Can. J. Math. Vol. 48 (2), 1996 pp. 330–342

SPECTRA FOR INFINITE TENSOR PRODUCT TYPE ACTIONS OF COMPACT GROUPS

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ABSTRACT. We present explicit calculations of the Arveson spectrum, the strong Arveson spectrum, the Connes spectrum, and the strong Connes spectrum, for an infinite tensor product type action of a compact group. Using these calculations and earlier results (of the authors and C. Peligrad) relating the various spectra to the ideal structure of the crossed product algebra, we prove that the topology of G influences the ideal structure of the crossed product algebra, in the following sense: if G contains a non-trivial connected group as a direct summand, then the crossed product algebra may be prime, but it is never simple; while if G is discrete, the crossed product algebra is simple if and only if it is prime. These results extend to compact groups analogous results of Bratteli for abelian groups. In addition, we exhibit a class of examples illustrating that for compact groups, unlike the case for abelian groups, the Connes spectrum and strong Connes spectrum need not be stable.

1. **Introduction and preliminaries.** In [5], notions were developed for the spectrum and strong spectrum of an action α of a compact group G on a C^*-algebra A, which led to analogues of the main results of [9] and [6] on abelian group actions, relating the Connes type spectra of the action to the ideal structure of the crossed product algebra. Specifically, for $\pi \in \hat{G}$, the space of unitary equivalence classes of irreducible representations of G, we denote by H_π the finite-dimensional Hilbert space on which π acts, and define the π-spectral subspace $A(\pi)$ by

DEFINITION 1.1. $A(\pi) \equiv \{T \in A \otimes B(H_\pi) : (\alpha_g \otimes \mathrm{id})(T) = T(1_A \otimes \pi_g), g \in G\}$.

Note that $A(\pi)^*A(\pi)$ is a two-sided ideal in $\big(A \otimes B(H_\pi)\big)^{\alpha\otimes \mathrm{ad}\,\pi}$, the fixed-point subalgebra of $A \otimes B(H_\pi)$ under the tensor product action of G. Then

DEFINITION 1.2. a) The Arveson spectrum

$$\mathrm{Sp}(\alpha) \equiv \Big\{\pi \in \hat{G} : A(\pi)^*A(\pi) \text{ is an essential ideal in } \big(A \otimes B(H_\pi)\big)^{\alpha\otimes \mathrm{ad}\,\pi}\Big\}.$$

b) The strong Arveson spectrum

$$\widetilde{\mathrm{Sp}}(\alpha) \equiv \Big\{\pi \in \hat{G} : A(\pi)^*A(\pi) = \big(A \otimes B(H_\pi)\big)^{\alpha\otimes \mathrm{ad}\,\pi}\Big\}.$$

The definitions of the Arveson type spectra can be reformulated in terms of the representation theory of the crossed product C^*-algebra $G \times_\alpha A$. We denote by $\langle V, \tau\rangle$ a

Received by the editors July 4, 1994.
AMS subject classification: Primary: 46L55; secondary: 22D25.

covariant pair of representations of the non-commutative dynamical system (G, A, α), corresponding to a representation L of $G \times_\alpha A$ by the well-known formula

$$L(f) = \int_G \tau\big(f(s)\big) V(s)\, ds, \quad f \in L^1(G, A).$$

Then with id denoting the trivial one dimensional representation of G, and $\bar{\pi}$ the conjugate of the representation π, we have

LEMMA 1.3. *Let $\pi \in \hat{G}$. Then*
a) $\bar{\pi} \in \mathrm{Sp}(\alpha)$ if and only if $\{L = \langle V, \tau\rangle \in (G \times_\alpha A)^\wedge : V \supseteq \pi$ and $V \supseteq \mathrm{id}\}$ is dense in $\{L = \langle V, \tau\rangle \in (G \times_\alpha A)^\wedge : V \supseteq \pi\}$, and
b) $\bar{\pi} \in \widetilde{\mathrm{Sp}}(\alpha)$ if and only if $\{L = \langle V, \tau\rangle \in (G \times_\alpha A)^\wedge : V \supseteq \pi$ and $V \supseteq \mathrm{id}\} = \{L = \langle V, \tau\rangle \in (G \times_\alpha A)^\wedge : V \supseteq \pi\}$.

PROOF. Lemma 3.1 of [5]. ■

Corresponding to the above two Arveson type spectra for the action α, we have two Connes type spectra. Let $H^\alpha(A)$ denote the family of all non-zero α-invariant hereditary C^*-subalgebras B of A.

DEFINITION 1.4. (a) The Connes spectrum $\Gamma(\alpha) \equiv \bigcap\{\mathrm{Sp}(\alpha|_B) : B \in H^\alpha(A)\}$.
(b) The strong Connes spectrum $\tilde{\Gamma}(\alpha) \equiv \bigcap\{\widetilde{\mathrm{Sp}}(\alpha|_B) : B \in H^\alpha(A)\}$.

With the above Connes type spectra, we have, analogous to the main results of [9] and [6], the following

THEOREM 1.5. *Let (G, A, α) be a C^*-dynamical system with G compact. Then*
(a) $G \times_\alpha A$ is prime if and only if A is G-prime and $\Gamma(\alpha) = \hat{G}$, while
(b) $G \times_\alpha A$ is simple if and only if A is G-simple and $\tilde{\Gamma}(\alpha) = \hat{G}$.

PROOF. See [5], Theorems 2.2 and 2.5, or Theorems 3.4 and 3.8. ■

The present paper was motivated by results of Bratteli [1] on product type actions of an abelian group. For any group G, a product type action is an action of the form $\alpha = \otimes_{i=1}^\infty \operatorname{ad} V_i$ on $A = \otimes_{i=1}^\infty M_i$, where V_i is a finite-dimensional unitary representation of G of dimension d_i, and M_i is the full $d_i \times d_i$ matrix algebra. For G abelian, Bratteli obtains necessary and sufficient conditions for $G \times_\alpha A$ to be prime or simple. Bratteli also shows that the ideal structure of the crossed product algebra is influenced by the topology of G, in the following sense: if G has a non-trivial connected group as a direct summand, then the crossed product is never simple, although it may be prime; if G is totally disconnected but not discrete, then for some product type actions the crossed product may be prime but not simple, while for others the crossed product is simple; and finally, if G is countable discrete, then the crossed product algebra is prime if and only if it is simple. Bratteli proves his results, not by a computation of Arveson and Connes type spectra, but rather by a careful analysis of the structure of $G \times_\alpha A$ as an inductive limit of the crossed product algebra $G \times (\otimes_{i=1}^n M_i)$. In fact, Bratteli's paper predates Kishimoto's [6], where the strong spectra are first defined, and their relationship to the simplicity of the crossed product is first proven.

Viewing product type actions as a rich class of actions, but of sufficiently specific structure to be studied in detail, we attempted to replicate Bratteli's results for the case of compact group actions. However, we do this by first explicitly computing the Arveson and Connes type spectra of the actions, and then applying Theorem 1.5. It turns out that in the compact case also, we obtain results similar to Bratteli's, so that the topology of the group influences whether or not it is possible for the crossed product to be prime but not simple.

