} \alpha_{j+2} \alpha_{j+2}^* \alpha_{j+2} \alpha_{j+3} \cdots \alpha_{j+k} \\ &\vdots \\ &= \alpha_{j+1} \cdots \alpha_{j+k} \alpha_{j+k}^* \alpha_{j+k} \\ &= \alpha_{j+1} \cdots \alpha_{j+k+1} \alpha_{j+k+1}^*. \end{aligned}$$

By induction on the number of star arrows, it follows that we can push all star arrows to the right. ■

The invariant theory of preprojective algebras was studied by Weispfenning, with particular interest toward a version of the Shephard–Todd–Chevalley theorem [21, 22]. Our interest is in a version of Auslander’s theorem for group actions on the projective algebra $R = \Pi_{\widetilde{A_{n-1}}}$. Particularly relevant to the present investigation is the following theorem due to Bao, He, and Zhang.

Theorem 2.3 [4, Theorem 3.5] *Let R be a Noetherian locally finite graded algebra, and let G be a finite subgroup of $\text{Aut}_{\text{gr}}(R)$. Assume further that R is GKdim-CM of global dimension 2 with $\text{GKdim } R \geq 2$. Then $\eta_{R,G}$ is a graded algebra isomorphism if and only if $\text{p}(R, G) \geq 2$.*

Let R' be the image of R in the composition

$$R \hookrightarrow R \# G \rightarrow (R \# G) / (f_G).$$

We call R' the identity component of $(R\#G)/(f_G)$, which we can associate with R . By [4, Lemma 5.2], $\text{GKdim}(R') = \text{GKdim}((R\#G)/(f_G))$.

Theorem 2.4 *Let $n \geq 3$, let $R = \Pi_{\widetilde{A}_{n-1}}$, let G be a finite subgroup of $\text{Aut}_{\text{gr}}(R)$, and let R' be the identity component of $(R\#G)/(f_G)$. Then $\eta_{R,G}$ is a graded algebra isomorphism if and only if $\dim_{\mathbb{k}}(R') < \infty$.*

Proof By Proposition 2.1, R satisfies the conditions of Theorem 2.3. That is, $\eta_{R,G}$ is an isomorphism if and only if $\text{p}(R, G) \geq 2$. Since $\text{GKdim}(R) = 2$, then $\text{p}(R, G) \geq 2$ if and only if $\text{GKdim}((R\#G)/(f_G)) = 0$, which is equivalent to $(R\#G)/(f_G)$ being a finite-dimensional \mathbb{k} -vector space. ■

Since R is locally finite, $\mathbb{k} \cup \bigcup_{\ell < m} R_\ell$ is finite-dimensional for all $m \in \mathbb{N}$. Consequently, if there exists $m \in \mathbb{N}$ such that every path of length at least m is in (f_G) , then R' has a complete set of coset representatives in $\mathbb{k} \cup \bigcup_{\ell < m} R_m$. That is, $\dim_{\mathbb{k}}(R') < \infty$.

We conclude this section with a discussion of graded automorphisms of $\Pi_{\widetilde{A}_{n-1}}$.

Let Q be a quiver. A quiver automorphism $\sigma = (\sigma_0, \sigma_1)$ of Q is a pair of bijections $\sigma_0 : Q_0 \rightarrow Q_0$ and $\sigma_1 : Q_1 \rightarrow Q_1$ such that for all $\alpha \in Q_1$, $s(\sigma_1(\alpha)) = \sigma_0(s(\alpha))$ and $t(\sigma_1(\alpha)) = \sigma_0(t(\alpha))$. Every quiver automorphism extends to a graded automorphism of $\mathbb{k}Q$ which, by an abuse of notation, we again denote by σ (see [16]).

Conversely, if σ is a graded automorphism of $\Pi_{\widetilde{A}_{n-1}}$, then σ_0 permutes the set Q_0 . Since Q is Schurian, then for any $\alpha \in Q_1$, $\sigma(\alpha)$ is necessarily a nonzero scalar multiple of the unique arrow from $\sigma_0(s(\alpha))$ to $\sigma_0(t(\alpha))$. First, we will consider automorphisms which fix the vertices of \widetilde{A}_{n-1} . In Section 3, we study automorphisms corresponding to dihedral automorphisms on \widetilde{A}_{n-1} .

For the remainder of this section, let $Q = \widetilde{A}_{n-1}$ and $R = \Pi_{\widetilde{A}_{n-1}}$. Let

$$F = \{ \sigma \in \text{Aut}_{\text{gr}}(R) : |\sigma| < \infty \text{ and } \sigma(e_i) = e_i \text{ for all } i = 0, \dots, n-1 \}.$$

Fixed subrings of R under automorphisms in F were studied by Weispfenning [21, 22].

Let $\sigma \in F$. By the above discussion, $\sigma(\alpha) \in \text{Span}_{\mathbb{k}}\{\alpha\}$ for each $\alpha \in Q_1$. Thus, there exists $\xi_i, \xi_i^* \in \mathbb{k}^\times$ such that $\sigma(\alpha_i) = \xi_i \alpha_i$ and $\sigma(\alpha_i^*) = \xi_i^* \alpha_i^*$ for each $i = 0, \dots, n-1$. Since σ is of finite order, each ξ_i, ξ_i^* must be a root of unity. On the other hand, $\sigma(\Omega) \in \text{Span}_{\mathbb{k}}(\Omega)$ and so $\sigma(\Omega) = \omega \Omega$ for some $\omega \in \mathbb{k}^\times$. It is not difficult to show using the preprojective relation that $\xi_i \xi_i^* = \omega$ for all i . The value ω in this case is the homological determinant of the σ -action on R [15, 22]. We will consider cases in which $\omega = 1$.

Our primary tool for studying the Auslander map for cyclic subgroups of F is the following result of He and Zhang, which we have rephrased for our purpose.

Lemma 2.5 [14, Lemma 3.4] *Let $\sigma \in F$, let $G = \langle \sigma \rangle$, and let $|\sigma| = m$. Assume that there are elements $a_0, \dots, a_{m-1} \in R$ such that $\sigma(a_i) = \zeta a_i$ for $i = 0, \dots, m-1$, where ζ is an m th primitive root of unity. Then $a_0 a_1 \cdots a_{m-1} \# 1 \in (f_G)$.*

We now apply Lemma 2.5 to establish an isomorphism of the Auslander map for certain scalar automorphisms.

