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On Universal Schauder Bases in Non-Archimedean Fréchet Spaces

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Abstract. It is known that any non-archimedean Fréchet space of countable type is isomorphic to a subspace of $c_0^{\mathbb{N}}$. In this paper we prove that there exists a non-archimedean Fréchet space U with a basis (u_n) such that any basis (x_n) in a non-archimedean Fréchet space X is equivalent to a subbasis (u_{k_n}) of (u_n). Then any non-archimedean Fréchet space with a basis is isomorphic to a complemented subspace of U. In contrast to this, we show that a non-archimedean Fréchet space X with a basis (x_n) is isomorphic to a complemented subspace of $c_0^{\mathbb{N}}$ if and only if X is isomorphic to one of the following spaces: c_0, $c_0 \times \mathbb{K}^{\mathbb{N}}$, $\mathbb{K}^{\mathbb{N}}$, $c_0^{\mathbb{N}}$. Finally, we prove that there is no nuclear non-archimedean Fréchet space H with a basis (h_n) such that any basis (y_n) in a nuclear non-archimedean Fréchet space Y is equivalent to a subbasis (h_{k_n}) of (h_n).

Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot|\colon \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [5], [7] and [6]. Bases in locally convex spaces are studied in [1]–[4].

Any infinite-dimensional Banach space E of countable type is isomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero (with the sup-norm $\|\cdot\|$)([6], Theorem 3.16), so E has a basis which is equivalent to the coordinate basis in c_0.

There exist Fréchet spaces of countable type without bases (see [9, Theorem 3]). However, any infinite-dimensional Fréchet space E contains a closed subspace X with a basis (x_n) (see [8, Theorem 2]). Moreover, any infinite-dimensional Fréchet space G of finite type is isomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}}$ of all sequences in $\mathbb{K}$ with the topology of pointwise convergence (see [3, Theorem 3.5]), so G has a basis which is equivalent to the coordinate basis in $\mathbb{K}^{\mathbb{N}}$.

Let $\mathcal{F}$ be the family of all bases in Fréchet spaces and let $\mathcal{F}_0 \subset \mathcal{F}$. A basis (u_n) is *universal* (respectively *quasi-universal*) for $\mathcal{F}_0$ if $(u_n) \in \mathcal{F}_0$ and any basis $(x_n) \in \mathcal{F}_0$ is equivalent (respectively quasi-equivalent) to a subbasis (u_{k_n}) of (u_n).

In this paper we study the existence and the uniqueness of universal bases for some important subfamilies of $\mathcal{F}$.

First, we show that there exists a universal basis for the family $\mathcal{F}_b$ of all bases in Banach spaces and any two universal bases for $\mathcal{F}_b$ are permutatively equivalent (Proposition 1).

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Next, we prove that there is a universal basis for the family $\mathcal{F}_c$ of all bases in Fréchet spaces with continuous norms and any two universal bases for $\mathcal{F}_c$ are permutatively equivalent (Theorem 2). A similar result we also show for the family $\mathcal{F}$ (Theorem 6).

It is known that the Fréchet space $c_0^{\mathbb{N}}$ is universal for the family of all Fréchet spaces of countable type, that is, $c_0^{\mathbb{N}}$ is of countable type and any Fréchet space of countable type is isomorphic to a subspace of $c_0^{\mathbb{N}}$ (see [3, Remark 3.6]). We prove that a Fréchet space E with a basis (x_n) is isomorphic to a complemented subspace of $c_0^{\mathbb{N}}$ if and only if E is isomorphic to one of the following spaces: c_0, $c_0 \times \mathbb{K}^{\mathbb{N}}$, $\mathbb{K}^{\mathbb{N}}$, $c_0^{\mathbb{N}}$ (Theorem 7). In contrast to this, if U is a Fréchet space with a basis which is universal for $\mathcal{F}$, then any Fréchet space with a basis is isomorphic to a complemented subspace of U. It is unknown whether there exists a Fréchet space F of countable type such that any Fréchet space of countable type is isomorphic to a complemented subspace of F. By Remark 9 there is a Fréchet space X of countable type that is not isomorphic to any complemented subspace of a Fréchet space with a basis.

Finally, we prove that there exists no quasi-universal basis for the family $\mathcal{F}_n$ of all bases in nuclear Fréchet spaces or for the family $\mathcal{F}_{nc}$ of all bases in nuclear Fréchet spaces with continuous norms (Theorem 10). In particular, there is no universal basis for $\mathcal{F}_n$ or $\mathcal{F}_{nc}$.

Preliminaries

We will denote by $\mathbb{N}$, Z, Q and $\mathbb{R}$ the sets of all positive integers, all integers, all rational numbers and all real numbers, respectively.

The linear span of a subset A of a linear space E is denoted by $\operatorname{lin} A$.

Let E, F be locally convex spaces. A map $T\colon E \to F$ is called a *linear homeomorphism* if T is linear, one-to-one, surjective and the maps T, T^{-1} are continuous. E is *isomorphic* to F ($E \simeq F$) if there exists a linear homeomorphism $T\colon E \to F$.

A *Fréchet space* is a metrizable complete lcs. A *Banach space* is a normable Fréchet space. Every n-dimensional lcs is isomorphic to the Banach space $\mathbb{K}^n$.

Let (x_n) be a sequence in a Fréchet space E. The series $\sum_{n=1}^{\infty} x_n$ is convergent in E if and only if $\lim x_n = 0$.