One further motivation for our study is the phenomenon of spectral instability. For G abelian, there is a dual action $\hat{\alpha}$ of the dual group $\hat{G}$ on $G \times_\alpha A$, and likewise a double dual action $\hat{\hat{\alpha}}$ of $\hat{\hat{G}} = G$ on the double crossed product $\hat{G} \times_{\hat{\alpha}} (G \times_\alpha A)$ such that the spectra are stable, *i.e.*, $\Gamma(\alpha) = \Gamma(\hat{\hat{\alpha}})$ and $\tilde{\Gamma}(\alpha) = \tilde{\Gamma}(\hat{\hat{\alpha}})$. Furthermore, this fact is critical in the proofs relating the Connes type spectra to the ideal structure of $G \times_\alpha A$ (see [9] and [6] for details). For non-abelian G, there is a dual coaction $\hat{\alpha}$ of G on $G \times_\alpha A$ and a double-dual action $\hat{\hat{\alpha}}$ of G on the coproduct algebra $G \times_{\hat{\alpha}} (G \times_\alpha A)$ ([8], [12]). However, in the case of a non-abelian compact G, the Connes type spectra need not be stable, and the Connes type spectra of the double-dual action may be properly contained in the Connes type spectra of the original action ([5], Example 5.2). Of interest, however, is that by virtue of the results of [5], if the Connes type spectra of the original action are full (*i.e.*, equal to all of $\hat{G}$), then so are the Connes type spectra of the double-dual action. We feel that this phenomenon of spectral instability presents a serious obstacle to the goal of developing common proofs, valid for both the abelian and compact cases, of the results relating spectra to ideal structure in the crossed product algebra, and thus deserves to be studied further. One impediment was the lack of examples of spectral instability—we had only the one example in [5]—and thus a further motivation in studying product type actions of compact groups was the hope of constructing more examples of spectral instability. We have constructed in Section 4 a class of examples, and perhaps of note is that at least for our examples, the topology of G seems to play a role. Not only can our type of example manifestly not occur in the abelian case, but it also cannot occur in the case of compact connected groups either, which suggests, however slightly, that perhaps spectral instability is a discrete group phenomenon.

Section 2 contains the computation of the Arveson and strong Arveson spectrum of a product type action of a compact group. In Section 3 we show that in computing the Connes spectrum and strong Connes spectrum of a product type action of a compact group, it suffices to consider only hereditary subalgebras on which the restricted action is again of product type, so that the computations in Section 2 of the Arveson type spectra yield formulas for the Connes type spectra. Section 4 contains the analogue, for compact groups, of Bratteli's results in [1] on product type actions of abelian groups, as well as a class of examples of spectral instability.

Let α be an action of a compact group G on a C^*-algebra $A = \overline{\bigcup_{n=1}^{\infty} A_n}$, where $\{A_n\}$ is an increasing sequence of α-invariant subalgebras. It follows by invariant integration over G that the fixed-point subalgebra $A^\alpha = \overline{\bigcup_{n=1}^{\infty} A_n^\alpha}$, and by invariant integration and Lemma 2.2 of [11] that $A(\pi) = \overline{\bigcup_{n=1}^{\infty} A_n(\pi)}$, for $\pi \in \hat{G}$. Using α_n for the restricted action

$\alpha|_{A_n}$, as we henceforth do, it follows that for $\pi \in \hat{G}$,

$$\big(A \otimes B(H_\pi)\big)^{\alpha\otimes \mathrm{ad}\,\pi} = \overline{\bigcup_{n=1}^{\infty} \big(A_n \otimes B(H_\pi)\big)^{\alpha_n \otimes \mathrm{ad}\,\pi}}.$$

It also follows that

$$A(\pi)^* A(\pi) = \overline{\bigcup_{n=1}^{\infty} A_n(\pi)^* A_n(\pi)},$$

where for subsets S, T of a C^*-algebra, we denote by S^*T the closure of the linear span of elements of the form s^*t, $s \in S$, $t \in T$. We use id for the trivial one-dimensional identity representation of a group.

2. **Arveson type spectra.** Let G be a compact group. For V a unitary representation of G, we denote by $\mathrm{sp}(V)$ the spectrum of V, that is, $\{\pi \in \hat{G} : \pi \subseteq V\}$, and by $\bar{V}$ the contragredient of V. For $H \subseteq \hat{G}$, let

$$S(H) \equiv \{\pi \in \hat{G} : \forall \sigma \in H, \mathrm{sp}(\pi \otimes \sigma) \subseteq H\}.$$

The above $S(H)$ is a non-commutative analogue of the set $S(H)$ defined in Proposition 2.1 of [6] for H a subset of the dual of an abelian group, and we shall show below that it can be used to characterize the strong Arveson spectrum of an action of a compact group, thus providing a characterization of $\widetilde{\mathrm{Sp}}(\alpha)$ in the compact case similar to that given in [6], for G abelian.

PROPOSITION 2.1. *Let (G, A, α) be a C^*-dynamical system, with G compact. Then* $\widetilde{\mathrm{Sp}}(\alpha) = \bigcap \big\{S\big(\mathrm{sp}(V)\big) : \langle V, \tau\rangle \in (G \times_\alpha A)^\wedge\big\}$.

PROOF. Let $\pi \in \widetilde{\mathrm{Sp}}(\alpha)$ and let $\langle V, \tau\rangle \in (G \times_\alpha A)^\wedge$. If $\sigma \in \hat{G}$ with $\sigma \subseteq V \otimes \pi$, then $\bar{\pi} \subseteq \bar{\sigma} \otimes V$. By Proposition 2.1 of [4], the covariant pair $\langle \bar{\sigma} \otimes V, I \otimes \tau\rangle$ can be written as a finite direct sum $\sum \oplus \langle V_i, \tau_i\rangle$ of irreducible covariant pairs for (G, A, α), so that $\bar{\pi} \subseteq V_i$ for some i. Thus, by Lemma 1.3(b), id $\subseteq V_i$, so that id $\subseteq \bar{\sigma} \otimes V$. Thus $\sigma \subseteq V$ and $\mathrm{sp}(\pi \otimes V) \subseteq \mathrm{sp}(V)$, so that $\pi \in S\big(\mathrm{sp}(V)\big)$. Conversely, let $\pi \in \bigcap \big\{S\big(\mathrm{sp}(V)\big) : \langle V, \tau\rangle \in (G \times_\alpha A)^\wedge\big\}$, and let $\langle V_0, \tau_0\rangle \in (G \times_\alpha A)^\wedge$ with $\bar{\pi} \subseteq V_0$. Then id $\subseteq \bar{\pi} \otimes \pi \subseteq V_0 \otimes \pi$. Thus id $\subseteq V_0$, and $\pi \in \widetilde{\mathrm{Sp}}(\alpha)$ by Lemma 1.3(b). ∎

PROPOSITION 2.2. *For G compact, let V be a finite-dimensional representation of G, of dimension n, and let α be the action* $\mathrm{ad}(V)$ *of G on M_n. Then* $\widetilde{\mathrm{Sp}}(\alpha) = \mathrm{Sp}(\alpha) = S\big(\mathrm{sp}(V)\big)$.