Theorem 2.6 *Let $\sigma \in F$ with $m = |\sigma|$, $1 < m < \infty$, and let $G = \langle \sigma \rangle$. As above, write $\sigma(\alpha_i) = \xi_i \alpha_i$ and $\sigma(\alpha_i^*) = \xi_i^* \alpha_i^*$, $i = 0, \dots, n - 1$, with $\xi_1 \xi_1^* = 1$. In each of the following cases, $\eta_{R,G}$ is an isomorphism.*

- (1) *There is some primitive m th root of unity ζ such that $\xi_i = \zeta$ for $i = 0, \dots, n - 1$.*
- (2) *There is some primitive m th root of unity ζ such that $\xi_0 \xi_1 \dots \xi_{n-1} = \zeta$.*
- (3) *For all $i, j = 0, \dots, n - 1$ with $i \neq j$, we have $\gcd(|\xi_i|, |\xi_j|) = 1$.*

Proof First, suppose that there exists a pure nonstar path q of length ℓ such that $q \# 1 \in (f_G)$. Then let p be a path containing at least 2ℓ nonstar arrows. Pushing all nonstar arrows to the left using the Structure Lemma (Lemma 2.2), it follows that p contains q , so $p \# 1 \in (f_G)$. The same argument applies if q, p are pure star paths. Hence, if (f_G) contains both a pure nonstar path and a pure star path of length ℓ , then $\dim_{\mathbb{K}}(R') < \infty$ and $\eta_{R,G}$ is an isomorphism by Theorem 2.4. Thus, in each case, we will attempt to produce such paths.

(1) Assuming $\xi_i = \zeta$ as in the hypothesis, take $a_i = \alpha_i$ and apply Lemma 2.5. Taking indices mod n , it follows that $a_0 \dots a_{m-1} \in (f_G)$ is a pure nonstar path. One similarly obtains a pure star path. Hence, $\eta_{R,G}$ is an isomorphism by the above argument.

(2) Let p be any pure nonstar path of length n , and let q be any pure star path of length n . Since p contains each nonstar arrow exactly once, $\sigma(p) = \zeta p$. No power of p is zero and so we apply Lemma 2.5 with $a_i = p$ to obtain $p^m \# 1 \in (f_G)$. Similarly, we obtain $q^m \# 1 \in (f_G)$. Hence, $\eta_{R,G}$ is an isomorphism.

(3) The order of σ is determined by its image on R_1 , which in turn is determined by its image on $\alpha_0, \dots, \alpha_{n-1}$. That is, $|\sigma| = k$, where k is the least positive integer such that $\sigma^k(\alpha_i) = \alpha_i$ for all $i = 0, \dots, n - 1$. Since the orders of the scalars ξ_i are relatively prime, then we have $|\zeta| = |\xi_0| \cdot |\xi_1| \dots |\xi_{n-1}| = |\sigma|$. The result now follows from (2). ■

3 Dihedral actions on $\Pi_{\widetilde{A_{n-1}}}$

In this section, we establish our main theorem regarding the Auslander map for dihedral actions on $\Pi_{\widetilde{A_{n-1}}}$.

Let σ be a quiver automorphism of a Schurian quiver Q . As discussed above, if $\alpha \in Q_1$, then $\sigma(\alpha)$ is a scalar multiple of the unique arrow from $\sigma_0(s(\alpha))$ to $\sigma_0(t(\alpha))$. Throughout this section, we assume that scalar is 1.

Proposition 3.1 *Let Q be a quiver such that \overline{Q} is Schurian, let $R = \Pi_Q$, and let $\sigma \in \text{Aut}_{\text{gr}}(\mathbb{K}\overline{Q})$ be induced from a quiver automorphism of \overline{Q} as above. If one of the following holds, then $\sigma \in \text{Aut}_{\text{gr}}(R)$:*

- (1) *σ is star-preserving: $\sigma(Q_1) = Q_1$ and $\sigma(Q_1^*) = Q_1^*$;*
- (2) *σ is star-inverting: $\sigma(Q_1) = Q_1^*$ and $\sigma(Q_1^*) = Q_1$.*

Proof Given a nonstar arrow α with source e_i and target e_j , α^* is the unique arrow with source e_j and target e_i . In particular, this holds for $\sigma(\alpha)$, so in case 1, we must have $\sigma(\alpha^*) = \sigma(\alpha)^*$. Then $\sigma(\alpha\alpha^*) = \sigma(\alpha)\sigma(\alpha)^*$ and $\sigma(\alpha^*\alpha) = \sigma(\alpha)^*\sigma(\alpha)$, so σ

permutes the summands of both $\sum_{\alpha \in Q_1} \alpha \alpha^*$ and $\sum_{\alpha \in Q_1} \alpha^* \alpha$. That is,

$$\sigma(\Omega) = \sigma\left(\sum_{\alpha \in Q_1} \alpha \alpha^*\right) - \sigma\left(\sum_{\alpha \in Q_1} \alpha^* \alpha\right) = \sum \alpha \alpha^* - \sum \alpha^* \alpha = \Omega.$$

The argument is similar in case 2 except we obtain $\sigma(\Omega) = -\Omega$, so again it preserves the ideal (Ω) . ■

For the remainder of this section, let $Q = \widetilde{A_{n-1}}$ and $R = \Pi_{\widetilde{A_{n-1}}}$. We will show that there is a group of quiver automorphisms of \overline{Q} that is isomorphic to the dihedral group on n vertices. We first identify two quiver automorphisms of \overline{Q} which extend to automorphisms of R .

- (1) Define $\rho : \overline{Q} \rightarrow \overline{Q}$ by $\rho(e_i) = e_{i+1}$, where the index is taken mod n . Then $\rho(\alpha_i)$ is the unique arrow with source $\rho(s(\alpha_i)) = e_{i+1}$ and target $\rho(t(\alpha_i)) = e_{i+2}$, which is α_{i+1} . Consequently, $\rho(\alpha_i^*) = \rho(\alpha_i)^* = \alpha_{i+1}^*$. Thus, ρ is a star-preserving automorphism of \overline{Q} , and has order n .
- (2) Define $r : \overline{Q} \rightarrow \overline{Q}$ by $r(e_i) = e_{n-i}$. Since $r(s(\alpha_i)) = e_{n-i}$ and $r(t(\alpha_i)) = e_{n-i-1}$, we must have $r(\alpha_i) = \alpha_{n-i-1}^*$ and $r(\alpha_i^*) = \alpha_{n-i-1}$. Thus, r is a star-inverting automorphism of \overline{Q} order 2.

