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$\mathscr{N}$-SEMIGROUPS AND THEIR TRANSLATION SEMIGROUPS

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Abstract

If h is a homomorphism of an $\mathscr{N}$-semigroup onto an $\mathscr{N}$-semigroup S', then h induces a homomorphism $\bar{h}$ of the translation semigroup $\Lambda(S)$ of S into $\Lambda(S')$ of S'. We will study the relations between the structures of S, S' and $\Lambda(S)$, $\Lambda(S)'$, and will introduce the specialized concepts of $\mathscr{N}$-semigroups S. In particular, we will be interested in power-joined steady or endless $\mathscr{N}$-semigroups. Finally, we will consider admissibility of torsion abelian groups, that is, consider what torsion abelian group can be a structure group of a power-joined steady $\mathscr{N}$-semigroup.

1. Introduction

By an $\mathscr{N}$-semigroup we mean a commutative cancellative archimedean semigroup without idempotent. Petrich (1973) introduced the concept of steadiness of $\mathscr{N}$-semigroups, that is, an $\mathscr{N}$-semigroup is called steady if it cannot be embedded into another $\mathscr{N}$-semigroup as a proper ideal. The author (1973) proved that a finitely generated $\mathscr{N}$-semigroup is steady if and only if it is isomorphic to the direct product of a finite abelian group and the positive integer semigroup under addition. The study of steadiness or similar concepts of $\mathscr{N}$-semigroups is of significance in the study of extensions of $\mathscr{N}$-semigroups to commutative cancellative semigroups. Accordingly the problem is related to their translation semigroups.

In this paper we will investigate the relation between $\mathscr{N}$-semigroups S and their translation semigroups $\Lambda(S)$, and will specialize S by means of $\Lambda(S)$ in the natural ways, so that we will introduce a few concepts of $\mathscr{N}$-semigroups, steadiness, endlessness, permutation-freeness (denoted by "per-free") and middle-freeness. In Section 4, we will prove that every $\mathscr{N}$-semigroup is a spined product of a per-free $\mathscr{N}$-semigroup and an abelian group. We will discuss how the $\mathscr{N}$-homomorphisms of $\mathscr{N}$-semigroups affect the homomorphisms of their translation semigroups. We are most interested in steady or endless $\mathscr{N}$-semigroups, in particular, those in the power-joined case. We will show that every power-joined steady $\mathscr{N}$-semigroup is a spined product of a power-joined endless $\mathscr{N}$-semigroup and a torsion abelian group. Therefore the study of steady $\mathscr{N}$-semigroups is reduced to the study of endless $\mathscr{N}$-semigroups. However, the structure of endless $\mathscr{N}$-semigroups is too complicated to be simply described even if power-joinedness

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is assumed. Really we shall have the large class of power-joined endless $\mathcal{N}$-semigroups, but it will be interesting that every $\mathcal{N}$-semigroup can be embedded into an endless $\mathcal{N}$-semigroup. Finally, we will consider "admissibility" of torsion abelian groups. What torsion abelian group can be a structure group of some endless $\mathcal{N}$-semigroup?

2. Preliminaries for $\mathcal{N}$-semigroups

Throughout this paper, Z denotes the group of all integers, Z_+ the semigroup of all positive integers, Z_+^0 the semigroup of all non-negative integers, $\mathbf{R}_+$ the semigroup of all positive real numbers, R_+ the semigroup of all positive rational numbers. The operation in each is addition.

The following is due to the author (1957) (also see Petrich (1973a) and Clifford and Preston (1961)).

PROPOSITION 2.1. *Let G be an abelian group and I: $G\times G\to Z_+^0$ be a function satisfying*

(2.1.1) $I(\alpha,\beta)=I(\beta,\alpha)$ *for all* $\alpha,\beta\in G$.

(2.1.2) $I(\alpha,\beta)+I(\alpha\beta,\gamma)=I(\alpha,\beta\gamma)+I(\beta,\gamma)$ *for all* $\alpha,\beta,\gamma\in G$.