PROOF. Let i denote the identity representation of M_n on the space of V. By the Mackey machine, $(G \times_\alpha M_n)^\wedge = \{\langle \tau \otimes V, I \otimes i\rangle : \tau \in \hat{G}\}$. In particular, $\langle V, i\rangle \in (G \times_\alpha M_n)^\wedge$ so that $\widetilde{\mathrm{Sp}}(\alpha) \subseteq S\big(\mathrm{sp}(V)\big)$ by Proposition 2.1. Assume $\pi \in S\big(\mathrm{sp}(V)\big)$, let $\tau \in \hat{G}$, and suppose that for $\sigma \in \hat{G}$, $\sigma \subseteq \tau \otimes V \otimes \pi$. Then the representation $\sigma \otimes \bar{\tau}$ is not disjoint from the representation $V \otimes \pi$, and thus for some $\rho \in \hat{G}$, we have $\rho \subseteq \sigma \otimes \bar{\tau}$ and $\rho \subseteq V \otimes \pi$. As $\pi \in S\big(\mathrm{sp}(V)\big)$, $\rho \subseteq V$. Now $\rho \subseteq \sigma \otimes \bar{\tau} \Rightarrow \sigma \subseteq \tau \otimes \rho \subseteq \tau \otimes V$. Thus $\mathrm{sp}(\tau \otimes V \otimes \pi) \subseteq \mathrm{sp}(\tau \otimes V)$

for all $\tau \in \hat{G}$, and $\pi \in \widetilde{\mathrm{Sp}}(\alpha)$. Finally, as the action $\mathrm{ad}(V)$ of G on M_n is inner, $G \times_\alpha M_n$ is isomorphic to $C^*(G) \otimes M_n$ and thus has a discrete dual. By Lemma 1.3, $\mathrm{Sp}(\alpha) = \widetilde{\mathrm{Sp}}(\alpha)$. ■

Henceforth, when we say that B is a unital subalgebra of a unital algebra A, we shall mean that they share the same unit.

The next result relates the strong Arveson spectrum of an inductive limit $A = \overline{\bigcup A_n}$ of invariant subalgebras, to the strong Arveson spectra of the action on the subalgebras, when all the A_n's are unital subalgebras of A.

PROPOSITION 2.3. *Let (G, A, α) be a C^*-dynamical system, with G compact, and $A = \overline{\bigcup_{n=1}^\infty A_n}$ a unital algebra which is an inductive limit of increasing, unital and invariant subalgebras A_n. Then $\widetilde{\mathrm{Sp}}(\alpha) = \bigcup_{n=1}^\infty \widetilde{\mathrm{Sp}}(\alpha_n)$.*

PROOF. Let $\pi \in \hat{G}$. As A_n is a unital, invariant subalgebra of A, clearly $\big(A_n \otimes B(H_\pi)\big)^{\alpha_n \otimes \mathrm{ad}\,\pi}$ is a unital subalgebra of $\big(A \otimes B(H_\pi)\big)^{\alpha \otimes \mathrm{ad}\,\pi}$. If $\pi \in \widetilde{\mathrm{Sp}}(\alpha_n)$ then $A_n(\pi)^* A_n(\pi) = \big(A_n \otimes B(H_\pi)\big)^{\alpha_n \otimes \mathrm{ad}\,\pi}$. Thus $A_n(\pi)^* A_n(\pi)$ contains the unit of $\big(A \otimes B(H_\pi)\big)^{\alpha \otimes \mathrm{ad}\,\pi}$, and so does the larger algebra $A(\pi)^* A(\pi)$. As this latter ideal in $\big(A \otimes B(H_\pi)\big)^{\alpha \otimes \mathrm{ad}\,\pi}$ contains the unit, it equals the whole algebra, and $\pi \in \widetilde{\mathrm{Sp}}(\alpha)$. Conversely, if $\pi \in \widetilde{\mathrm{Sp}}(\alpha)$, then $A(\pi)^* A(\pi) = \big(A \otimes B(H_\pi)\big)^{\alpha \otimes \mathrm{ad}\,\pi}$, and the unit of $\big(A \otimes B(H_\pi)\big)^{\alpha \otimes \mathrm{ad}\,\pi}$ can be approximated by elements of $A_n(\pi)^* A_n(\pi)$. It follows that for some n, $A_n(\pi)^* A_n(\pi)$ contains an invertible element, and thus equals all of $\big(A_n \otimes B(H_\pi)\big)^{\alpha_n \otimes \mathrm{ad}\,\pi}$. Therefore $\pi \in \widetilde{\mathrm{Sp}}(\alpha_n)$. ■

COROLLARY 2.4. *Let G be compact, and let $\alpha = \bigotimes_{i=1}^\infty \mathrm{ad}\, V_i$ on $A = \bigotimes_{i=1}^\infty M_i$, where V_i is a finite-dimensional unitary representation of G of dimension d_i, and M_i is the full $d_i \times d_i$ matrix algebra. Then the strong Arveson spectrum*

$$\widetilde{\mathrm{Sp}}(\alpha) = \bigcup_{n=1}^{\infty} S\big(\mathrm{sp}(V_1 \otimes \cdots \otimes V_n)\big).$$

PROOF. Apply Propositions 2.2 and 2.3. ■

Note that a statement analogous to Proposition 2.3 for the Arveson spectrum $\mathrm{Sp}(\alpha)$ need not hold (see Remark 4.5). Due to this fact, the computation of the Arveson spectrum is more complicated.

LEMMA 2.5. *Let $A = \overline{\bigcup_n A_n}$ be an inductive limit of a sequence of increasing, finite-dimensional C^*-algebras. For each n, let I_n be an ideal of A_n with $I_n \subseteq I_{n+1}$, and let $I = \overline{\bigcup_n I_n}$. Then I is essential in A if and only if, for each k, there exists $n > k$ such that if $x \in A_k$ and $x \neq 0$, then $I_n x \neq (0)$.*

PROOF. Assume I is essential in A, and let $x \in A_k$, $\|x\| = 1$. Clearly there exists $n_x > k$ and $j \in I_{n_x}$ such that $jx \neq 0$. Thus $jy \neq 0$ for all y in some neighborhood of x, and a simple compactness argument, applied to the unit ball of A_k, proves one direction of the lemma. For the converse, let $J \equiv \{x \in A : Ix = 0\}$. Then J is a closed two-sided ideal of A, with $J = \overline{\bigcup_k (J \cap A_k)}$. As for each k, $I_n \cdot (J \cap A_k) = (0)$ for all n, it follows by hypothesis that for each k, $J \cap A_k = (0)$, and thus $J = (0)$. ■

COROLLARY 2.6. *Let α be an action of the compact group G on A, where $A = \overline{\bigcup_n A_n}$ is the inductive limit of an increasing sequence of finite-dimensional α-invariant subalgebras. Then $\pi \in \mathrm{Sp}(\alpha)$ if and only if, for each k, there exists $n > k$ such that if*

$$x \in (G \times_{\alpha_k} A_k)\chi_{\bar{\pi}}(G \times_{\alpha_k} A_k)$$

and $x \neq 0$, then

$$x(G \times_{\alpha_n} A_n)\chi_{\mathrm{id}}(G \times_{\alpha_n} A_n)\chi_{\bar{\pi}}(G \times_{\alpha_n} A_n)\chi_{\mathrm{id}}(G \times_{\alpha_n} A_n) \neq (0)$$

in $(G \times_{\alpha_n} A_n)\chi_{\bar{\pi}}(G \times_{\alpha_n} A_n)$.