By Proposition 3.1, $G = \langle \rho, r \rangle$ extends to a subgroup of $\text{Aut}_{\text{gr}}(R)$ where $R = \Pi_Q$. It is clear that $G \cong D_n$, and so we identify D_n with the group G acting on R by graded automorphisms.

Theorem 3.2 *Let G be a subgroup of D_n . If there exists a reflection $\tau \in D_n$ that fixes a vertex and $\tau \notin G$, then $\dim_{\mathbb{k}}(R') < \infty$.*

Proof Let τ be the reflection that fixes e_i and suppose $\tau \notin G$. Since τ is the only nontrivial element of D_n that fixes e_i , we have $e_i g(e_i) = 0$ for all $g \in G \setminus \{1\}$. Consequently, $e_i(f_G)e_i = e_i \# 1$. Let p be a path of length at least $2n + 1$, so p contains at least $n + 1$ nonstar arrows or at least $n + 1$ star arrows. Without loss of generality, suppose p has at least $n + 1$ nonstar arrows. By the Structure Lemma (Lemma 2.2), we may push all star arrows to the right, so that

$$p = \alpha_j \alpha_{j+1} \cdots \alpha_{j+n-1} \alpha_{j+n} p'$$

for some path p' and some $j = 0, \dots, n - 1$ where the indices are taken mod n . Then, for some $0 \leq k \leq n - 1$, $i = j + k + 1 \pmod n$, so

$$p = (\alpha_j \cdots \alpha_{j+k}) e_i (\alpha_{j+k+1} \cdots \alpha_j p').$$

Hence, $p \# 1 \in (f_G)$ and so $q \# 1 \in (f_G)$ for all paths q of length at least $2m + 1$. Thus, $\dim_{\mathbb{k}}(R') < \infty$. ■

Theorem 3.2 shows that the Auslander map is an isomorphism for the pair (R, G) so long as G is missing a reflection which fixes some vertex. In case n is odd, this includes all proper subgroups of D_n . However, in the case that n is even, there is one additional subgroup, W_n . It remains to show that the Auslander map fails to be an isomorphism in the case of W_n and the full dihedral group D_n .

3.1 The D_n case

For $x \in R$, we denote by the $\mathcal{O}(x)$ the orbit of x under D_n . We begin by describing the orbits of R under the D_n action so as to find a \mathbb{k} -basis of R^{D_n} .

Recall that for $k \geq 0$, we let Q_k (resp. Q_k^*) denote the set of paths of length k containing only nonstar (resp. star) arrows, and $Q_0 = Q_0^*$ is the set of trivial paths. Furthermore, let $Q_\ell Q_k^*$ denote the set of paths containing exactly ℓ nonstar arrows followed by k star arrows. Then, in the double quiver, we have

$$\overline{Q_\ell} = \bigcup_{\substack{i+j=\ell \\ i,j \geq 0}} Q_i Q_j^*.$$

Clearly, $\overline{Q_\ell}$ is a generating set for the graded piece R_ℓ of R . Finally, for $\ell \geq k \geq 0$, set $B_{\ell,k} = Q_\ell Q_k^* \cup Q_k Q_\ell^*$.

Lemma 3.3 For any $p \in B_{\ell,k}$, $\mathcal{O}(p) = B_{\ell,k}$.

Proof By Structure Lemma (Lemma 2.2), a path p is uniquely determined by its source along with the number of nonstar and star arrows it contains. Consequently, each $p \in Q_\ell Q_k^*$ is uniquely determined by its source, as is each $q \in Q_k Q_\ell^*$. Thus, for each $i = 0, \dots, n-1$, let p_i (resp. q_i) denote the unique path in $Q_\ell Q_k^*$ (resp. $Q_k Q_\ell^*$) with source e_i . Then $B_{\ell,k} = \{p_0, \dots, p_{n-1}, q_0, \dots, q_{n-1}\}$.

Let $x \in B_{\ell,k}$ and $y \in \mathcal{O}(x)$, so $y = g(x)$ for some $g \in D_n$. If g is a rotation, then g bijectively maps Q_1 to Q_1 and Q_1^* to Q_1^* . Consequently, $g(x)$ has the same number of nonstar arrows as x , and the same number of star arrows as x . That is, if $x \in Q_\ell Q_k^*$, then $y \in Q_\ell Q_k^*$. Thus, $y \in B_{\ell,k}$. If g is a reflection, then g bijectively maps Q_1 to Q_1^* and Q_1^* to Q_1 . Hence, $g(x)$ has the same number of nonstar arrows as x has star arrows, and the same number of star arrows as x has nonstar arrows. That is, if $x \in Q_\ell Q_k^*$, then $y \in Q_k Q_\ell^*$. Once again, $y \in B_{\ell,k}$, so $\mathcal{O}(x) \subseteq B_{\ell,k}$.