Let (x_n) and (y_n) be sequences in locally convex spaces E and F, respectively. We say that:

> (x_n) is *equivalent* to (y_n) if there is a linear homeomorphism $T\colon \operatorname{lin}(x_n) \to \operatorname{lin}(y_n)$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$; (x_n) is *permutatively equivalent* to (y_n) if (x_n) is equivalent to a permutation $(y_{\pi(n)})$ of (y_n); (x_n) is *semi equivalent* to (y_n) if (x_n) is equivalent to $(\alpha_n y_n)$ for some $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$; (x_n) is *quasi-equivalent* to (y_n) if (x_n) is semi equivalent to a permutation $(y_{\pi(n)})$ of (y_n).

A sequence (x_n) in a Fréchet space E is equivalent to a sequence (y_n) in a Fréchet space F if and only if there is a linear homeomorphism T between the closed linear spans of (x_n) and (y_n) such that $Tx_n = y_n$, $n \in \mathbb{N}$.

A sequence (x_n) in a lcs E is a *basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$. If additionally the coefficient functionals $f_n\colon E \to \mathbb{K}$, $x \to \alpha_n$ ($n \in \mathbb{N}$) are continuous, then (x_n) is a *Schauder basis* in E.

A subsequence (x_{k_n}) of a basis (x_n) in a lcs E is a *subbasis* of (x_n).

As in the real or complex case any basis in a Fréchet space is a Schauder basis (see [4, Corollary 4.2]).

By a *seminorm* on a linear space E we mean a function $p\colon E \to [0,\infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and $p(x+y) \le \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$.

Two norms p, q on a linear space E are *equivalent* if there exist positive numbers a, b such that $ap(x) \le q(x) \le bp(x)$ for each $x \in E$. Every two norms on a finite-dimensional linear space are equivalent.

The set of all continuous seminorms on a metrizable lcs E is denoted by $\mathcal{P}(E)$. A non-decreasing sequence $(p_k) \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $k \in \mathbb{N}$ with $p \le p_k$. A sequence (p_k) of norms on E is a *base of norms* in $\mathcal{P}(E)$ if it is a base in $\mathcal{P}(E)$.

Any metrizable lcs E possesses a base (p_k) in $\mathcal{P}(E)$. Every metrizable lcs E with a continuous norm has a base of norms (p_k) in $\mathcal{P}(E)$.

A metrizable lcs E is *of finite type* if $\dim(E/\ker p) < \infty$ for any $p \in \mathcal{P}(E)$, and *of countable type* if E contains a linearly dense countable set.

Let p be a seminorm on a linear space E. A sequence $(x_n) \subset E$ is 1-*orthogonal with respect to* p if $p(\sum_{i=1}^n \alpha_i x_i) = \max_{1\le i\le n} p(\alpha_i x_i)$ for all $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in \mathbb{K}$.

A sequence (x_n) in a metrizable lcs E is 1-*orthogonal with respect to* $(p_k) \subset \mathcal{P}(E)$ if (x_n) is 1-orthogonal with respect to p_k for any $k \in \mathbb{N}$.

A sequence (x_n) in a metrizable lcs E is *orthogonal* if it is 1-orthogonal with respect to a base (p_k) in $\mathcal{P}(E)$. (In [6] a sequence (x_n) in a normed space $(E, |||\cdot|||)$ is called *orthogonal* if it is 1-orthogonal with respect to the norm $|||\cdot|||$.)

A linearly dense orthogonal sequence of non-zero elements in a metrizable lcs E is an *orthogonal basis* in E.

Every orthogonal basis in a metrizable lcs E is a Schauder basis in E (see [3, Proposition 1.4]) and every Schauder basis in a Fréchet space F is an orthogonal basis in F (see [3, Proposition 1.7]).

Let $B = (b_{k,n})$ be an infinite real matrix such that $\forall k, n \in \mathbb{N} : 0 < b_{k,n} \le b_{k+1,n}$.

The space $K(B) = \{(\alpha_n) \subset \mathbb{K} : \lim_n |\alpha_n| b_{k,n} = 0 \text{ for all } k \in \mathbb{N}\}$ with the base of norms (p_k): $p_k\big((\alpha_n)\big) = k \max_n |\alpha_n| b_{k,n}$, $k \in \mathbb{N}$, is called the *Köthe space* associated with the matrix B. $K(B)$ is a Fréchet space and the sequence (e_n) of coordinate vectors forms a basis in it (see [2, Proposition 2.2]). The coordinate basis (e_n) is 1-orthogonal with respect to the base of norms (p_k).

Let E be a Fréchet space with a Schauder basis (x_n) which is 1-orthogonal with respect to a base of norms (p_k) in $\mathcal{P}(E)$. Then E is nuclear if and only if

$$\forall k \in \mathbb{N}\ \exists m > k : \lim_n [p_k(x_n)/p_m(x_n)] = 0$$

(see [2, Propositions 2.4 and 3.5]).

Results

We start with the following proposition.

Proposition 1 *There exists a universal basis (w_n) for the family $\mathcal{F}_b$ of all bases in Banach spaces. A basis $(x_n) \in \mathcal{F}_b$ is universal for $\mathcal{F}_b$ if and only if it is permutatively equivalent to (w_n).*

Proof It is easy to check that an orthogonal sequence (x_n) in a Banach space $(X, \|\cdot\|_X)$ is equivalent to an orthogonal sequence (y_n) in a Banach space $(Y, \|\cdot\|_Y)$ if and only if there exists a number $A \geq 1$ such that

$$\forall n \in \mathbb{N} : A^{-1}|||x_n|||_X \leq |||y_n|||_Y \leq A|||x_n|||_X.$$

Let $\{N_t : t \in Z\}$ be a family of pairwise disjoint infinite subsets of $\mathbb{N}$ such that $\bigcup\{N_t : t \in Z\} = \mathbb{N}$. Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. Denote by (e_n) the coordinate basis in c_0. Put $w_n = \alpha^t e_n$ for all $n \in N_t, t \in Z$. Clearly, (w_n) is a basis in c_0.