PROOF. It follows from Definition 1.2(a) and Lemma 2.5 that $\pi \in \mathrm{Sp}(\alpha)$ if and only if, for each k, there exists $n > k$ such that if

$$x \in \big(A_k \otimes B(H_\pi)\big)^{\alpha_k \otimes \mathrm{ad}\,\pi}$$

and $x \neq 0$, then

$$xA_n(\pi)^* A_n(\pi) \neq (0).$$

By Lemma 3 of [7], $\big(A_n \otimes B(H_\pi)\big)^{\alpha_n \otimes \mathrm{ad}\,\pi}$ is $*$-isomorphic to the C^*-algebra C of G-central elements of $\chi_{\bar{\pi}}(G \times_{\alpha_n} A)\chi_{\bar{\pi}}$, where a function $f: G \longrightarrow A$ is G-central if $f(t) = \alpha_g f(g^{-1}tg)$, $g, t \in G$. It is clear that the isomorphism of [7] maps the subalgebra $\big(A_k \otimes B(H_\pi)\big)^{\alpha_k \otimes \mathrm{ad}\,\pi}$ onto the C^*-algebra D of G-central elements of $\chi_{\bar{\pi}}(G \times_{\alpha_k} A_k)\chi_{\bar{\pi}}$, and it follows from the proof of Lemma 2.10 of [11] that this isomorphism also maps the ideal $A_n(\pi)^* A_n(\pi)$ onto the ideal I of G-central elements of

$$\chi_{\bar{\pi}}(G \times_{\alpha_n} A_n)\chi_{\mathrm{id}}(G \times_{\alpha_n} A_n)\chi_{\bar{\pi}}.$$

As in the remarks following Lemma 1 of [7], $\chi_{\bar{\pi}}(G \times_{\alpha_n} A_n)\chi_{\bar{\pi}}$ is $*$-isomorphic to $M \otimes C$, where M is a finite dimensional matrix algebra, and routine computations verify that this $*$-isomorphism maps $\chi_{\bar{\pi}}(G \times_{\alpha_k} A_k)\chi_{\bar{\pi}}$ onto $M \otimes D$, and

$$\chi_{\bar{\pi}}(G \times_{\alpha_n} A_n)\chi_{\mathrm{id}}(G \times_{\alpha_n} A_n)\chi_{\bar{\pi}}$$

onto $M \otimes I$. As clearly $xI \neq (0)\ \forall x \neq 0$ in D if and only if $x'(M \otimes I) \neq (0)\ \forall x' \neq 0$ in $M \otimes D$, we have $\pi \in \mathrm{Sp}(\alpha)$ if and only if, for each k, $\exists n > k$ such that if

$$x \in \chi_{\bar{\pi}}(G \times_{\alpha_k} A_k)\chi_{\bar{\pi}}$$

and $x \neq 0$, then

$$x\chi_{\bar{\pi}}(G \times_{\alpha_n} A_n)\chi_{\mathrm{id}}(G \times_{\alpha_n} A_n)\chi_{\bar{\pi}} \neq (0).$$

To finish the corollary, we must pass from $\chi_{\bar{\pi}}(G \times_{\alpha_n} A_n)\chi_{\bar{\pi}}$ to the Morita equivalent algebra $(G \times_{\alpha_n} A_n)\chi_{\bar{\pi}}(G \times_{\alpha_n} A_n)$. The computations are straightforward, but as we have not found any explicit reference, we include them in the following:

LEMMA 2.7. *Let A be a C^*-algebra, B a C^*-subalgebra, and I an ideal of A. Let p be a projection in the multiplier algebra of A with pB, $Bp \subseteq B$. Then the following are equivalent:*

(a) $xpIp \neq (0)$, for all $x \neq 0$ in pBp.

(b) $xIpI \neq (0)$, for all $x \neq 0$ in BpB.

PROOF. (a) $\Rightarrow$ (b). Let $x \in BpB$, and suppose that $xIpI = (0)$. Then $(AxA) \cdot (IpI) = (0)$, so $(pBxBp) \cdot (IpI) = (0)$, and, by using the fact that $Ip \subseteq I$ so that $Ip \subseteq IpI$, we have $(pBxBp) \cdot (pIp) = (0)$. By hypothesis $pBxBp = (0)$, so $(BpB)x(BpB) = (0)$ and $x = 0$ as $x \in BpB$.

(b) $\Rightarrow$ (a). Let $x \in pBp$, and suppose that $xpIp = (0)$. Then $xpIpI = (0)$, and as $xp \in pBp \subseteq BpB$, $xp = 0$. But $x = xp$. ∎

Of course, Corollary 2.6 is proven by applying Lemma 2.7 with $A = G \times_{\alpha_n} A_n$, $B = G \times_{\alpha_k} A_k$, $I = (G \times_{\alpha_n} A_n)\chi_{\mathrm{id}}(G \times_{\alpha_n} A_n)$, and $p = \chi_{\bar{\tau}}$. ∎

A representation-theoretic interpretation of Corollary 2.6 is provided by

PROPOSITION 2.8. *Let A be a C^*-algebra, B a C^*-subalgebra of A with discrete dual $\hat{B}$, and I an ideal in A. Then the following are equivalent:*

(a) $Ix \neq (0)$, $\forall x \in B$, $x \neq 0$

(b) $\forall \pi \in \hat{B}$, $\exists \sigma \in \hat{A}$ such that $\sigma|_I \neq (0)$ and $\sigma|_B \supseteq \pi$.

PROOF. (a) $\Rightarrow$ (b). Let $\pi \in \hat{B}$, and let $B_1 \equiv \pi^{-1}(0)$. Then, as $\hat{B}$ is discrete, there is an ideal B_2 in B such that $B = B_1 \oplus B_2$ and $\ker(\rho) \supseteq B_2$ $\forall \rho \in \hat{B}$ with $\rho \neq \pi$. Let $x \in B_2$ with $x \neq 0$. As $Ix \neq (0)$, $\exists \sigma \in \hat{I} \subseteq \hat{A}$ such that $\sigma(x) \neq 0$. It follows that $\sigma|_B \supseteq \pi$, and we are done.