We have $|\mathcal{O}(x)| = |D_n|/|\text{stab}(x)|$, and $g \in \text{stab}(x)$ only if g fixes the source of x . Hence, g is the identity or the unique reflection r fixing $s(x)$. Now, if $\ell \neq k$, then $Q_\ell Q_k^* \neq Q_k Q_\ell^*$ so $|B_{\ell,k}| = 2n$, and r inverts the number of star and nonstar arrows, so $r(x) \neq x$. Consequently, $\text{stab}(x) = \{1\}$, so $|\mathcal{O}(x)| = |G| = 2n$. Thus, $\mathcal{O}(x) = B_{\ell,k}$. If, on the other hand, $\ell = k$, then $B_{\ell,k} = B_{\ell,\ell} = Q_\ell Q_\ell^*$, and $r(x) = x$, so $\text{stab}(x) = \{1, r\}$. In this case, we have $|B_{\ell,k}| = n = |D_n|/|\text{stab}(x)| = |\mathcal{O}(x)|$, so $\mathcal{O}(x) = B_{\ell,k}$. ■

Set

$$\mathbb{O}(\ell, k) = \sum_{p \in B_{\ell,k}} p.$$

By Lemma 3.3, these are exactly the orbit sums of homogeneous elements in R , and hence form a \mathbb{k} -basis for R^{D_n} . This shows that R^{D_n} has Hilbert series

$$\begin{aligned} H_{R^{D_n}}(t) &= 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + \dots \\ &= (1+t) \sum_{k=0}^{\infty} (k+1)(t^2)^k = \frac{1}{(1-t)(1-t^2)}. \end{aligned}$$

Lemma 3.4 The orbit sums $\mathbb{O}(\ell, k)$ satisfy the following relations:

$$(3) \quad \mathbb{O}(1, 0)\mathbb{O}(\ell, k) = \begin{cases} \mathbb{O}(\ell + 1, k) + \mathbb{O}(\ell, k + 1), & \text{if } \ell > k, \\ \mathbb{O}(\ell + 1, k), & \text{if } \ell = k, \end{cases}$$

$$(4) \quad \mathbb{O}(1, 1)^m = \mathbb{O}(m, m).$$

Proof To prove (3), we suppose that $\ell > k$ and then

$$\begin{aligned} & \mathbb{O}(1, 0)\mathbb{O}(\ell, k) \\ &= \left(\sum_{i=0}^{n-1} \alpha_i + \alpha_i^* \right) \left(\sum_{i=0}^{n-1} \alpha_i \cdots \alpha_{i+\ell-1} a_{i+\ell-1}^* \cdots \alpha_{i+\ell-k}^* + \alpha_i \cdots \alpha_{i+k-1} \alpha_{i+k-1}^* \cdots \alpha_{i+k-\ell}^* \right) \\ &= \sum_{i=0}^{n-1} \alpha_i \cdots \alpha_{i+\ell} a_{i+\ell}^* \cdots \alpha_{i+\ell-k+1}^* + \alpha_i \cdots \alpha_{i+k} \alpha_{i+k}^* \cdots \alpha_{i+k-(\ell-1)}^* \\ &\quad + \alpha_i^* \alpha_i \cdots \alpha_{i+\ell-1} a_{i+\ell-1}^* \cdots \alpha_{i+\ell-k}^* + \alpha_i^* \alpha_i \cdots \alpha_{i+k-1} \alpha_{i+k-1}^* \cdots \alpha_{i+k-\ell}^* \\ &= \sum_{i=0}^{n-1} \alpha_i \cdots \alpha_{i+\ell} a_{i+\ell}^* \cdots \alpha_{i+(\ell+1)-k}^* + \alpha_i \cdots \alpha_{i+k} \alpha_{i+k}^* \cdots \alpha_{i+(k+1)-\ell}^* \\ &\quad + \alpha_i^* \alpha_i \cdots \alpha_{i+\ell-1} a_{i+\ell-1}^* \cdots \alpha_{i+(\ell+1)-k}^* + \alpha_i^* \alpha_i \cdots \alpha_{i+k-1} \alpha_{i+k-1}^* \cdots \alpha_{i+(k+1)-\ell}^* \\ &= \mathbb{O}(\ell + 1, k) + \mathbb{O}(\ell, k + 1). \end{aligned}$$

On the other hand, if $\ell = k$, then

$$\begin{aligned} \mathbb{O}(1, 0)\mathbb{O}(\ell, \ell) &= \left(\sum_{i=0}^{n-1} \alpha_i + \alpha_i^* \right) \left(\sum_{i=0}^{n-1} \alpha_i \cdots \alpha_{i+\ell-1} a_{i+\ell-1}^* \cdots \alpha_i^* \right) \\ &= \sum_{i=0}^{n-1} \alpha_i \cdots \alpha_{i+\ell} a_{i+\ell}^* \cdots \alpha_{i+1}^* + \alpha_i^* \alpha_i \cdots \alpha_{i+\ell-1} a_{i+\ell-1}^* \cdots \alpha_i^* \\ &= \sum_{i=0}^{n-1} \alpha_i \cdots \alpha_{i+\ell} a_{i+\ell}^* \cdots \alpha_{i+1}^* + \alpha_{i+1} \cdots \alpha_{i+\ell} \alpha_{i+\ell}^* \cdots \alpha_i^* \\ &= \mathbb{O}(\ell + 1, \ell). \end{aligned}$$

For (4), the result is obvious if $m = 1$. Suppose that it holds for some m , then

$$\begin{aligned} \mathbb{O}(1, 1)^{m+1} &= \mathbb{O}(1, 1)\mathbb{O}(m, m) \\ &= \left(\sum_{i=0}^{n-1} \alpha_i \alpha_i^* \right) \left(\sum_{i=0}^{n-1} \alpha_i \cdots \alpha_{i+m-1} \alpha_{i+m-1}^* \alpha_{i+m-2}^* \cdots \alpha_i^* \right) \\ &= \sum_{i=0}^{n-1} \alpha_i \alpha_i^* \alpha_i \cdots \alpha_{i+m-1} \alpha_{i+m-1}^* \cdots \alpha_i^* \\ &= \sum_{i=0}^{n-1} \alpha_i \cdots \alpha_{i+m} \alpha_{i+m}^* \cdots \alpha_i^* \\ &= \mathbb{O}(m + 1, m + 1). \end{aligned}$$

The result now follows by induction. ■

Set $s_0 = \mathbb{O}(0, 0) = 1$, $s_1 = \mathbb{O}(1, 0)$, and $s_2 = \mathbb{O}(2, 0)$. We claim that $R^{D_n} = \mathbb{k}[s_1, s_2]$.