Let π be a permutation of $\mathbb{N}$ and let (y_n) be a basis in a Banach space $(Y, |||\cdot|||)$. We can choose an increasing sequence $(k_n) \subset \mathbb{N}$ such that

$$\forall n \in \mathbb{N}\, \forall t \in Z : [k_n \in \pi^{-1}(N_t) \Leftrightarrow |\alpha|^t \leq |||y_n||| < |\alpha|^{t+1}].$$

Then $\forall n \in \mathbb{N} : \|w_{\pi(k_n)}\| \leq |||y_n||| < |\alpha|\, \|w_{\pi(k_n)}\|$. It follows that (y_n) is equivalent to $(w_{\pi(k_n)})$. Thus for any permutation π of $\mathbb{N}$ the basis $(w_{\pi(n)})$ in c_0 is universal for $\mathcal{F}_b$. Hence any basis $(x_n) \in \mathcal{F}_b$ which is permutatively equivalent to (w_n) is universal for $\mathcal{F}_b$.

Now, let us assume that a basis (x_n) in a Banach space $(X, |||\cdot|||)$ is universal for $\mathcal{F}_b$. Then the basis (w_n) is equivalent to a subbasis (x_{s_n}) of (x_n). Thus there exists $t_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N} : |\alpha|^{-t_0}\|w_n\| \leq |||x_{s_n}||| < |\alpha|^{t_0}\|w_n\|.$$

Hence $\forall t \in Z\, \forall n \in N_t : |\alpha|^{t-t_0} \leq |||x_{s_n}||| < |\alpha|^{t+t_0}$. Therefore for any $t \in Z$ the set $M_t = \{n \in \mathbb{N} : |\alpha|^{2t_0 t} \leq |||x_n||| < |\alpha|^{2t_0(t+1)}\}$ is infinite. Let π be a permutation of $\mathbb{N}$ such that

$$\pi(M_t) = \{n \in \mathbb{N} : |\alpha|^{2t_0 t} \leq \|w_n\| < |\alpha|^{2t_0(t+1)}\}, \quad t \in Z.$$

Then $\forall n \in \mathbb{N} : |\alpha|^{-2t_0}|||x_n||| < \|w_{\pi(n)}\| < |\alpha|^{2t_0}|||x_n|||$, so (x_n) is equivalent to $(w_{\pi(n)})$. Thus (x_n) is permutatively equivalent to (w_n). ■

Now, we prove the following.

Theorem 2 *There exists a universal basis (v_n) for the family $\mathcal{F}_c$ of all bases in Fréchet spaces with continuous norms. A basis $(x_n) \in \mathcal{F}_c$ is universal (respectively quasi-universal) for $\mathcal{F}_c$ if and only if (x_n) is permutatively equivalent (respectively quasi-equivalent) to (v_n).*

Proof For any $m \in \mathbb{N}$ the set $A_m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : 0 = x_0 < \cdots < x_m\}$ is open in $\mathbb{R}^m$. Moreover, $\bigcup_{m=1}^{\infty} A_m \cap Q^m = \{d_n : n \in \mathbb{N}\}$ for some sequence (d_n). Assume that $d_n \in Q^{m(n)}, n \in \mathbb{N}$. For every $n \in \mathbb{N}$ there exists an increasing sequence $(b^0_{k,n})_{k=1}^{\infty} \subset Q$ such that $(b^0_{1,n}, \dots, b^0_{m(n),n}) = d_n$. Put $B = (b^0_{k,n})$. Let

(v_n) be the coordinate basis in the Köthe space $V = K(B)$. For any $m \in \mathbb{N}$ the set $D_m = \{(b^0_{1,n}, \dots, b^0_{m,n}) : n \in \mathbb{N}\}$ is dense in A_m, since $D_m = A_m \cap Q^m$.

Let F be a Fréchet space with a continuous norm and with a basis (y_n). Then (y_n) is 1-orthogonal with respect to a base of norms (p_k) in $\mathcal{P}(F)$. Put $b_{k,n} = p_k(y_n)$ for $k, n \in \mathbb{N}$. For every $m \in \mathbb{N}$ the set $S_m = A_m \cap \prod_{k=1}^m (b_{k,m}, 2b_{k,m})$ is open in A_m. Moreover, $S_m \neq \varnothing$, since $(tb_{1,m}, \dots, t^m b_{m,m}) \in S_m$ for any $t \in (1, 2^{1/(m+1)})$. Hence $D_m \cap S_m$ is a dense subset of S_m for all $m \in \mathbb{N}$.

Let π be a permutation of $\mathbb{N}$ (and $(c_n) \subset \mathbb{N}$). Then there exists an increasing sequence $(k_n) \subset \mathbb{N}$ (with $k_{n+1} > c_{k_n}, n \in \mathbb{N}$) such that $(b^0_{1,\pi(k_n)}, \dots, b^0_{n,\pi(k_n)}) \in S_n$, $n \in \mathbb{N}$.

Then $\forall n \in \mathbb{N}\ \forall 1 \leq i \leq n : b_{i,n} < b^0_{i,\pi(k_n)} < 2b_{i,n}$. Hence we get $a_j := \sup_n (b_{j,n}/b^0_{j,\pi(k_n)}) < \infty$ and $b_j := \sup_n (b^0_{j,\pi(k_n)}/b_{j,n}) < \infty$ for any $j \in \mathbb{N}$.