(b) $\Rightarrow$ (a). Let $x \in B$ with $Ix = (0)$. Let $\pi \in \hat{B}$ and choose $\sigma \in \hat{A}$ with $\sigma|_I \neq (0)$ and $\sigma|_B \supseteq \pi$. As $\sigma(Ix) = \sigma(I)\sigma(x) = 0$ but $\sigma|_I$ is irreducible, $\sigma(x) = 0$ and thus $\pi(x) = 0$. It follows that $x = 0$. ∎

THEOREM 2.9. *Let G be compact, and let $\alpha = \otimes_{i=1}^{\infty} \mathrm{ad}\, V_i$ on $A = \otimes_{i=1}^{\infty} M_i$, where V_i is a finite-dimensional unitary representation of G of dimension d_i, and M_i is the full $d_i \times d_i$ matrix algebra. Let W_n be the representation $V_1 \otimes \cdots \otimes V_n$. Then*

$$\begin{aligned}\mathrm{Sp}(\alpha) = \{\tau \in \hat{G} :\ & \text{for each } k, \text{ there exists } n > k \text{ such that if } \pi \in \hat{G} \\ & \text{and } \pi \subseteq \tau \otimes W_k, \text{ then there exists } \sigma \in \hat{G} \text{ with } \sigma \subseteq \\ & \tau \otimes W_n, \sigma \subseteq W_n \text{ and } \pi \subseteq \sigma \otimes \overline{(V_{k+1} \otimes \cdots \otimes V_n)}\}.\end{aligned}$$

PROOF. Denote $M_1 \otimes \cdots \otimes M_n$ by A_n, and let $\alpha_n = \mathrm{ad}\, W_n$. For ease of notation, we denote $G \times_{\alpha_k} A_k$ simply by C_k. By Corollary 2.6 and Proposition 2.8, $\tau \in \mathrm{Sp}(\alpha)$ if and only if, for each k, $\exists n > k$ such that for each irreducible representation L of $C_k\chi_{\bar{\tau}}C_k$, there exists an irreducible representation R of $C_n\chi_{\bar{\tau}}C_n$ such that

$$R|_{C_n\chi_{\mathrm{id}}C_n\chi_{\bar{\tau}}C_n\chi_{\mathrm{id}}C_n} \neq (0)$$

and $R|_{C_k\chi_{\bar{\tau}}C_k} \supseteq L$. Now the irreducible representations R of $C_n\chi_{\bar{\tau}}C_n$ correspond to the irreducible covariant pairs $\langle \eta, \rho \rangle$ of (G, A_n, α_n) for which $\eta \supseteq \bar{\tau}$. By the Mackey machine, these are precisely the irreducible covariant pairs of the form $\langle \sigma\otimes W_n, I\otimes i\rangle$ such that $\sigma\otimes W_n \supseteq \bar{\tau}$, where $\sigma \in \hat{G}$, I is the identity operator on H_σ, and i is the identity representation of A_n. Letting $R = \langle \sigma\otimes W_n, I\otimes i\rangle$, we have that

$$R|_{C_n\chi_{\mathrm{id}}C_n\chi_{\bar{\tau}}C_n\chi_{\mathrm{id}}C_n} \neq (0)$$

if and only if $\sigma\otimes W_n \supseteq$ id also (id being the identity representation of G), *i.e.*, if and only if $\overline{W_n} \supseteq \sigma$. A trivial calculation also shows that if $L = \langle \pi\otimes W_k, I\otimes i\rangle$ is an irreducible representation of $C_k\chi_{\bar{\tau}}C_k$, for $\pi \in \hat{G}$ with (*a priori*) $\pi\otimes W_k \supseteq \bar{\tau}$, then $R|_{C_k\chi_{\bar{\tau}}C_k} \supseteq L$ if and only if $\sigma\otimes(V_{k+1}\otimes\cdots\otimes V_n) \supseteq \pi$. We thus have that $\tau \in \mathrm{Sp}(\alpha)$ if and only if, for each k, $\exists\, n > k$ such that if $\pi \in \hat{G}$ and $\bar{\tau} \subseteq \pi\otimes W_k$, then for some $\sigma \in \hat{G}$,

$$\bar{\tau} \subseteq \sigma\otimes W_n, \quad \mathrm{id} \subseteq \sigma\otimes W_n \quad \text{and} \quad \pi \subseteq \sigma\otimes(V_{k+1}\otimes\cdots\otimes V_n).$$

To finish the proof, replace π by $\bar{\pi}$, σ by $\bar{\sigma}$, and observe that $\bar{\tau} \subseteq \bar{\pi}\otimes W_k$ iff $\pi \subseteq \tau\otimes W_k$, $\bar{\tau} \subseteq \bar{\sigma}\otimes W_n$ iff $\sigma \subseteq \tau\otimes W_n$, id $\subseteq \bar{\sigma}\otimes W_n$ iff $\sigma \subseteq W_n$, and $\bar{\pi} \subseteq \bar{\sigma}\otimes(V_{k+1}\otimes\cdots\otimes V_n)$ iff $\pi \subseteq \sigma\otimes(\overline{V_{k+1}\otimes\cdots\otimes V_n})$. ∎

REMARK 2.10. To better understand Theorem 2.9, observe that if $\pi \subseteq \tau\otimes W_k$, then

$$\pi = \pi\otimes\mathrm{id} \subseteq (\tau\otimes W_k)\otimes(V_{k+1}\otimes\cdots\otimes V_n)\otimes(\overline{V_{k+1}\otimes\cdots\otimes V_n}) \tag{1}$$
$$= (\tau\otimes W_n)\otimes(\overline{V_{k+1}\otimes\cdots\otimes V_n}), \tag{2}$$

so that $\pi \subseteq \sigma\otimes(\overline{V_{k+1}\otimes\cdots\otimes V_n})$ for some $\sigma \in \hat{G}$ with $\sigma \subseteq \tau\otimes W_n$. Theorem 2.9 states that for τ to be in $\mathrm{Sp}(\alpha)$, we must be able to choose $\sigma \subseteq W_n$ also. Note that if $\tau \in S\big(\mathrm{sp}(W_n)\big)$, this follows automatically.

3. **Connes type spectra.**

To pass from computing Arveson type spectra to Connes type spectra, we must be able to handle hereditary subalgebras. Thus we must consider the case of non-unital inductive limits, for which we have the following result, much weaker than Proposition 2.3.