Lemma 3.5 *The orbit sums s_1 and s_2 commute.*

Proof We recall first that, for every arrow α , there is exactly one nonstar arrow β and one star arrow γ such that $\alpha\beta \neq 0$ and $\alpha\gamma \neq 0$. Using this fact and the preprojective relation, we have

$$\begin{aligned} s_1 s_2 &= \left(\sum_{i=0}^{n-1} \alpha_i + \alpha_i^* \right) \left(\sum_{i=0}^{n-1} \alpha_i \alpha_{i+1} + \alpha_i^* \alpha_{i-1}^* \right) \\ &= \sum_{i=0}^{n-1} \alpha_i \alpha_{i+1} \alpha_{i+2} + \alpha_i \alpha_i^* \alpha_{i-1}^* + \alpha_i^* \alpha_i \alpha_{i+1} + \alpha_i^* \alpha_{i-1}^* \alpha_{i-2}^* \\ &= \sum_{i=0}^{n-1} \alpha_i \alpha_{i+1} \alpha_{i+2} + \alpha_{i-1}^* \alpha_{i-1} \alpha_{i-1}^* + \alpha_{i+1} \alpha_{i+1}^* \alpha_{i+1} + \alpha_i^* \alpha_{i-1}^* \alpha_{i-2}^* \\ &= \sum_{i=0}^{n-1} \alpha_i \alpha_{i+1} \alpha_{i+2} + \alpha_{i-1}^* \alpha_{i-2}^* \alpha_{i-2} + \alpha_{i+1} \alpha_{i+2} \alpha_{i+2}^* + \alpha_i^* \alpha_{i-1}^* \alpha_{i-2}^* \\ &= \left(\sum_{i=0}^{n-1} \alpha_i \alpha_{i+1} + \alpha_i^* \alpha_{i-1}^* \right) \left(\sum_{i=0}^{n-1} \alpha_i + \alpha_i^* \right) = s_2 s_1. \quad \blacksquare \end{aligned}$$

Lemma 3.6 *For all $\ell \geq k \geq 0$, $\mathbb{O}(\ell, k) \in \mathbb{k}[s_1, s_2]$.*

Proof We already have $\mathbb{O}(0, 0), \mathbb{O}(1, 0), \mathbb{O}(2, 0) \in \mathbb{k}[s_1, s_2]$. Then

$$\begin{aligned} s_1^2 &= \left(\sum_{i=0}^{n-1} \alpha_i + \alpha_i^* \right) \left(\sum_{i=0}^{n-1} \alpha_i + \alpha_i^* \right) \\ &= \left(\sum_{i=0}^{n-1} \alpha_i \alpha_{i+1} + \alpha_i^* \alpha_{i-1}^* \right) + \left(\sum_{i=0}^{n-1} \alpha_i \alpha_i^* + \alpha_i^* \alpha_i \right) \\ &= s_2 + 2\mathbb{O}(1, 1). \end{aligned}$$

Hence, $\mathbb{O}(1, 1) \in \mathbb{k}[s_1, s_2]$. Suppose inductively that $\mathbb{O}(\ell, k) \in \mathbb{k}[s_1, s_2]$ for all ℓ, k with $\ell \geq k \geq 0$ and $\ell + k \leq d$ for some $d \geq 2$. First, assume that d is even, so that $\mathbb{O}(\frac{d}{2}, \frac{d}{2}) \in \mathbb{k}[s_1, s_2]$. Then, by (3), $\mathbb{O}(\frac{d}{2} + 1, \frac{d}{2}) = \mathbb{O}(1, 0)\mathbb{O}(\frac{d}{2}, \frac{d}{2}) \in \mathbb{k}[s_1, s_2]$. Furthermore, since $\mathbb{O}(\frac{d}{2} + 1, \frac{d}{2} - 1) \in \mathbb{k}[s_1, s_2]$, then

$$\mathbb{O}\left(\frac{d}{2} + 2, \frac{d}{2} - 1\right) = \mathbb{O}(1, 0)\mathbb{O}\left(\frac{d}{2} + 1, \frac{d}{2} - 1\right) - \mathbb{O}\left(\frac{d}{2} + 1, \frac{d}{2}\right).$$

By another induction, we have $\mathbb{O}(\ell, k) \in \mathbb{k}[s_1, s_2]$ with $\ell + k = d + 1$.

Now, assume that d is odd. Then $d + 1$ is even, and since $\mathbb{O}(1, 1) \in \mathbb{k}[s_1, s_2]$, then, by (4), $\mathbb{O}(1, 1)^{(d+1)/2} = \mathbb{O}(\frac{d+1}{2}, \frac{d+1}{2})$. Now, the argument proceeds as in the even case. ■

We now proceed to our main result for this section.

Theorem 3.7 *The Auslander map is not an isomorphism for the pair (R, D_n) .*

Proof Combining the previous two lemmas, there is a surjective map $\mathbb{k}[s_1, s_2] \rightarrow R^{D_n}$. Since both algebras have the same Hilbert series, then it follows that this map is an isomorphism. It now suffices to show that the set

$$S = \{e_0, \dots, e_{n-1}, \alpha_0, \dots, \alpha_{n-1}\}$$

is a basis for R over R^{D_n} . That is, R is a rank $2n$ free module over R^{D_n} . Then we have

$$R \cong \bigoplus_{i=0}^{n-1} (e_i R^{D_n} \oplus \alpha_i R^{D_n})$$

as R^{D_n} -modules. Since $\alpha_{n-1} R^{D_n} \cong e_0 R^{D_n}(-1)$, then $\text{End}_{R^{D_n}} R$ contains a map of negative degree and so the Auslander map is not an isomorphism for (R, D_n) .

First, we show that the set S generates R as an R^{D_n} -module. Clearly, $R_0 \subset \text{Span}_{R^{D_n}} S$. Moreover, for all $i = 0, \dots, n-1$, $\alpha_{i-1}^* = e_i(s_1) - \alpha_i(1)$. Hence, $R_1 \subset \text{Span}_{R^{D_n}} S$.

Note that there are exactly three paths of degree 2 for each vertex. Consider the degree 2 paths based at vertex 0. We have

$$e_0 \circ (1, 1) = \alpha_0 \alpha_0^*, \quad e_0 \circ (2, 0) = \alpha_0 \alpha_1 + \alpha_{n-1}^* \alpha_{n-2}^*, \quad \alpha_0 \circ (1, 0) = \alpha_0 \alpha_1 + \alpha_0 \alpha_0^*.$$

Hence, $\{\alpha_0 \alpha_0^*, \alpha_0 \alpha_1, \alpha_{n-1}^* \alpha_{n-2}^*\} \subset \text{Span}_{R^{D_n}} S$. A similar argument for the remaining vertices shows that $R_2 \subset \text{Span}_{R^{D_n}} S$.