It follows that the basis (y_n) is equivalent to the subbasis $(v_{\pi(k_n)})$ of $(v_{\pi(n)})$. Indeed, let $T\colon \operatorname{lin}(y_n) \to \operatorname{lin}(v_{\pi(k_n)})$ be a linear map with $Ty_n = v_{\pi(k_n)}$, $n \in \mathbb{N}$. Clearly, T is a bijection. Let (q_j) be the standard base of norms in $\mathcal{P}(V)$. Let $j \in \mathbb{N}$ and $y \in \operatorname{lin}(y_n)$. Then $y = \sum_{i=1}^m \alpha_i y_i$ for some $m \in \mathbb{N}$, $\alpha_1, \dots, \alpha_m \in \mathbb{K}$. We have

$$p_j(y) = \max_{1\leq i\leq m} |\alpha_i| p_j(y_i) \leq a_j \max_{1\leq i\leq m} |\alpha_i| q_j(v_{\pi(k_i)}) = a_j q_j(Ty),$$

$$q_j(Ty) = \max_{1\leq i\leq m} |\alpha_i| q_j(v_{\pi(k_i)}) \leq b_j \max_{1\leq i\leq m} |\alpha_i| p_j(y_i) = b_j p_j(y).$$

Therefore T is a linear homeomorphism.

Thus for any permutation π of $\mathbb{N}$ the basis $(v_{\pi(n)})$ in V is universal for $\mathcal{F}_c$. Hence any basis $(x_n) \in \mathcal{F}_c$ which is permutatively equivalent (respectively quasi-equivalent) to (v_n) is universal (respectively quasi-universal) for $\mathcal{F}_c$.

Now, let us assume that a basis (x_n) in a Fréchet space X with a continuous norm is universal for $\mathcal{F}_c$. We shall show that (x_n) is permutatively equivalent to (v_n).

Suppose that (x_n) is 1-orthogonal with respect to a base of norms (r_k) in $\mathcal{P}(X)$. The basis (v_n) is equivalent to a subbasis (x_{k_n}) of (x_n). Put $M = \{k_n : n \in \mathbb{N}\}$ and $L = (\mathbb{N} \setminus M)$. Clearly, it is enough to consider the case when $L \neq \varnothing$.

Denote by G the closed linear span of $\{x_n : n \in L\}$. It is easy to show that the linear space $c_0(G) = \{(y_n) \subset G : \lim y_n = 0\}$ with the base of norms $(r^0_k) : r^0_k\big((y_n)\big) = \max_n r_k(y_n), k \in \mathbb{N}$ is a Fréchet space.

For an infinite countable set A we will denote by $S(A)$ an arbitrary sequence (a_n) with $\{a_n : n \in \mathbb{N}\} = A$ such that $a_n \neq a_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

For $i, j \in \mathbb{N}$, $n \in L$ put $x^n_{i,j} = 0$ if $j \neq i$, and $x^n_{i,j} = x_n$ if $j = i$. Let $x^n_i = (x^n_{i,j})^\infty_{j=1}$ for all $i \in \mathbb{N}$, $n \in L$. Set $(s_m) = S(\{x^n_i : i \in \mathbb{N}, n \in L\})$. It is easy to check that (s_m) is a basis in the Fréchet space $c_0(G)$ which is permutatively equivalent to the basis $S\big(\{(x_n, 0); n \in L\} \cup \{(0, s_n) : n \in \mathbb{N}\}\big)$ in $G \times c_0(G)$.

The basis (s_n) is equivalent to a subbasis (v_{m_n}) of (v_n). Hence (s_n) is equivalent to $(x_{k_{m_n}})$. Put $W = \{k_{m_n} : n \in \mathbb{N}\}$ and $K = (M \setminus W)$. Since $W = [\mathbb{N} \setminus (L \cup K)]$, then (x_n) is permutatively equivalent to the basis $S\big(\{(x_n, 0, 0) : n \in L\} \cup \{(0, s_n, 0) : n \in \mathbb{N}\} \cup \{(0, 0, x_n) : n \in K\}\big)$ in $G \times c_0(G) \times H$, where H is the closed linear span of $\{x_n : n \in K\}$. But $S\big(\{(x_n, 0) : n \in L\} \cup \{(0, s_n) : n \in \mathbb{N}\}\big)$ is permutatively equivalent to $(x_{k_{m_n}})$, so (x_n) is permutatively equivalent to $S\big(\{(x_n,0) : n \in W\} \cup \{(0,x_n) : n \in K\}\big)$. Thus (x_n) is permutatively equivalent to $S(\{x_n : n \in M\})$. Hence (x_n) is permutatively equivalent to (v_n).

Similarly, one can show the following. If a basis (x_n) in a Fréchet space X with a continuous norm is quasi-universal for $\mathcal{F}_c$, then (x_n) is quasi-equivalent to (v_n). (In this case (v_n) is quasi-equivalent to a subbasis (x_{k_n}) of (x_n) and (s_n) is semi equivalent to $(x_{k_{\pi(m_n)}})$ for some increasing sequence $(m_n) \subset \mathbb{N}$ and some permutation π of $\mathbb{N}$; instead of W we take $W' = \{k_{\pi(m_n)} : n \in \mathbb{N}\}$.) ■

From now on, (v_n) is a universal basis for $\mathcal{F}_c$ and V is a Fréchet space with the basis (v_n).

By the proof of Theorem 2 we obtain

Remark 3 For any $(z_n) \in \mathcal{F}_c$ and any sequence $(c_n) \subset \mathbb{N}$ there exists an increasing sequence $(k_n) \subset \mathbb{N}$ with $k_{n+1} > c_{k_n}$, $n \in \mathbb{N}$ such that (z_n) is equivalent to (v_{k_n}).

Clearly, any Fréchet space with a continuous norm and with a basis is isomorphic to a complemented subspace of V. The following is also true.