LEMMA 3.1. *Let α be an action of the compact group G on $B = \overline{\bigcup_k B_k}$, and suppose that the algebras B_k are α-invariant and increasing. Then $\bigcap_k \mathrm{Sp}(\alpha_k) \subseteq \mathrm{Sp}(\alpha)$, $\bigcap_k \widetilde{\mathrm{Sp}}(\alpha_k) \subseteq \widetilde{\mathrm{Sp}}(\alpha)$.*

PROOF. Let $\pi \in \bigcap_k \mathrm{Sp}(\alpha_k)$, let J be the annihilator of $B(\pi)^*B(\pi)$ in $\big(B\otimes B(H_\pi)\big)^{\alpha\otimes\mathrm{ad}\,\pi}$, and let $J_k = J\cap\big(B_k\otimes B(H_\pi)\big)^{\alpha_k\otimes\mathrm{ad}\,\pi}$. As

$$\big(B\otimes B(H_\pi)\big)^{\alpha\otimes\mathrm{ad}\,\pi} = \overline{\bigcup_k \big(B_k\otimes B(H_\pi)\big)^{\alpha_k\otimes\mathrm{ad}\,\pi}},$$

$J = \overline{\bigcup_k J_k}$. As J_k annihilates $B(\pi)^*B(\pi)$, it annihilates the subalgebra $B_k(\pi)^*B_k(\pi)$, and thus $J_k = (0)$ for all k, as $\pi \in \mathrm{Sp}(\alpha_k)$ for all k. Thus $J = (0)$ and $\pi \in \mathrm{Sp}(\alpha)$.

A similar, but even easier, proof works for the strong Arveson spectrum $\widetilde{\mathrm{Sp}}$. ∎

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LEMMA 3.2. *Let α be an action of the compact group G on the algebra A, where $A = \overline{\bigcup_n A_n}$, $\{A_n\}$ being an increasing sequence of α-invariant finite dimensional subalgebras. Then any α-invariant hereditary C^*-subalgebra of A has an increasing approximate unit consisting of α-invariant projections.*

PROOF. Let B be an α-invariant hereditary C^*-subalgebra of A and let h be a strictly positive element of B. By integrating over G, we may suppose that h is α-invariant. By the proof of [10, Section 3.10.5], B thus has an approximate unit consisting of α-invariant elements. It follows easily that any approximate unit for B^α is also an approximate unit for B. As B^α is AF, it has an increasing approximate unit of projections, and we are done. ∎

PROPOSITION 3.3. *Let $A = \overline{\bigcup_n A_n}$, G, α be as above. Denote by $\mathcal{P}$ the family of all α-invariant projections in $\bigcup_n A_n$. Then*

$$\Gamma(\alpha) = \bigcap_{p\in\mathcal{P}} \mathrm{Sp}(\alpha|_{pAp}),$$

and

$$\tilde{\Gamma}(\alpha) = \bigcap_{p\in\mathcal{P}} \widetilde{\mathrm{Sp}}(\alpha|_{pAp}).$$

PROOF. Clearly $\Gamma(\alpha) \subseteq \bigcap_{p\in\mathcal{P}} \mathrm{Sp}(\alpha|_{pAp})$. Let B be an α-invariant hereditary C^*-subalgebra of A. We want to show that $\mathrm{Sp}(\alpha|_B) \supseteq \bigcap_{p\in\mathcal{P}} \mathrm{Sp}(\alpha|_{pAp})$. Now B has an approximate unit $\{q_k\}$ of α-invariant projections (Lemma 3.2) and $\mathrm{Sp}(\alpha|_B) \supseteq \bigcap_k \mathrm{Sp}(\alpha|_{q_kBq_k})$ by Lemma 3.1. As $q_k \in A^\alpha$ and, by integration over G, $A^\alpha = \overline{\bigcup_n A_n^\alpha}$, we have from [3, Lemma 1.6] that there is a projection $p_k \in \bigcup_n A_n^\alpha$ as close as we wish to q_k. Applying now [3, Lemma 1.8] we get a partial isometry $w_k \in A^\alpha$ from q_k to p_k. Conjugation by w_k is an α-equivariant $*$-isomorphism of $q_kBq_k = q_kAq_k$ onto p_kAp_k, so that $\mathrm{Sp}(\alpha|_{q_kBq_k}) = \mathrm{Sp}(\alpha|_{p_kAp_k})$. Thus

$$\mathrm{Sp}(\alpha|_B) \supseteq \bigcap_k \mathrm{Sp}(\alpha|_{p_kAp_k}) \supseteq \bigcap_{p\in\mathcal{P}} \mathrm{Sp}(\alpha|_{pAp}).$$

The proof relating the strong Arveson and Connes spectra is identical, and we omit it. ∎

Now for the action $\alpha = \otimes_{i=1}^\infty \operatorname{ad} V_i$ on $A = \otimes_{i=1}^\infty M_i$, where V_i is a finite-dimensional representation of dimension d_i, and M_i is the full $d_i \times d_i$ matrix algebra, the above proposition shows that to compute both the Connes spectrum $\Gamma(\alpha)$ and the strong Connes spectrum $\tilde{\Gamma}(\alpha)$, we need only intersect the Arveson and strong Arveson spectra, respectively, for actions of the form $\alpha' = \operatorname{ad} W \otimes (\otimes_{i=n+1}^\infty \operatorname{ad} V_i)$ on $B = M_W \otimes (\otimes_{i=n+1}^\infty M_i)$, where $n < \infty$, W is a subrepresentation of $V_1 \otimes \cdots \otimes V_n$ (of dimension $d_W > 0$), and M_W is the full $d_W \times d_W$ matrix algebra. In other words, we need consider only the Arveson and strong Arveson spectra, respectively, of actions which are again of tensor product form. We thus have the following

THEOREM 3.4. *Let G be compact, and let $\alpha = \otimes_{i=1}^{\infty} \operatorname{ad} V_i$ on $A = \otimes_{i=1}^{\infty} M_i$, where V_i is a finite-dimensional unitary representation of G of dimension d_i, and M_i is the full $d_i \times d_i$ matrix algebra. Then*

$$\tilde{\Gamma}(\alpha) = \bigcap_{1 \leq n < \infty} \left[\bigcap_{W \subseteq V_1 \otimes \cdots \otimes V_n} \bigcup_{k=n+1}^{\infty} S\left(\operatorname{sp}\left(W \otimes \left(\bigotimes_{i=n+1}^{k} V_i \right) \right) \right) \right].$$

PROOF. The proof follows from Proposition 3.3, the above remarks, and Corollary 2.4. ∎

THEOREM 3.5. *Let G be compact, and let $\alpha = \otimes_{i=1}^{\infty} \operatorname{ad} V_i$ on $A = \otimes_{i=1}^{\infty} M_i$, where V_i is a finite-dimensional unitary representation of G of dimension d_i, and M_i is the full $d_i \times d_i$ matrix algebra. Then the following conditions are equivalent:*

a) $\tau \in \Gamma(\alpha)$.

b) For each $\ell \geq 1$ and each subrepresentation $W \subseteq V_1 \otimes \cdots \otimes V_\ell$, $\exists n > \ell$ such that if $\pi \in \hat{G}$ and $\pi \subseteq \tau \otimes W$, then there exists $\sigma \in \hat{G}$ with $\sigma \subseteq \tau \otimes W \otimes V_{\ell+1} \otimes \cdots \otimes V_n$, $\sigma \subseteq W \otimes V_{\ell+1} \otimes \cdots \otimes V_n$, and $\pi \subseteq \sigma \otimes (\overline{V_{\ell+1} \otimes \cdots \otimes V_n})$.