In particular, the above argument shows that $R_2 = S_1 R_1^{D_n} + S_0 R_2^{D_n}$, which implies that $R_2 = R_1 R_1^{D_n} + R_0 R_2^{D_n}$. Multiplying by R_1 on the left gives $R_3 = R_2 R_1^{D_n} + R_1 R_2^{D_n}$, and by induction, $R_{m+1} = R_m R_1^{D_n} + R_{m-1} R_2^{D_n}$ for all m . Thus, R is generated as a right R^{D_n} -module by R_0 and R_1 . It follows that $R \subset \text{Span}_{R^{D_n}} S$. That is, S is a generating set for R as an R^{D_n} -module.

For independence, we note that, for every element of S , there is exactly one other element in S with the same source. Hence, it suffices to prove that $e_i R^{D_n} \cap \alpha_i R^{D_n} = \{0\}$. We do this computation for $i = 0$, and the other vertices follow similarly.

Suppose that $a \in e_i R^{D_n} \cap \alpha_i R^{D_n}$. We may assume without loss of generality that a is homogeneous of degree d . Suppose first that d is even. Then there exist scalars $k_i, k'_i \in \mathbb{k}$ such that

$$\begin{aligned} a &= e_0 (k_0 \circ (d, 0) + k_1 \circ (d-1, 1) + \dots + k_{d/2} \circ (d/2, d/2)), \\ a &= \alpha_0 (k'_0 \circ (d-1, 0) + k'_1 \circ (d-2, 1) + \dots + k'_{d/2-1} \circ (d/2, d/2-1)). \end{aligned}$$

From the second expression, we note that every path summand of a must contain at least one nonstarred arrow. Hence, $k_0 = 0$. But then, from the first expression, we note that every path summand of a must contain at least one starred arrow, so $k'_0 = 0$. Continuing in this way, we see that $a = 0$. ■

3.2 The W_n case

In case n is odd, $\eta_{R,G}$ is an isomorphism if and only if G is a proper subgroup of D_n . In case n is even, there is one additional instance when $\eta_{R,G}$ fails to be an isomorphism, namely for the subgroup W_n defined as

$$W_n = \langle \tau \in D_n : \tau(e_i) = e_i \text{ for some } i = 0, \dots, n-1 \rangle.$$

That is, W_n is generated by the reflections in D_n that pass through a vertex. If n is odd, then every reflection fixes a vertex, so W_n contains every reflection and $W_n = D_n$. If n is even, only half of the reflections fix a vertex, so W_n is a proper subgroup of D_n . Since W_n is of index 2 in D_n , W_n is a maximal subgroup of D_n .

Throughout this section, we assume that n is even. Our strategy will be similar to the previous section. The key difference is that the invariant ring is no longer connected graded. In particular, there are exactly twice as many orbits in each graded piece as in the D_n case.

As in the previous section, for $\ell \geq k \geq 0$, set $B_{\ell,k} = Q_\ell Q_k^* \cup Q_k Q_\ell^*$. Then define

$$B_{\ell,k}^{\text{even}} = e_i B_{\ell,k} \text{ for } i \text{ even} \quad \text{and} \quad B_{\ell,k}^{\text{odd}} = e_i B_{\ell,k} \text{ for } i \text{ odd}.$$

Lemma 3.8 For any $p \in B_{\ell,k}^{\text{even}}$ (resp. $p \in B_{\ell,k}^{\text{odd}}$), $\mathcal{O}(p) = B_{\ell,k}^{\text{even}}$ (resp. $\mathcal{O}(p) = B_{\ell,k}^{\text{odd}}$).

Proof This is similar to the proof of Lemma 3.3. In particular, the $B_{\ell,k}$ partition the paths of Q . However, since $g \in W_n$ preserves the parity of the idempotents e_i , it follows that $g(B_{\ell,k}^{\text{even}}) \subset B_{\ell,k}^{\text{even}}$. Because g is bijective, then, in fact, we have equality.

It remains to show that we have an equivalence with the orbits. If $p \in B_{\ell,k}^{\text{even}}$, then clearly $\mathcal{O}(p) \subset B_{\ell,k}^{\text{even}}$. Since $|B_{\ell,k}^{\text{even}}| = \frac{1}{2}|B_{\ell,k}|$ and $|W_n| = \frac{1}{2}|D_n|$, then it follows from the argument in Lemma 3.3 that $|B_{\ell,k}^{\text{even}}| = |\mathcal{O}(p)|$. A similar argument applies to $|B_{\ell,k}^{\text{odd}}|$. ■

Set

$$\mathbb{O}(\ell, k)^{\text{even}} = \sum_{p \in B_{\ell,k}^{\text{even}}} p \quad \text{and} \quad \mathbb{O}(\ell, k)^{\text{odd}} = \sum_{p \in B_{\ell,k}^{\text{odd}}} p.$$

These form a \mathbb{k} -basis for R^{W_n} . Thus, R^{W_n} has total Hilbert series

$$H_{R^{W_n}}^{\text{tot}}(t) = \frac{2}{(1-t)(1-t^2)}.$$

However, since $(R^{W_n})_0 = \mathbb{k}^2$, then we can also record the matrix-valued Hilbert series. Let M be the 2×2 matrix defined by

$$\begin{aligned} M_{0,0} &= \#\{p \in B_{\ell,k}^{\text{even}} \text{ with target } e_i, i \text{ even}\}, \\ M_{0,1} &= \#\{p \in B_{\ell,k}^{\text{even}} \text{ with target } e_i, i \text{ odd}\}, \\ M_{1,0} &= \#\{p \in B_{\ell,k}^{\text{odd}} \text{ with target } e_i, i \text{ even}\}, \\ M_{1,1} &= \#\{p \in B_{\ell,k}^{\text{odd}} \text{ with target } e_i, i \text{ odd}\}. \end{aligned}$$

Note that for $p \in B_{\ell,k}$, the parity of the target depends on the source and the parity of $\ell + k$. Hence, it follows that the matrix-valued Hilbert series of R^{W_n} is

$$\begin{aligned} H_{R^{W_n}} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} t^2 + \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} t^3 + \dots \\ &= \left(I - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t \right)^{-1} \left(I - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t^2 \right)^{-1}. \end{aligned}$$

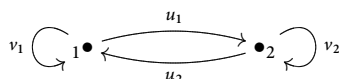
Proofs of the relations in the next lemma are similar to the corresponding proofs in Lemma 3.4.

