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ON THE COMMUTATIVITY OF SEMI-SIMPLE ASSOCIATIVE ALGEBRAS

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Recall that an automorphism Φ of an algebra A is called regular if, for all $x \in A$, $\Phi(x) = x$ implies $x = 0$. For various types of algebras it is known that the existence of a regular automorphism implies that the algebra is radical. Here, for associative algebras, we relax the restrictions on the fixed points so that all automorphisms of commutative algebras immediately satisfy the hypothesis on the fixed points. Of course, we can no longer expect the conclusion to be that the algebra is radical. Instead our conclusions, always for semi-simple algebras, will be that the algebra is commutative. Our method of proof will be to pass to the associated Lie algebra and to then apply a known result on regular automorphisms of Lie algebras.

THEOREM 1. *Let A be an associative algebra over a field of characteristic $p \neq 2$. Let Φ be an automorphism of A of finite period n whose fixed point set F is contained in the center of A. Suppose further that if $p \neq 0$, then p and n are relatively prime. Let B be a semi-simple subalgebra of A. Then B is commutative.*

Proof. Φ is a Lie automorphism of A^-, the associated Lie algebra of A. Furthermore F^- is an ideal of A^- contained in the center and Φ induces an automorphism $\bar{\Phi}$ of A^-/F^- which we claim is regular. For suppose that $\bar{\Phi}(\bar{x}) = \bar{x}$. Then $\Phi(x) = x + y$ for some $y \in F$. Then for each positive integer k, $\Phi^k(x) = x + ky$. In particular, $x = \Phi^n(x) = x + ny$. Hence under the assumptions on n and p, $y = 0$. Therefore $x \in F$ and $\bar{\Phi}$ is regular. Then by [4, Theorem], A^-/F^- is solvable. Hence A^- and B^- are solvable also. The result will now follow from the following.

LEMMA. *Let B be a semi-simple associative algebra over a field k of characteristic $p \neq 2$. Then B is commutative if and only if B^- is solvable.*

Proof. If B is commutative, then the result is clear. Suppose that B^- is solvable. If B is a division algebra, then the Lie subring of B generated by all commutators of B is its own derived ring by [1, Lemma 3]. Since B^- is solvable as a Lie algebra, the only possibility is that each commutator is zero and the result holds in this case.

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Now suppose that B is a primitive algebra, that M is a faithful irreducible B-module and that Γ is the centralizer of M. As is well-known, Γ is a division algebra over k and M is a vector space over Γ. Suppose $\dim_\Gamma M > 1$ and let u and v be linearly independent vectors in M. By the Jacobson–Chevalley density theorem, there exist E_1, F_1, $H_1 \in B$ such that

$$uE_1 = v \qquad uF_1 = 0 \qquad uH_1 = u$$
$$vE_1 = 0 \qquad vF_1 = u \qquad vH_1 = -v.$$

For $i > 1$, define inductively $H_i = [E_{i-1}, F_{i-1}]$, $E_i = \frac{1}{2}[H_{i-1}, E_{i-1}]$, and $F_i = \frac{1}{2}[F_{i-1}, H_{i-1}]$. We see that H_i and H_1 are the same on $((u, v))$. A similar comment holds for E_i and E_1 and also for F_i and F_1. Furthermore H_i, E_i, and F_i are in the ith term in the derived series of B^-, hence there must exist an integer $n > 0$ such that $E_n = F_n = H_n = 0$ since B^- is solvable. This contradicts, for instance, that $uH_n = u$. Therefore $\dim_\Gamma M = 1$ and $B \simeq \Gamma$. By the preceding paragraph, B is commutative.

Finally if B is semi-simple, then B is the subdirect sum of primitive algebras, each of which is the homomorphic image of B, hence must have solvable Lie structure. Therefore B is the subdirect sum of commutative algebras and the result holds.

Note that the lemma fails at characteristic 2. The example of 2 by 2 matrices with elements from a field of characteristic 2 shows this.

If we require that our automorphism have prime period p, we obtain a similar result for rings.

THEOREM 2. *Let A be an associative ring. Let Φ be an automorphism of A of prime period p such that all the fixed points of Φ are contained in the center of A. Suppose further that for all $x \in A$, $px = 0$ implies $x = 0$. Let B be a semi-simple subring of A. Then B is commutative.*

To see this we note that B^- is nilpotent by [2, Theorem 1]. One can complete the argument in the same manner as in the above lemma. The only exception is in the primitive case at characteristic 2. Then instead of E_i, F_i, and H_i we use

$$uK_1 = u \qquad uJ_1 = v$$
$$vK_1 = 0 \qquad vJ_1 = 0$$

and argue in the same manner replacing the derived series of B^- by the lower central series.

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