PROOF. By Proposition 3.3, the remarks preceding Theorem 3.4, and Theorem 2.9, $\tau \in \Gamma(\alpha)$ if and only if

(∗) for each $\ell \geq 1$, each subrepresentation $W \subseteq V_1 \otimes \cdots \otimes V_\ell$, and each integer $k \geq \ell$, $\exists n > k$ such that if $\pi \in \hat{G}$ and $\pi \subseteq \tau \otimes W \otimes V_{\ell+1} \otimes \cdots \otimes V_k$ (where $\tau \otimes W \otimes V_{\ell+1} \otimes \cdots \otimes V_k$ simply means $\tau \otimes W$ when $k = \ell$), then there exists $\sigma \in \hat{G}$ with $\sigma \subseteq \tau \otimes W \otimes V_{\ell+1} \otimes \cdots \otimes V_n$, $\sigma \subseteq W \otimes V_{\ell+1} \otimes \cdots \otimes V_n$, and $\pi \subseteq \sigma \otimes (\overline{V_{k+1} \otimes \cdots \otimes V_n})$.

If $\tau \in \Gamma(\alpha)$, apply (∗) with $\ell = k$ to obtain (b). Now assume (b), and let $\ell \geq 1$, W a subrepresentation of $V_1 \otimes \cdots \otimes V_\ell$, and $k \geq \ell$. Apply (b) to the integer k and the subrepresentation $W \otimes V_{\ell+1} \otimes \cdots \otimes V_k$ of $V_1 \otimes \cdots \otimes V_k$, to obtain an integer $n > k$ such that if

$$\pi \in \hat{G} \quad \text{with} \quad \pi \subseteq \tau \otimes W \otimes V_{\ell+1} \otimes \cdots \otimes V_k,$$
$$\exists \sigma \in \hat{G} \quad \text{with} \quad \sigma \subseteq (\tau \otimes W \otimes V_{\ell+1} \otimes \cdots \otimes V_k) \otimes (V_{k+1} \otimes \cdots \otimes V_n),$$
$$\sigma \subseteq (W \otimes V_{\ell+1} \otimes \cdots \otimes V_k) \otimes (V_{k+1} \otimes \cdots \otimes V_n),$$

and

$$\pi \subseteq \sigma \otimes (\overline{V_{k+1} \otimes \cdots \otimes V_n}).$$

Hence (∗) holds and $\tau \in \Gamma(\alpha)$. ∎

REMARK 3.6. If G is in addition abelian, it is easy to see that Theorem 3.5 simplifies to give the formula for $\Gamma(\alpha)$ presented in Section 4 of [2].

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4. **Examples.** Our first goal is to show, as was done in [1] for abelian groups, that if G contains a non-trivial connected group as a direct summand, then $G\times_\alpha A$ is not simple, where $\alpha = \otimes_{i=1}^\infty \operatorname{ad} V_i$ on $A = \otimes_{i=1}^\infty M_i$.

LEMMA 4.1. *Let G be a compact connected group, and let V be a finite-dimensional unitary representation of G. Then $S\big(\operatorname{sp}(V)\big) = \{\mathrm{id}\}$.*

PROOF. As a first step, let G be a compact connected Lie group, and let $\tau \in \hat{G}$ with $\tau \in S\big(\operatorname{sp}(V)\big)$. Denoting by χ_ρ the character of $\rho \in \hat{G}$, we have, $\forall \sigma \in \operatorname{sp}(V)$, that

$$\chi_\tau\chi_\sigma = \sum_{\rho\in\operatorname{sp}(V)} m(\rho,\sigma)\chi_\rho,$$

where $m(\rho,\sigma)$ are non-negative integers, not all of them zero. Let T^m be a maximal torus in G, and for $\rho \in \hat{G}$ let $\xi_\rho = \chi_\rho|_{T^m}$. Then $\xi_\rho(z_1,\ldots,z_m)$ is a linear combination, with non-negative integer coefficients, not all zero, of monomials $\prod_{i=1}^m z_i^{k_i}$, $k_i \in Z$. Let ℓ_1 be the highest power of z_1 appearing in ξ_τ and pick $\sigma \in \operatorname{sp}(V)$ which maximizes, among all $\rho \in \operatorname{sp}(V)$, the highest power of z_1 appearing in ξ_ρ. As

$$\xi_\tau\xi_\sigma = \sum_{\rho\in\operatorname{sp}(V)} m(\rho,\sigma)\xi_\rho,$$

clearly $\ell_1 \leq 0$. By picking $\sigma' \in \operatorname{sp}(V)$ which minimizes, among all $\rho \in \operatorname{sp}(V)$, the lowest power of z_1 appearing in ξ_ρ, we get $\ell_1 \geq 0$ also. Arguing analogously with the lowest power of z_1 appearing in ξ_τ, and with each of the other variables, we have that ξ_τ is a natural number. As τ is determined by ξ_τ, $\tau = \mathrm{id}$. For G an arbitrary compact connected group, we use the fact that G is a projective limit of compact connected Lie groups. Thus, for some closed normal subgroup N of G, G/N is a compact connected Lie group, and τ and each $\sigma \in \operatorname{sp}(V)$ can be factored through G/N, so that the above discussion applies. ∎

The following lemmas show how to easily compute $\Gamma(\alpha)$ and $\tilde{\Gamma}(\alpha)$ in special cases. (*cf.* Proposition 4.1 of [2]).

LEMMA 4.2. *Let G be compact, and let $\alpha = \otimes_{i=1}^\infty \operatorname{ad} V_i$ on $A = \otimes_{i=1}^\infty M_i$, where V_i is a finite-dimensional unitary representation of G of dimension d_i, and M_i is the full $d_i \times d_i$ matrix algebra. Let*

$$\Gamma_1(\alpha) \equiv \{\tau \in \hat{G} : \textit{for each } \ell \geq 1, \exists n > \ell \textit{ with } \tau \subseteq (V_{\ell+1}\otimes\cdots\otimes V_n)\otimes\overline{(V_{\ell+1}\otimes\cdots\otimes V_n)}\}.$$

Then $\Gamma_1(\alpha) \subseteq \Gamma(\alpha)$ and, if $\mathrm{id} \subseteq V_1\otimes\cdots\otimes V_\ell$ for all ℓ, $\Gamma_1(\alpha) = \Gamma(\alpha)$.