Proposition 4 *Let E be a Fréchet space with a continuous norm and with a basis. If any Fréchet space with a continuous norm and with a basis is isomorphic to a complemented subspace of E, then E is isomorphic to V.*

Proof It is clear that the Fréchet spaces $c_0(E)$ and $c_0(V)$ have continuous norms and bases (see the proof of Theorem 2). Moreover, $E \times c_0(E)$ and $V \times c_0(V)$ are isomorphic to $c_0(E)$ and $c_0(V)$, respectively. Thus there exist Fréchet spaces G and H such that $V \simeq c_0(E) \times H \simeq E \times c_0(E) \times H \simeq E \times V$ and $E \simeq c_0(V) \times G \simeq V \times c_0(V) \times G \simeq V \times E$. Hence E is isomorphic to V. ■

By the closed graph theorem (see [5, Theorem 2.49]) we get

Remark 5 Let (x_n) be a basis in a Fréchet space E and (y_n) a basis in a Fréchet space F. Then the following conditions are equivalent:

(1) (x_n) is equivalent to (y_n);

(2) for any $(\beta_n) \subset \mathbb{K}$ the sequence $(\beta_n x_n)$ is convergent to 0 in E if and only if the sequence $(\beta_n y_n)$ is convergent to 0 in F;

(3) for any $(\beta_n) \subset \mathbb{K}$ the series $\sum_{n=1}^{\infty} \beta_n x_n$ is convergent in E if and only if the series $\sum_{n=1}^{\infty} \beta_n y_n$ is convergent in F.

Using Remark 5 we shall prove the following.

Theorem 6 *There exists a universal basis (u_n) for the family $\mathcal{F}$ of all bases in Fréchet spaces. A basis $(x_n) \in \mathcal{F}$ is universal (respectively quasi-universal) for $\mathcal{F}$ if and only if (x_n) is permutatively equivalent (respectively quasi-equivalent) to (u_n).*

Proof Put $U = V^{\mathbb{N}}$. Assume that (v_n) is 1-orthogonal with respect to a base of norms (q_k) in $\mathcal{P}(V)$. Set $p_k\big((x_n)\big) = k \max_{1 \le n \le k} q_k(x_n)$ for $(x_n) \in U$, $k \in \mathbb{N}$. Clearly, (p_k) is a base in $\mathcal{P}(U)$. For $n, i, j \in \mathbb{N}$ we put $v^n_{i,j} = 0$ if $j \neq i$, and $v^n_{i,j} = v_n$ if $j = i$. Let $v^n_i = (v^n_{i,j})_{j=1}^{\infty}$ for all $n, i \in \mathbb{N}$. It is easy to see that there exists a bijection $\varphi\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the sequence $\big(\varphi(i,n)\big)_{n=1}^{\infty}$ is increasing for any $i \in \mathbb{N}$. Let $u_{\varphi(i,n)} = v^n_i$ for $i, n \in \mathbb{N}$. Of course, (u_m) is a basis in U. Moreover, for any $i \in \mathbb{N}$ the subbasis $(u_{\varphi(i,n)})_{n=1}^{\infty}$ of (u_m) is equivalent to (v_n), so it is a universal basis for $\mathcal{F}_c$. Put $M_i = \{\varphi(i,n) : n \in \mathbb{N}\}$, $i \in \mathbb{N}$. Clearly, $M_i = \{m \in \mathbb{N} : u_m \in (\ker p_{i-1} \setminus \ker p_i)\}$ for $i \in \mathbb{N}$, where $p_0(x) = 0$ for $x \in U$.

Now, let us assume that (y_n) is a basis in a Fréchet space Y. Then (y_n) is 1-orthogonal with respect to a base (r_k) in $\mathcal{P}(Y)$. Let $r_0(y) = 0$ for $y \in Y$. Put $N_i = \{n \in \mathbb{N} : y_n \in (\ker r_{i-1} \setminus \ker r_i)\}$ for $i \in \mathbb{N}$. Denote by W the set of all $i \in \mathbb{N}$ for which the set N_i is infinite. For any $i \in W$ the sequence $(y_m)_{m \in N_i}$ is a basis in the closed linear span Y_i of $\{y_m : m \in N_i\}$ and $r_i | Y_i$ is a continuous norm on Y_i.

Let π be a permutation of $\mathbb{N}$. By the proof of Theorem 2 we can construct inductively an increasing sequence $(k_n) \subset \mathbb{N}$ with $\{\pi(k_n) : n \in N_i\} \subset M_i$, $i \in \mathbb{N}$, such that $(y_n)_{n \in N_i}$ is equivalent to $(u_{\pi(k_n)})_{n \in N_i}$ for any $i \in W$. We shall prove that (y_n) is equivalent to $(u_{\pi(k_n)})$.

Let $(\beta_n) \subset \mathbb{K}$. Assume that $\lim_n \beta_n y_n = 0$. Then $\lim_{n \in N_i} \beta_n y_n = 0$ for any $i \in W$. By Remark 5, $\lim_{n \in N_i} \beta_n u_{\pi(k_n)} = 0$ for any $i \in W$. We show that $\lim_n \beta_n u_{\pi(k_n)} = 0$. Suppose, by contradiction, that there exists a neighborhood M of 0 in U and an increasing sequence $(d_m) \subset \mathbb{N}$ such that $(\beta_{d_m} u_{\pi(k_{d_m})}) \subset (U \setminus M)$. Then for any $i \in \mathbb{N}$ the set $\{m \in \mathbb{N} : d_m \in N_i\}$ is finite. Therefore for every $i \in \mathbb{N}$ there is $m_i \in \mathbb{N}$ with $(d_m)_{m=m_i}^{\infty} \subset \bigcup_{j=i+1}^{\infty} N_j$. Hence $\big(\pi(k_{d_m})\big)_{m=m_i}^{\infty} \subset \bigcup_{j=i+1}^{\infty} M_j$, so $p_i(\beta_{d_m} u_{\pi(k_{d_m})}) = 0$ for all $m, i \in \mathbb{N}$ with $m \ge m_i$. It follows that $\lim_m \beta_{d_m} u_{\pi(k_{d_m})} = 0$, a contradiction. Thus $\lim_n \beta_n u_{\pi(k_n)} = 0$.