(2.1.3) $I(\varepsilon,\alpha)=1$ (ε *being the identity of* G) *for all* $\alpha\in G$.

(2.1.4) *For each* $\alpha\in G$ *there is an* $m\in Z_+$ *such that* $I(\alpha,\alpha^m)>0$.

Let $S=\{(x,\alpha): x\in Z_+^0, \alpha\in G\}$. *Define the operation by*

$$(x,\alpha)(y,\beta)=(x+y+I(\alpha,\beta),\alpha\beta).$$

Then S is an $\mathcal{N}$-semigroup, denoted by $S=(G;I)$. Every $\mathcal{N}$-semigroup can be obtained in this manner.

A function I: $G\times G\to Z_+^0$ which satisfies (2.1.1) through (2.1.4) is called an $\mathcal{I}$-function on G.

Let S be an $\mathcal{N}$-semigroup and let $a\in S$. Define a relation ρ_a on S by $x\rho_a y$ if and only if $a^m x=a^n y$ for some $m,n\in Z_+$. Then ρ_a is a congruence and $G_a=S/\rho_a$ is an abelian group and there exists an $\mathcal{I}$-function, I_a: $G_a\times G_a\to Z_+^0$ such that $S\cong(G_a;I_a)$. The group G_a is called the *structure group* of S with respect to a. The element a is called a *standard element* of the representation. Every element of S is expressed as $a^n p$ where p is a prime element relative to a, namely, $p\notin aS$; and there is a one-to-one correspondence between the prime elements relative to a and the elements of G_a.

PROPOSITION 2.2 (the author (1974a)). *Let φ: $G\to\mathbf{R}_+$ be a function satisfying*

(2.2.1) $\varphi(\varepsilon)=1$, ε *the identity of* G.

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(2.2.2) *$\varphi(\alpha)+\varphi(\beta)-\varphi(\alpha\beta)\in Z_+^0$ for all $\alpha,\beta\in G$.*

(2.2.3) *For every $\alpha\in G$, $n\varphi(\alpha)-\varphi(\alpha^n)\in Z_+$ for some $n\in Z_+$.*

Define $((G;\varphi))=\{((x,\alpha)): x\in \mathbf{R}_+, x-\varphi(\alpha)\in Z_+^0, \alpha\in G\}$ with a binary operation

$$((x,\alpha))((y,\beta))=((x+y,\alpha\beta)).$$

Let

(2.2.4) $$I(\alpha,\beta)=\varphi(\alpha)+\varphi(\beta)-\varphi(\alpha\beta).$$

Then I satisfies (2.1.1) *through* (2.1.4), *and* $(G;I)\cong((G;\varphi))$ *under*

$$(x,\alpha)\mapsto((x+\varphi(\alpha),\alpha)).$$

(2.2.3) is equivalent to (2.2.3′) below:

(2.2.3′) *For every $\alpha\in G, \varphi(\alpha)+\varphi(\alpha^m)-\varphi(\alpha^{m+1})>0$ for some $m\in Z_+$.*

The function φ which satisfies (2.2.1) through (2.2.3) is called a *defining function* on G. Let $S=(G;I)$. If we define $h: S\to\mathbf{R}_+$ by $h(x,\alpha)=r(x+\varphi(\alpha))$ where r is a fixed element of $\mathbf{R}_+$ then h is a homomorphism of S into $\mathbf{R}_+$. Conversely if h is a homomorphism of S into $\mathbf{R}_+$ and if we define φ by $\varphi(\alpha)=h(0,\alpha)/h(0,\varepsilon)$ where $(0,\varepsilon)$ is the standard element of $(G;I)$ and $(0,\alpha)$ is any prime element of $(G;I)$ relative to $(0,\varepsilon)$. Then φ satisfies (2.2.4). The existence of a homomorphism h of S into $\mathbf{R}_+$ was proved by the author (1974) or Kobayashi (1973). If φ is a defining function on G, define $\mathscr{R}(\varphi)$ by $\mathscr{R}(\varphi)=\{x+\varphi(\alpha): x\in Z_+^0, \alpha\in G\}$. Then $\mathscr{R}(\varphi)$ is an $\mathscr{N}$-subsemigroup of $\mathbf{R}_+$. Proposition 2.2 says $S=((G;\varphi))$ is isomorphic to a subdirect product of $\mathscr{R}(\varphi)$ and G.