Lemma 3.9 *Let \bullet, \dagger denote opposite parities. The orbit sums $\mathbb{O}(\ell, k)$ satisfy the following relations:*

$$(5) \quad \mathbb{O}(1, 0)^\bullet \mathbb{O}(\ell, k)^\dagger = \begin{cases} \mathbb{O}(\ell + 1, k)^\dagger + \mathbb{O}(\ell, k + 1)^\dagger, & \text{if } \ell + k \text{ is even and } \ell > k, \\ \mathbb{O}(\ell + 1, k)^\dagger, & \text{if } \ell + k \text{ is even and } \ell = k, \\ \mathbb{O}(\ell + 1, k)^\bullet + \mathbb{O}(\ell, k + 1)^\bullet, & \text{if } \ell + k \text{ is odd and } \ell > k, \\ \mathbb{O}(\ell + 1, k)^\bullet, & \text{if } \ell + k \text{ is odd and } \ell = k, \end{cases}$$

$$(6) \quad (\mathbb{O}(1, 1)^\bullet)^m = \mathbb{O}(m, m)^\bullet.$$

We set $s_0 = \mathbb{O}(0, 0)^{\text{even}}$, $s_1 = \mathbb{O}(1, 0)^{\text{even}}$, and $s_2 = \mathbb{O}(2, 0)^{\text{even}}$. Similarly, we set $s'_0 = \mathbb{O}(0, 0)^{\text{odd}}$, $s'_1 = \mathbb{O}(1, 0)^{\text{odd}}$, and $s'_2 = \mathbb{O}(2, 0)^{\text{odd}}$. Let C denote the subalgebra of R^{W_n} generated by these elements. Let Q be the following quiver:



and let $\mathbb{k}Q$ denote its path algebra. We assign degree 1 to the arrows u_1, u_2 and degree 2 to v_1, v_2 . We will show that $C = R^{W_n}$ and that $R^{W_n} \cong \mathbb{k}Q / (v_1 u_1 - u_1 v_2, v_2 u_2 - u_2 v_1)$.

Remark 3.10 The algebra $\mathbb{k}Q / (v_1 u_1 - u_1 v_2, v_2 u_2 - u_2 v_1)$ is a quotient-derivation algebra appearing in the classification of graded twisted Calabi–Yau algebras of global dimension 2 [19]. In particular, the matrix corresponding to the Nakayama automorphism μ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the μ -twisted superpotential is $v_1 u_1 - u_2 v_1 + v_2 u_2 - u_1 v_2$.

Lemma 3.11 *The relations $s_2 s_1 = s_1 s'_2$ and $s'_2 s'_1 = s'_1 s_2$ hold in C .*

Proof We prove the first relation. The second is similar.

$$\begin{aligned} s_2 s_1 &= \left(\sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i} \alpha_{2i+1} + \alpha_{2i+1}^* \alpha_{2i}^*) \right) \left(\sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i} + \alpha_{2i+1}^*) \right) \\ &= \sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i} \alpha_{2i+1} \alpha_{2i+2} + \alpha_{2i} \alpha_{2i+1} \alpha_{2i+1}^* + \alpha_{2i+2} \alpha_{2i+2}^* \alpha_{2i+1}^* + \alpha_{2i+1}^* \alpha_{2i}^* \alpha_{2i-1}^*) \\ &= \sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i} \alpha_{2i+1} \alpha_{2i+2} + \alpha_{2i} \alpha_{2i}^* \alpha_{2i-1}^* + \alpha_{2i+2} \alpha_{2i+3} \alpha_{2i+3}^* + \alpha_{2i+1}^* \alpha_{2i}^* \alpha_{2i-1}^*) \\ &= \sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i} \alpha_{2i+1} \alpha_{2i+2} + \alpha_{2i} \alpha_{2i}^* \alpha_{2i-1}^* + \alpha_{2i+1}^* \alpha_{2i+1} \alpha_{2i+2} + \alpha_{2i+1}^* \alpha_{2i}^* \alpha_{2i-1}^*) \\ &= \left(\sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i} + \alpha_{2i+1}^*) \right) \left(\sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i+1} \alpha_{2i+2} + \alpha_{2i+2}^* \alpha_{2i+1}^*) \right) = s'_1 s'_2. \quad \blacksquare \end{aligned}$$

Lemma 3.12 We have $C = R^{W_n}$.

Proof Clearly, $C \subset R^{W_n}$. We claim $R^{W_n} \subset C$. By definition, $(R^{W_n})_0 \subset C$ and $(R^{W_n})_1 \subset C$. Now,

$$\begin{aligned} s_1 s'_1 &= \left(\sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i} + \alpha_{2i+1}^*) \right) \left(\sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i+1} + \alpha_{2i}^*) \right) \\ &= \sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i} \alpha_{2i+1} + \alpha_{2i} \alpha_{2i}^* + \alpha_{2i+1}^* \alpha_{2i+1} + \alpha_{2i+1}^* \alpha_{2i}^*) \\ &= \sum_{i=0}^{\frac{n-2}{2}} (\alpha_{2i} \alpha_{2i+1} + \alpha_{2i+1}^* \alpha_{2i}^* + 2\alpha_{2i} \alpha_{2i}^*) \\ &= s_2 + 2\mathbb{O}(1, 1)^{\text{even}}. \end{aligned}$$

Thus, $\mathbb{O}(1, 1)^{\text{even}} \in C$. A similar proof with $s'_1 s_1$ shows that $\mathbb{O}(1, 1)^{\text{odd}} \in C$ so that $(R^{W_n})_2 \subset C$. The remainder of the proof follows similarly to Lemma 3.6 with proper respect shown toward parity. In particular, we use (5) and (6). ■

Theorem 3.13 The Auslander map is not an isomorphism for the pair (R, W_n) .