Similarly, assuming that $\lim_n \beta_n u_{\pi(k_n)} = 0$ we get $\lim_n \beta_n y_n = 0$. By Remark 5, (y_n) is equivalent to $(u_{\pi(k_n)})$. Thus the basis $(u_{\pi(n)})$ in U is universal for $\mathcal{F}$. Hence any basis $(x_n) \in \mathcal{F}$, which is permutatively equivalent (respectively quasi-equivalent) to (u_n), is universal (respectively quasi-universal) for $\mathcal{F}$.

Now, let us assume that a basis (x_n) in a Fréchet space X is universal (respectively quasi-universal) for $\mathcal{F}$. As in the proof of Theorem 2 one can show that (x_n) is permutatively equivalent (respectively quasi-equivalent) to (u_n). ∎

The Fréchet spaces $c_0^{\mathbb{N}}$ and $V^{\mathbb{N}}$ are universal for the family of all Fréchet spaces of countable type. By Theorem 6 any Fréchet space with a basis is isomorphic to a complemented subspace of $V^{\mathbb{N}}$. In contrast to this, we shall prove the following result for $c_0^{\mathbb{N}}$.

Theorem 7 *A Fréchet space X with a basis (x_n) is isomorphic to a complemented subspace of $c_0^{\mathbb{N}}$ if and only if X is isomorphic to one of the following spaces: c_0, $c_0 \times \mathbb{K}^{\mathbb{N}}$, $\mathbb{K}^{\mathbb{N}}$, $c_0^{\mathbb{N}}$.*

Proof Clearly, any Fréchet space, which is isomorphic to one of the following spaces: c_0, $c_0 \times \mathbb{K}^{\mathbb{N}}$, $\mathbb{K}^{\mathbb{N}}$, $c_0^{\mathbb{N}}$, is isomorphic to a complemented subspace of $c_0^{\mathbb{N}}$.

To prove the converse, let us denote by P a linear continuous projection from $c_0^{\mathbb{N}}$ onto a complemented subspace Y of $c_0^{\mathbb{N}}$ which is isomorphic to X. Let (e_n) be the coordinate basis in c_0. For $n, i, j \in \mathbb{N}$ we put $e_{i,j}^n = 0$ if $j \neq i$, and $e_{i,j}^n = e_n$ if $j = i$. Set $e_i^n = (e_{i,j}^n)_{j=1}^{\infty}$ for $n, i \in \mathbb{N}$. Let $\varphi\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection. Put $z_{\varphi(k,n)} = e_k^n$ for all $n, k \in \mathbb{N}$. Clearly, (z_n) is a basis in $c_0^{\mathbb{N}}$. Let (y_n) be a basis in Y. Denote by (f_n) and (h_n) the sequences of coefficient functionals associated with the bases (z_n) and (y_n), respectively. Put $g_n(z) = h_n(Pz)$ for $n \in \mathbb{N}$ and $z \in c_0^{\mathbb{N}}$. Since

$$1 = |g_n(y_n)| = \Big| g_n\Big(\sum_{k=1}^{\infty} f_k(y_n) z_k \Big) \Big|$$

$$= \Big| \sum_{k=1}^{\infty} f_k(y_n) g_n(z_k) \Big| \leq \max_k |f_k(y_n) g_n(z_k)|, \quad n \in \mathbb{N},$$

then for any $n \in \mathbb{N}$ there exists $t_n \in \mathbb{N}$ with $|f_{t_n}(y_n) g_n(z_{t_n})| \geq 1$.

Put $p_k\big((\alpha_n)\big) = k \max_{1 \leq n \leq k} \|\alpha_n\|$ for all $k \in \mathbb{N}$ and $(\alpha_n) \in c_0^{\mathbb{N}}$. Clearly, (p_k) is a base in $\mathcal{P}(c_0^{\mathbb{N}})$ and the basis (z_n) is 1-orthogonal with respect to the base (p_k).

For any $k \in \mathbb{N}$ there exist $q_k \in \mathcal{P}(c_0^{\mathbb{N}})$ and $s_k \in \mathbb{N}$ with $p_k \leq q_k \leq p_{s_k}$ and $q_k \circ P \leq p_{s_k}$ such that (y_n) is 1-orthogonal with respect to q_k. For all $n, k \in \mathbb{N}$ we obtain

$$p_k\big(f_{t_n}(y_n) z_{t_n} \big) \leq \max_m p_k\big(f_m(y_n) z_m \big) = p_k(y_n) \leq |g_n(z_{t_n})|^{-1} \max_m q_k\big(g_m(z_{t_n}) y_m \big)$$

$$= |g_n(z_{t_n})|^{-1} q_k(P z_{t_n}) \leq p_{s_k}\big(f_{t_n}(y_n) z_{t_n} \big).$$

Hence

$$(*) \qquad p_k\big(f_{t_n}(y_n) z_{t_n} \big) \leq p_k(y_n) \leq p_{s_k}\big(f_{t_n}(y_n) z_{t_n} \big) \quad \text{for all } k, n \in \mathbb{N}.$$

Put $r_k(y) = \max_n |h_n(y)| p_k\big(f_{t_n}(y_n) z_{t_n} \big)$ for $k \in \mathbb{N}$, $y \in Y$.

By $(*)$, we get $r_k(y) \leq \max_n |h_n(y)| q_k(y_n) = q_k(y) \leq p_{s_k}(y)$, and $p_k(y) \leq \max_n |h_n(y)| p_k(y_n) \leq \max_n |h_n(y)| p_{s_k}\big(f_{t_n}(y_n) z_{t_n} \big) = r_{s_k}(y)$.