A semigroup S is called power-joined if, for every x, $y\in S$, there are positive integers m,n such that $x^m=y^n$.

See the following in Chrislock (1966, 1969), Higgins (1969, 1969a) and Sasaki and Tamura (1971).

PROPOSITION 2.3. *An $\mathscr{N}$-semigroup $S\cong(G_a;I_a)$ is power-joined (finitely generated) if and only if G_a is torsion (finite) for some $a\in S$, equivalently for all $a\in S$. A power-joined (finitely generated) S is isomorphic to a subdirect product of a subsemigroup of $R_+(Z_+)$ and a torsion (finite) abelian group. Given I, φ is determined by*

(2.3.1) $$\varphi(\alpha)=\frac{1}{n}\sum_{i=1}^{n} I(\alpha,\alpha^i)$$

where n is the order of α. In particular if G is finite,

$$\varphi(\alpha)=\frac{1}{|G|}\sum_{\xi\in G} I(\alpha,\xi).$$

Thus, if G is torsion, there is a one-to-one correspondence between I's and φ's on the same G; and if G is torsion, neither (2.1.4) nor (2.2.3) is required since each one of these is automatically satisfied.

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Spined products. Let A, B and C be semigroups. Let $f_1: A \to C$ and $f_2: B \to C$ be homomorphisms of A and B onto C respectively. Define S by

$$S = \{(a,b): f_1(a) = f_2(b)\}$$

and define the binary operation in S by

$$(a,b)(c,d) = (ac, bd).$$

Then S is a semigroup. S is called a spined product of A and B with respect to f_1, f_2 and C, denoted by

$$A \underset{C;f_1,f_2}{\bowtie} B.$$

(See Kimura (1958) or Yamada (1962).) It is uniquely determined by f_1, f_2 and C up to isomorphism. In other words, $D = A \bowtie_{C;f_1,f_2} B$ is the *pull back* in the theory of categories. By a spined product $D = A \bowtie B$ we mean $D = A \bowtie_{C;f_1,f_2} B$ for some C, f_1 and f_2.

See the following in books of universal algebra, for example Gratzer (1968).

PROPOSITION 2.4. *Let S, S_1 and S_2 be semigroups. Then*

$$S \cong S_1 \underset{C;f_1,f_2}{\bowtie} S_2$$

for some semigroup C and some surjective homomorphisms $f_1: S_1 \to C$ and $f_2: S_2 \to C$ if and only if there are congruences ρ_1 and ρ_2 on S such that

(2.4.1) $S/\rho_1 \cong S_1$, $S/\rho_2 \cong S_2$.

(2.4.2) $\rho_1 \cap \rho_2 = \iota$, $\quad$ *ι the equality relation.*

(2.4.3) $\rho_1 \cdot \rho_2 = \rho_2 \cdot \rho_1$.

PROPOSITION 2.5 (the author (1975)). *Let A be an $\mathcal{N}$-semigroup, B a commutative cancellative archimedean semigroup and C a commutative cancellative semigroup such that there exist surjective homomorphisms $f: A \to C$ and $g: B \to C$. Then $A \bowtie_{C;f,g} B$ is an $\mathcal{N}$-semigroup.*

3. Preliminaries for translations

Let S be a commutative cancellative semigroup, and $Q(S)$ denote the quotient group of S, that is, the group of quotients of S. Define $\mathcal{I}_Q(S)$ by

$$\mathcal{I}_Q(S) = \{a \in Q(S): aS \subseteq S\}.$$

$\mathcal{I}_Q(S)$ is a subsemigroup of $Q(S)$ containing S. $\mathcal{I}_Q(S)$ is called the idealizer of S in $Q(S)$. On the other hand, let $\Lambda(S)$ denote the semigroup of translations of S, and $\Gamma(S)$ the semigroup of inner translations of S. See the definition of translation in Clifford and Preston (1961). In this case, there is no distinction between left and right translations, and $\Lambda(S)$ is isomorphic to the translational hull of S.