PROOF. Let $\tau \in \Gamma_1(\alpha)$, $\ell \geq 1$ and $W \subseteq V_1\otimes\cdots\otimes V_\ell$. For some $n > \ell$ and $\rho,\eta \in \hat{G}$ with $\rho,\eta \subseteq V_{\ell+1}\otimes\cdots\otimes V_n$, we have $\tau \subseteq \rho\otimes\bar{\eta}$. If $\pi \in \hat{G}$ with $\pi \subseteq \tau\otimes W$, then $\pi \subseteq \rho\otimes\bar{\eta}\otimes W$, and $\pi \subseteq \sigma\otimes\bar{\eta}$ for some $\sigma \in \hat{G}$ with $\sigma \subseteq \rho\otimes W$. Clearly $\pi \subseteq \sigma\otimes\overline{(V_{\ell+1}\otimes\cdots\otimes V_n)}$ and $\sigma \subseteq W\otimes(V_{\ell+1}\otimes\cdots\otimes V_n)$. To show, by Theorem 3.5, that $\tau \in \Gamma(\alpha)$, we need only show that $\sigma \subseteq \tau\otimes W\otimes V_{\ell+1}\otimes\cdots\otimes V_n$ also. As $\tau \subseteq \rho\otimes\bar{\eta}$, $\rho \subseteq \tau\otimes\eta$. Thus, as $\sigma \subseteq \rho\otimes W$,

$$\sigma \subseteq \tau\otimes\eta\otimes W \subseteq \tau\otimes W\otimes V_{\ell+1}\otimes\cdots\otimes V_n,$$

and $\tau \in \Gamma(\alpha)$. Now assume $\tau \in \Gamma(\alpha)$ and that $\mathrm{id} \subseteq V_1 \otimes \cdots \otimes V_\ell$ for all ℓ. That $\tau \in \Gamma_1(\alpha)$ follows by applying Theorem 3.5 with $W = \mathrm{id} \subseteq V_1 \otimes \cdots \otimes V_\ell$. ∎

In a similar vein, we have

LEMMA 4.3. *Let G, A, α and V_i be as in Lemma 4.2. Let*

$$\tilde{\Gamma}_1(\alpha) \equiv \big\{\tau \in \hat{G} : \textit{for each } \ell \geq 1, \exists n > \ell \textit{ with } \tau \in S\big(\mathrm{sp}(V_{\ell+1} \otimes \cdots \otimes V_n)\big)\big\}.$$

Then $\tilde{\Gamma}_1(\alpha) \subseteq \tilde{\Gamma}(\alpha)$ and, if $\mathrm{id} \subseteq V_1 \otimes \cdots \otimes V_\ell$ for all ℓ, $\tilde{\Gamma}_1(\alpha) = \tilde{\Gamma}(\alpha)$.

PROOF. Similar to that of Lemma 4.2. ∎

PROPOSITION 4.4 (*cf.* [1, COROLLARY 2.4 AND THEOREM 3.1]). *Let G be compact, and let $\alpha = \otimes_{i=1}^{\infty} \mathrm{ad}\, V_i$ on $A = \otimes_{i=1}^{\infty} M_i$, where V_i is a finite-dimensional unitary representation of G of dimension d_i, and M_i is the full $d_i \times d_i$ matrix algebra. If G contains a non-trivial connected group as a direct summand, then $G \times_\alpha A$ is not simple, although it may be prime.*

PROOF. Let $G = A \times B$, with A connected and non-trivial. It follows easily from Lemma 4.1 that for any finite-dimensional representation V of G,

$$S\big(\mathrm{sp}(V)\big) \subseteq \{\mathrm{id}_A\} \times \hat{B},$$

and thus by Theorem 3.4 that $\tilde{\Gamma}(\alpha) \neq \hat{G}$, so that $G \times_\alpha A$ is not simple, by Theorem 1.5(b). However, if we choose V_i so that $\mathrm{id} \subseteq V_i$ for each i, and $\mathrm{sp}(V_i)$ is an increasing sequence of subsets of $\hat{G}$ whose union equals $\hat{G}$, it follows from Lemma 4.2 that $\Gamma(\alpha) = \hat{G}$ and thus, by Theorem 1.5(a), that $G \times_\alpha A$ is prime. ∎

REMARK 4.5. The above example shows that the analogue of Proposition 2.3 need not hold for $\mathrm{Sp}(\alpha)$. For if it did, then by Proposition 2.2 and Proposition 3.3, we would have $\Gamma(\alpha) = \tilde{\Gamma}(\alpha)$ for all $\alpha = \otimes_{i=1}^{\infty} \mathrm{ad}\, V_i$.

REMARK 4.6. At the other extreme, if G is compact and discrete, (*i.e.*, finite) then it follows from Proposition 4.2 of [5] that $\Gamma(\alpha) = \tilde{\Gamma}(\alpha)$ for all actions α of the type we have been considering, thus providing an analogue for compact groups of Theorem 3.8 of [1].

Our second class of examples involves the phenomenon of spectral instability, as discussed in Section 1. Let G be a finite group with a non-trivial one dimensional representation χ, and an irreducible representation π of dimension $n > 1$, such that $\chi \otimes \pi$ and π are unitarily equivalent. Examples of such groups are all the symmetric groups S_d with $d \geq 3$, and A_4. Note also, by Lemma 4.1, that no such example can be found in a compact connected group.

PROPOSITION 4.7. *Let G, χ, and π be as above, let α be the action* $\mathrm{ad}\,\pi$ *of G on M_n, and let $\hat{\hat{\alpha}}$ be the double-dual action of G on $G \times_{\hat{\alpha}} (G \times_\alpha M_n)$. Then*

$$\chi \in \Gamma(\alpha) = \tilde{\Gamma}(\alpha),$$

while

$$\Gamma(\hat{\alpha}) = \tilde{\Gamma}(\hat{\alpha}) = \{\mathrm{id}\}.$$

PROOF. That $\Gamma(\alpha) = \tilde{\Gamma}(\alpha)$, and $\Gamma(\hat{\alpha}) = \tilde{\Gamma}(\hat{\alpha})$, follow from Proposition 4.2 of [5]. As π is irreducible, M_n has no non-trivial α-invariant hereditary C^*-subalgebras, and thus $\Gamma(\alpha) = \mathrm{Sp}(\alpha)$, $\tilde{\Gamma}(\alpha) = \widetilde{\mathrm{Sp}}(\alpha)$. That $\chi \in \Gamma(\alpha) = \tilde{\Gamma}(\alpha)$ follows immediately from Proposition 2.2. For the second assertion of the proposition, we employ Vallin's duality theorem [12, Théorème 5.2] which states that $G \times_{\hat{\alpha}} (G \times_\alpha M_n)$ with the action $\hat{\hat{\alpha}}$ is equivariantly $*$-isomorphic to $B\big(\ell^2(G)\big) \otimes M_n$ with the action $\mathrm{ad}(\rho \otimes \pi)$, ρ being the right regular representation of G on $\ell^2(G)$. But $\rho \otimes \pi$ is a multiple of ρ, and $\mathrm{id} \subseteq \rho$, so there is an $\hat{\hat{\alpha}}$-invariant hereditary subalgebra $A \subseteq B\big(\ell^2(G)\big) \otimes M_n$ on which the action $\hat{\hat{\alpha}}$ equals $\mathrm{ad}\,\mathrm{id}$. By Proposition 2.2, $\mathrm{Sp}(\mathrm{ad}\,\mathrm{id}) = \{\mathrm{id}\}$, and thus $\Gamma(\hat{\alpha}) = \tilde{\Gamma}(\hat{\alpha}) = \{\mathrm{id}\}$. ∎

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