Thus (r_k) is a base in $\mathcal{P}(Y)$. Put $b_n = \big(y_n / f_{t_n}(y_n) \big)$, $n \in \mathbb{N}$. Clearly, (b_n) is a basis in Y which is 1-orthogonal with respect to (r_k). Let $k, n \in \mathbb{N}$. Since $r_k(b_n) = p_k(z_{t_n})$, then $r_k(b_n) = 0$ or $r_k(b_n) = k$. Set $r_0(y) = 0$ for $y \in Y$.

Put $N_k = \{ n \in \mathbb{N} : b_n \in (\ker r_{k-1} \setminus \ker r_k) \}$, $k \in \mathbb{N}$. Clearly, $\bigcup_{k=1}^{\infty} N_k = \mathbb{N}$ and $N_i \cap N_j = \varnothing$ for $i, j \in \mathbb{N}$ with $i \neq j$.

Consider four cases:

Case 1 For some $k_0 \in \mathbb{N}$ we have $\bigcup_{k=1}^{k_0} N_k = \mathbb{N}$: Then $\forall k \geq k_0\ \forall n \in \mathbb{N} : r_k(b_n) = k$. Hence Y is normable, so it is isomorphic to c_0.

Case 2 For any $k \in \mathbb{N}$ the set N_k is finite: Let $k \in \mathbb{N}$. Since $\{ b_n : n \in \bigcup_{i=k+1}^{\infty} N_i \} \subset \ker r_k$, then $\dim(Y / \ker r_k) < \infty$. Thus Y is of finite type, so it is isomorphic to $\mathbb{K}^{\mathbb{N}}$.

Case 3 For some increasing sequence $(i_n) \subset \mathbb{N}$ the sets N_{i_n}, $n \in \mathbb{N}$, are infinite: Let $n \in \mathbb{N}$ and $i_0 = 0$. Put $M_n = \bigcup_{k=i_{n-1}+1}^{i_n} N_k$. Denote by Y_n the closed linear span of

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$\{b_j : j \in M_n\}$. Since $r_k(b_j) = k$ for $j \in N_k$, $k \in \mathbb{N}$, then $r_k(b_j) = k$ for $j \in M_n$ and $k \geq i_n$. Hence Y_n is normable, so it is isomorphic to c_0.

For any $(a_n) \in \prod_{n=1}^{\infty} Y_n$ the series $\sum_{n=1}^{\infty} a_n$ is convergent in Y. Indeed, let $k \in \mathbb{N}$. Since $r_k(b_j) = 0$ for $j \in N_{k+1}$, then $r_k(b_j) = 0$ for $j \in M_n$, $n \in \mathbb{N}$ with $i_{n-1} \geq k$. Hence $r_k(a_n) = 0$ for any $n \in \mathbb{N}$ with $i_{n-1} \geq k$. Thus $\lim a_n = 0$.

Let T_n be the natural projection from Y onto Y_n, $n \in \mathbb{N}$. Clearly, $y = \sum_{n=1}^{\infty} T_n y$ for any $y \in Y$. By the open mapping theorem ([5], Corollary 2.74), the continuous map $T\colon Y \to \prod_{n=1}^{\infty} Y_n$, $y \to (T_n y)$ is an isomorphism. Thus Y is isomorphic to $c_0^{\mathbb{N}}$.

Case 4 For some $k_0 \in \mathbb{N}$ the sets $W_1 := \bigcup_{k=1}^{k_0} N_k$, $W_2 := \bigcup_{k=k_0+1}^{\infty} N_k$ are infinite and the sets N_k, $k > k_0$, are finite: The closed linear span Z_1 of $\{b_n : n \in W_1\}$ is normable, since $r_i(b_n) = i$ for $i \geq k_0$ and $n \in W_1$. Thus Z_1 is isomorphic to c_0. The closed linear span Z_2 of $\{b_n : n \in W_2\}$ is of finite type, since $\{b_n : n \in \bigcup_{k=i+1}^{\infty} N_k\} \subset Z_2 \cap \ker r_i$ for any $i \geq k_0$. Thus Z_2 is isomorphic to $\mathbb{K}^{\mathbb{N}}$. Hence Y is isomorphic to $c_0 \times \mathbb{K}^{\mathbb{N}}$. ■

For $V^{\mathbb{N}}$ we have the following result (see the proof of Proposition 4).

Proposition 8 *A Fréchet space E with a basis is isomorphic to $V^{\mathbb{N}}$ if and only if any Fréchet space with a basis is isomorphic to a complemented subspace of E.*

Remark 9 There exists a Fréchet space X of countable type which is is not isomorphic to any complemented subspace of $V^{\mathbb{N}}$. Indeed, there is a nuclear Fréchet space X with a continuous norm and without the bounded approximation property (see [9, Theorem 11]). Since any complemented subspace of a Fréchet space with a basis has the bounded approximation property, then X is not isomorphic to any complemented subspace of $V^{\mathbb{N}}$.

For bases in nuclear Fréchet spaces we shall prove the following.