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PROPOSITION 3.1. *Let* $a \in \mathscr{I}_Q(S)$. *If* $\lambda_a: S \to S$ *is defined by* $x\lambda_a = xa$, *then* $\lambda_a \in \Lambda(S)$. *Every element of* $\Lambda(S)$ *is obtained in this manner. Under* $a \mapsto \lambda_a$, $\mathscr{I}_Q(S) \cong \Lambda(S)$ *and* $S \cong \Gamma(S)$. *Hence* $\Lambda(S)$ *is a commutative cancellative semigroup.*

PROOF. Let λ be a translation of S. Define $\bar{\lambda}: Q(S) \to Q(S)$ as follows. If $x \in Q(S)$, $x = y^{-1}z$ where $y, z \in S$, then $x\bar{\lambda} = y^{-1}(z\lambda)$. It can be shown that $\bar{\lambda}$ is well defined, it is a translation of $Q(S)$ and extends λ. However, we know that every translation of $Q(S)$ is inner, hence $x\bar{\lambda} = xa$ for some fixed $a \in Q(S)$. Obviously a must satisfy $Sa \subseteq S$.

In particular, we consider $\Lambda(S)$ of an $\mathscr{N}$-semigroup $S = (G: I)$. By the author (1970), $Q(S)$ can be obtained as the abelian group extension of Z by G with respect to the factor system $f: G \times G \to Z$ defined by $f(\alpha, \beta) = I(\alpha, \beta) - 1$, that is, $Q(S) = Z \times G$ with the following operation: for $\{m, \alpha\}, \{n, \beta\} \in Z \times G$, let

$$\{m, \alpha\} \cdot \{n, \beta\} = \{m + n + f(\alpha, \beta), \alpha\beta\}.$$

S can be embedded into $Q(S)$ under $(m, \alpha) \to \{m+1, \alpha\}$. Expressing each element of $\mathscr{I}_Q(S)$ in terms of G and I, we obtain

PROPOSITION 3.2 (Dickinson, 1970; Hall, 1969, 1972). *If* $S = (G; I)$, *then* $\Lambda(S) \cong \{[m, \alpha]: m \in Z_+^0, \alpha \in G, m + I(\alpha, \xi) > 0$ *for all* $\xi \in G\}$ *where the binary operation is defined by*

$$[m, \alpha][n, \beta] = [m + n + I(\alpha, \beta) - 1, \alpha\beta]$$

and

$$(x, \xi)[m, \alpha] = (x + m + I(\xi, \alpha) - 1, \xi\alpha) \quad \text{for } (x, \xi) \in S.$$

From now on, we will identify $\Lambda(S)$ with the semigroup of $[m, \alpha]$'s above. Let $\Psi(S)$ denote the archimedean component of $\Lambda(S)$ containing $\Gamma(S)$ and $\mathscr{G}(S)$ the archimedean component containing the identity translation. $\Psi(S)$ is an ideal of $\Lambda(S)$, and $\mathscr{G}(S)$ is the group of units of $\Lambda(S)$ which consists of permutation translations of S.