Proof Denote the trivial paths of Q by f_0, f_1 . There is a map $\phi : \mathbb{k}Q \rightarrow R^{W_n}$ defined by setting

$$f_0 \mapsto s_0, f_1 \mapsto s'_0, u_1 \mapsto s_1, u_2 \mapsto s'_1, v_1 \mapsto s_2, v_2 \mapsto s'_2.$$

It is easy to verify that this determines a well-defined surjective map $\mathbb{k}Q \rightarrow R^{W_n}$ and

$$K = (v_1 u_1 - u_1 v_2, v_2 u_2 - u_2 v_1)$$

belongs to $\ker \phi$. By comparing the matrix-valued Hilbert series, it is clear that $\mathbb{k}Q/K \cong R^{W_n}$.

The remainder of the proof follows analogously to Theorem 3.7. In particular, R is a free R^{W_n} -module with basis $S = \{e_0, \dots, e_{n-1}, \alpha_0, \dots, \alpha_{n-1}\}$, and this gives rise to a map in $\text{End}_{R^{W_n}} R$ of negative degree. ■

Theorems 3.7 and 3.13 give instances of fixed rings of preprojective algebras which are graded Calabi–Yau. These examples are novel from those presented by Weispenfenning in that they do not fix pointwise the degree zero part of $\Pi_{\overline{A_{n-1}}}$.

Corollary 3.14 Let $G = D_n$ or $G = W_n$. Then $\text{p}(R, G) = 1$.

Proof Let $p = \alpha_0 \alpha_1 \cdots \alpha_{n-1}$ and $q = \alpha_{n-1}^* \alpha_{n-2}^* \cdots \alpha_0^*$. Set

$$f_1 = e_0 \# 1 + e_0 \# r_0 = e_0 (f_G) e_0 \in (f_G).$$

Then $(p - q) \# 1 = p f_1 - f_1 q \in (f_G)$. Consequently, $\text{p}(R, W_n) \geq 1$. By Theorems 2.3, 3.7, and 3.13, $\text{p}(R, G) < 2$. Thus, $\text{p}(R, G) = 1$ by Bergman’s Gap Theorem [5]. ■

Acknowledgment The authors appreciate the referee's comments and suggestions for improving this manuscript.

References

- [1] M. Auslander, *On the purity of the branch locus*. Amer. J. Math. **84**(1962), 116–125.
- [2] D. Baer, W. Geigle, and H. Lenzing, *The preprojective algebra of a tame hereditary Artin algebra*. Comm. Algebra **15**(1987), nos. 1–2, 425–457.
- [3] Y.-H. Bao, J.-W. He, and J. J. Zhang, *Noncommutative Auslander theorem*. Trans. Amer. Math. Soc. **370**(2018), no. 12, 8613–8638.
- [4] Y.-H. Bao, J.-W. He, and J. J. Zhang, *Pertinency of Hopf actions and quotient categories of Cohen–Macaulay algebras*. J. Noncommut. Geom. **13**(2019), no. 2, 667–710.
- [5] G. M. Bergman, *A note on growth functions of algebras and semigroups*. Research Note, University of California, Berkeley, 1978.
- [6] R. Bocklandt, *Graded Calabi Yau algebras of dimension 3*. J. Pure Appl. Algebra. **212**(2008), no. 1, 14–32.
- [7] K. Chan, E. Kirkman, C. Walton, and J. J. Zhang, *McKay correspondence for semisimple Hopf actions on regular graded algebras, I*. J. Algebra **508**(2018), 512–538.
- [8] K. Chan, A. Young, and J. J. Zhang, *Noncommutative cyclic isolated singularities*. Trans. Amer. Math. Soc. **373**(2020), no. 6, 4319–4358.
- [9] J. Chen, E. Kirkman, and J. J. Zhang, *Auslander's theorem for group coactions on Noetherian graded down-up algebras*. Transform. Groups **25**(2020), no. 4, 1037–1062.
- [10] S. Crawford, *Actions of small groups on two-dimensional Artin–Schelter regular algebras*. J. Algebra **560**(2020), 609–666.
- [11] J. Gaddis, E. Kirkman, W. F. Moore, and R. Won, *Auslander's theorem for permutation actions on noncommutative algebras*. Proc. Amer. Math. Soc. **147**(2019), no. 5, 1881–1896.
- [12] J. Gaddis and D. Yee, *Congenial algebras: extensions and examples*. Comm. Algebra **47**(2019), no. 10, 4347–4357.
- [13] V. Ginzburg, *Calabi–Yau algebras*. Preprint, 2007. [arXiv:math/0612139](https://arxiv.org/abs/math/0612139)
- [14] J.-W. He and Y. Zhang, *Local cohomology associated to the radical of a group action on a Noetherian algebra*. Israel J. Math. **231**(2019), no. 1, 303–342.
- [15] P. Jørgensen and J. J. Zhang, *Gourmet's guide to Gorensteinness*. Adv. Math. **151**(2000), no. 2, 313–345.
- [16] R. Kinser and C. Walton, *Actions of some pointed Hopf algebras on path algebras of quivers*. Algebra Number Theory **10**(2016), no. 1, 117–154.
- [17] I. Mori and K. Ueyama, *Ample group action on AS-regular algebras and noncommutative graded isolated singularities*. Trans. Amer. Math. Soc. **368**(2016), no. 10, 7359–7383.
- [18] X. Qin, Y. Wang, and J. Zhang, *Maximal Cohen–Macaulay modules over a noncommutative 2-dimensional singularity*. Front. Math. China **14**(2019), no. 5, 923–940.
- [19] M. L. Reyes and D. Rogalski, *Growth of graded twisted Calabi–Yau algebras*. J. Algebra **539**(2019), 201–259.
- [20] J. T. Stafford and J. J. Zhang, *Homological properties of (graded) Noetherian PI rings*. J. Algebra **168**(1994), no. 3, 988–1026.
- [21] S. Weispfenning, *Properties of the fixed ring of a preprojective algebra*. J. Algebra **517**(2019), 276–319.
- [22] S. Weispfenning, *Generalized Gorensteinness and a homological determinant for preprojective algebras*. Comm. Algebra **48**(2020), no. 7, 3035–3060.

Department of Mathematics, Miami University, Oxford, OH 45056, USA

e-mail: barahoja@miamioh.edu gaddisj@miamioh.edu