Theorem 10 *There is no quasi-universal basis for the family $\mathcal{F}_n$ of all bases in nuclear Fréchet spaces or for the family $\mathcal{F}_{nc}$ of all bases in nuclear Fréchet spaces with continuous norms.*

Proof Let E be a nuclear Fréchet space with a basis (x_n). Assume that E is not of finite type. Then (x_n) is 1-orthogonal with respect to a base (q_k) in $\mathcal{P}(E)$ with $\dim(E/\ker q_1) = \infty$. Let $i \in \mathbb{N}$. Put $N_i = \{n \in \mathbb{N} : q_i(x_n) > 0\}$. Clearly, the closed linear span X_i of $\{x_n : n \in N_i\}$ is an infinite-dimensional nuclear Fréchet space and $(x_n)_{n \in N_i}$ is a basis in X_i which is 1-orthogonal with respect to the base of norms $(q_k|X_i)_{k=i}^{\infty}$ in $\mathcal{P}(X_i)$. Therefore we have $\lim_{n \in N_i}[q_i(x_n)/q_j(x_n)] = 0$ for some $j > i$. Thus we can construct inductively an increasing sequence $(i_k) \subset \mathbb{N}$ such that $\lim_{n \in N_{i_k}}[q_{i_k}(x_n)/q_{i_{k+1}}(x_n)] = 0$ for any $k \in \mathbb{N}$. Put $p_k = q_{i_k}$ and $M_k = N_{i_k}$ for $k \in \mathbb{N}$. Let $a_{i,j}(n) = [p_j(x_n)/p_i(x_n)]$ for $k, i, j \in \mathbb{N}$ with $k \leq i < j$ and $n \in M_k$. Clearly, $\lim_{n \in M_k} a_{i,j}(n) = \infty$ for $k, i, j \in \mathbb{N}$ with $k \leq i < j$. Thus there exists an increasing sequence $(s_n) \subset \mathbb{N}$ such that for any $n > 1$ we have

$$(*) \qquad \max_{1 \leq i < j \leq n} \max_{m \in M(n)} a_{i,j}(m) < \min_{1 \leq k \leq n} \min_{m \in M_k(s_n)} a_{k,k+1}(m)$$

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where $M(n) = \{m \in M_1 : m \le n\}$ and $M_k(s_n) = \{i \in M_k : i \ge s_n\}$ (we assume that $\max \varnothing = 0$). Let $t_j = \min\{n \in M_1 : s_n \ge j\}$ for $j \in \mathbb{N}$. Put $b_{i,j} = p_i(x_{t_j})$ for all $i, j \in \mathbb{N}$. Then $0 < b_{i,j} \le b_{i+1,j}$ for all $i, j \in \mathbb{N}$ and $\lim_j (b_{i,j}/b_{i+1,j}) = \lim_j a_{i,i+1}^{-1}(t_j) = 0$ for any $i \in \mathbb{N}$. Thus for $B = (b_{i,j})$ the Köthe space $K(B)$ is nuclear. We shall show that the coordinate basis (e_n) in $K(B)$ is not quasi-equivalent to any subbasis of (x_n). Suppose, by contradiction, that (e_n) is equivalent to $(\alpha_n x_{k_{\pi(n)}})$ for some $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$, some increasing sequence $(k_n) \subset \mathbb{N}$ and some permutation π of $\mathbb{N}$. Then there exists a linear homeomorphism $T\colon \operatorname{lin}(e_n) \to \operatorname{lin}(x_{k_n})$ with $T(e_n) = \alpha_n x_{k_{\pi(n)}}$ for any $n \in \mathbb{N}$. By the continuity of T and T^{-1} we obtain

$$\forall m \in \mathbb{N}\ \exists u(m),\ v(m) \in \mathbb{N}\ \forall x \in \operatorname{lin}(e_n) : p_m(Tx) \le r_{u(m)}(x) \le p_{v(m)}(Tx)$$

where (r_k) is the standard base of norms in $\mathcal{P}\big(K(B)\big)$. Clearly, we can assume that $u(m) < u(m+1)$ for any $m \in \mathbb{N}$.

Then $\forall m, n \in \mathbb{N} : p_m(x_{k_{\pi(n)}}) \le r_{u(m)}(\alpha_n^{-1} e_n) \le p_{v(m)}(x_{k_{\pi(n)}})$. Thus

$$\forall m, n \in \mathbb{N} : [p_m(x_{k_{\pi(n)}})/r_{u(m)}(e_n)] \le |\alpha_n^{-1}| \le [p_{v(m)}(x_{k_{\pi(n)}})/r_{u(m)}(e_n)].$$

Hence $\forall i, j, n \in \mathbb{N} : [p_j(x_{k_{\pi(n)}})/r_{u(j)}(e_n)] \le [p_{v(i)}(x_{k_{\pi(n)}})/r_{u(i)}(e_n)]$.

Since $\forall n \in \mathbb{N} : p_{v(1)}(x_{k_{\pi(n)}}) \ge r_{u(1)}(\alpha_n^{-1} e_n) > 0$, then $\{k_{\pi(n)} : n \in \mathbb{N}\} \subset M_{v(1)}$ and $\forall i, j, n \in \mathbb{N} : [r_{u(j)}(e_n)/r_{u(i)}(e_n)] \ge [p_j(x_{k_{\pi(n)}})/p_{v(i)}(x_{k_{\pi(n)}})]$. Thus we have

$$(**) \qquad \forall i, j, n \in \mathbb{N} : a_{u(i),u(j)}(t_n) \ge a_{v(i),j}(k_{\pi(n)}).$$

Let $j = v(1) + 1$ and $n \in M_1$ with $n > u(j)$. Since $\max\{k_{\pi(b)} : 1 \le b \le s_n\} \ge s_n$, there exists $d \in \mathbb{N}$ with $d \le s_n \le k_{\pi(d)}$. Hence $t_d \le n$ and $k_{\pi(d)} \in M_{v(1)}(s_n)$. Thus by $(*)$ we get $a_{u(1),u(j)}(t_d) < a_{v(1),j}(k_{\pi(d)})$, contrary to $(**)$.

It follows that the basis (x_n) is not universal for $\mathcal{F}_n$ or $\mathcal{F}_{nc}$. This completes the proof, since any basis in a Fréchet space of finite type is not universal for $\mathcal{F}_n$. ∎

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