PROPOSITION 3.3 (Dickinson, 1970; Hall, 1969, 1972). *If* $S = (G; I)$ *then*

$$\begin{aligned}
\Gamma(S) &= \{[m, \alpha]: m \in Z_+, \alpha \in G\},\\
\Psi(S) &= \{\lambda \in \Lambda(S): \lambda^n \in \Gamma(S) \text{ for some } n \in Z_+\},\\
&= \Gamma(S) \cup \{[0, \alpha]: I(\alpha, \xi) > 0 \text{ for all } \xi \in G \text{ and } I(\alpha, \alpha^n) > 1 \text{ for some } n \in Z_+\},\\
\mathscr{G}(S) &= \{[0, \alpha]: I(\alpha, \xi) = 1 \text{ for all } \xi \in G\},\\
\Lambda'(S) &= \Lambda(S) \setminus (\Psi(S) \cup \mathscr{G}(S)) = \{[0, \alpha]: I(\alpha, \xi) > 0 \text{ for all } \xi \in G,\\
&\qquad I(\alpha, \alpha^n) = 1 \text{ for all } n \in Z_+ \text{ and}\\
&\qquad I(\alpha, \xi) > 1 \text{ for some } \xi \in G\}.
\end{aligned}$$

The subsemigroups $\Psi(S)$ *and* $\mathscr{G}(S)$ *are not empty, but the subset* $\Lambda'(S)$ *could be empty.* $\Gamma(S)$ *and* $\Psi(S)$ *are ideals of* $\Lambda(S)$.

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$\Lambda(S)$ is the largest commutative cancellative semigroup which contains S as an ideal in the following sense: If ι_2: $S \to \Lambda(S)$ is the inclusion map and if f: $S \to T$ is an embedding of S into a commutative cancellative semigroup T and $f(S)$ is an ideal of T, then there is an embedding τ: $T \to \Lambda(S)$ such that $\iota_2 = \tau f$.

If $\Lambda'(S)$ is not empty, we shall call each archimedean component of $\Lambda'(S)$ a *middle component*.

Now we introduce the following terminology by specializing the type of $\Lambda(S)$.

DEFINITION 3.4. Let S be an $\mathcal{N}$-semigroup and ε the identity map on S.

(3.4.1) S is *steady* if and only if $\Gamma(S) = \Psi(S)$.

(3.4.2) S is *permutation-free* (*per-free*) if and only if $\mathscr{G}(S) = \{\varepsilon\}$.

(3.4.3) S is *middle-free* (mid-free) if and only if $\Lambda(S) = \Psi(S) \cup \mathscr{G}(S)$.

(3.4.4) S is *endless* if and only if $\Lambda(S) = \Gamma(S) \cup \{\varepsilon\}$.

Since $\Lambda(S) \cong \mathscr{I}_Q(S)$, we have

PROPOSITION 3.5. *Let S be an* $\mathcal{N}$-semigroup.

(3.5.1) *The statements* (3.5.1.1) to (3.5.1.4) *are equivalent*:

(3.5.1.1) *S is steady.*

(3.5.1.2) *If $c \in \mathscr{I}_Q(S)$ and $c^2 \in S$, then $c \in S$.*

(3.5.1.3) *If $c \in \mathscr{I}_Q(S)$ and $c^n \in S$ for some $n \in Z_+$, then $c \in S$.*

(3.5.1.4) *If $c \in \mathscr{I}_Q(S) \setminus S$, then $c^n \in \mathscr{I}_Q(S) \setminus S$ for all $n \in Z_+$.*

(3.5.2) *S is per-free if and only if $c \in \mathscr{I}_Q(S)$ and $cS = S$ imply $c = 1$.*

(3.5.3) *S is mid-free if and only if $c \in \mathscr{I}_Q(S)$ implies $cS = S$ or $c^n \in S$ for some n.*

(3.5.4) *S is endless if and only if $c \in \mathscr{I}_Q(S)$ and $c \neq 1$ imply $c \in S$.*

Proposition 3.5 can be restated in terms of elements of S, for example (3.5.1.2) says: $aS \subseteq bS$ and $a^2 \in b^2 S$ imply $a \in bS$. The proof of the equivalence of the first two of (3.5.1) is due to Petrich (1973). It is easy to show that the last three of (3.5.1) are equivalent. To prove (3.5.2), use the fact that a translation of S is a bijection if and only if it is surjective.

By Proposition 3.3 we have

PROPOSITION 3.6. *Let* $S = (G; I)$.

(3.6.1) *S is steady if and only if $I(\alpha, \xi) > 0$ for all $\xi \in G$ implies $I(\alpha, \alpha^n) = 1$ for all $n \in Z_+$.*

(3.6.2) *S is per-free if and only if $I(\alpha, \xi) = 1$ for all $\xi \in G$ implies $\alpha = \varepsilon$.*

(3.6.3) *S is middle-free if and only if $I(\alpha, \xi) > 0$ for all $\xi \in G$ implies $I(\alpha, \alpha^n) > 1$ for some $n \in Z_+$ or $I(\alpha, \xi) = 1$ for all $\xi \in G$.*

(3.6.4) *S is endless if and only if $I(\alpha, \xi) > 0$ for all $\xi \in G$ implies $\alpha = \varepsilon$, that is, for each $\alpha \neq \varepsilon$, $I(\alpha, \eta) = 0$ for some $\eta \in G$.*

Describing (3.6.1) and (3.6.4) in terms of defining functions φ, we have

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PROPOSITION 3.7.
(3.7.1) *$S = ((G; \varphi))$ is steady if and only if, for each $\alpha \in G$, $\varphi(\alpha)+\varphi(\xi)-\varphi(\alpha\xi) > 0$ for all $\xi \in G$ implies $\varphi(\alpha)+\varphi(\alpha^m)-\varphi(\alpha^{m+1}) = 1$ for all $m \in Z_+$.*
(3.7.2) *$S = ((G; \varphi))$ is endless if and only if for each $\alpha \neq \varepsilon$, there is $\xi \in G$ such that $\varphi(\alpha)+\varphi(\xi)-\varphi(\alpha\xi) = 0$.*

4. Structures

Let $S = (G; I) = ((G; \varphi))$ be an $\mathcal{N}$-semigroup. Define K by

$$K = \{\alpha \in G: I(\alpha, \xi) = 1 \text{ for all } \xi \in G\}.$$

The K is a subgroup of G since $\mathcal{G}(S) \cong K$ under $[0, \alpha] \to \alpha$ by Proposition 3.3.

LEMMA 4.1. *Let $\xi, \eta, \zeta, \lambda \in G$, and let $S = (G; I) = ((G; \varphi))$. Then $\xi \equiv \eta \pmod K$ if and only if $I(\xi, \zeta) = I(\eta, \zeta)$ for all $\zeta \in G$. In this case, if $\xi \equiv \eta$ and $\lambda \equiv \zeta \pmod K$, then $I(\xi, \lambda) = I(\eta, \zeta)$.*

PROOF. Assume $\xi \equiv \eta \pmod K$. We prove that if $\alpha \in K$, $I(\alpha\xi, \zeta) = I(\xi, \zeta)$. First $I(\alpha, \xi)+I(\alpha\xi, \zeta) = I(\alpha, \xi\zeta)+I(\xi, \zeta)$ for all $\zeta \in G$. Since $I(\alpha, \xi) = I(\alpha, \xi\zeta) = 1$, we have $I(\alpha\xi, \zeta) = I(\xi, \zeta)$ for all $\zeta \in G$. As $\eta = \alpha\xi$ for some $\alpha \in K$, we have done the necessity. Conversely assume $I(\xi, \zeta) = I(\eta, \zeta)$ for all $\zeta \in G$. Now

$$I(\xi^{-1}\eta, \xi)+I(\eta, \zeta) = I(\xi^{-1}\eta, \xi\zeta)+I(\xi, \zeta)$$

for all $\zeta \in G$. Then $I(\eta, \zeta) = I(\xi, \zeta)$ implies $I(\xi^{-1}\eta, \xi) = I(\xi^{-1}\eta, \xi\zeta)$. Since ζ is arbitrary, we have

$$I(\xi^{-1}\eta, \xi\zeta) = I(\xi^{-1}\eta, \varepsilon) = 1 \quad \text{for all } \zeta \in G.$$

Therefore $\xi^{-1}\eta \in K$, that is, $\xi \equiv \eta \pmod K$.

THEOREM 4.2. *Let S be a commutative semigroup. S is an $\mathcal{N}$-semigroup if and only if S is isomorphic to a spined product of a per-free $\mathcal{N}$-semigroup S^* and an abelian group G.*

PROOF. Assume $S = (G; I)$. Let $G^* = G/K$ where K is defined at the beginning of this section; and let $g: G \to G^*$ be the natural homomorphism: $\xi g = \xi^*$, $\xi \in G$. Define $I^*: G^* \times G^* \to Z_+^0$ by

$$I^*(\alpha^*, \beta^*) = I(\alpha, \beta). \tag{4.2.1}$$

I^* is well defined because of Lemma 4.1. It is easy to see that I^* satisfies (2.1.1) through (2.1.4). Let $S^* = (G^*; I^*)$. Thus we have the $\mathcal{N}$-semigroup S^*. If $I^*(\alpha^*, \xi^*) = 1$ for all $\xi^* \in G^*$, then, by the definition of I^*, $I(\alpha, \xi) = 1$ for all $\xi \in G$, hence $\alpha \in K$, equivalently, $\alpha^* = \varepsilon^*$ where ε^* is the identity element of G^*. Therefore

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S^* is per-free. An element of S is denoted by (m, ξ), $m \in Z_+^0$, $\xi \in G$; an element of S^* is denoted by (n, η^*) $n \in Z_+^0$, $\eta^* \in G^*$. Define $h: S \to S^*$ by

$$(m, \xi) \mapsto (m, \xi^*).$$

Obviously h is onto, and also we have

$$\begin{aligned}((m, \xi)(n, \eta))h &= (m+n+I(\xi, \eta), \xi\eta)h \\ &= (m+n+I(\xi,\eta), (\xi\eta)^*) \\ &= (m+n+I^*(\xi^*, \eta^*), \xi^*\eta^*) \\ &= (m, \xi^*)(n, \eta^*) \\ &= (m, \xi)h \cdot (n, \eta)h.\end{aligned}$$

Thus h is a homomorphism of S onto S^*.

Now define $k: S \to G$ and $f: S^* \to G^*$ by

$$(m, \xi)k = \xi, \quad (m, \eta^*)f = \eta^*, \quad \text{respectively.}$$

Then k and f are surjective homomorphisms.

Let ρ, τ and σ denote the congruences on S induced by the homomorphisms h, k and kg, respectively, that is,

$(m, \xi)\,\rho(n, \eta)$ if and only if $m = n$ and $\xi \equiv \eta \pmod K$.
$(m, \xi)\,\tau(n, \eta)$ if and only if $\xi = \eta$.
$(m, \xi)\,\sigma(n, \eta)$ if and only if $\xi \equiv \eta \pmod K$.

Then $\rho \cap \tau = \iota$. To show $\rho \cdot \tau = \sigma$, assume $(m, \xi)\,\rho \cdot \tau(l, \zeta)$. Then $(m, \xi)\,\rho(n, \eta)\,\tau(l, \zeta)$ implies $m = n$, and $\xi \equiv \eta = \zeta$ by definition, so $\xi \equiv \zeta \pmod K$, hence $(m, \xi)\,\sigma(l, \zeta)$. Thus $\rho \cdot \tau \subseteq \sigma$. Assume $(m, \xi)\,\sigma(l, \zeta)$. By definition, $\xi \equiv \zeta \pmod K$. We get $(m, \xi)\,\rho(m, \zeta)\,\tau(l, \zeta)$ whence $\sigma \subseteq \rho \cdot \tau$. We have shown $\rho \cdot \tau = \sigma$. Immediately we can show $\tau \cdot \rho = \sigma$. By Proposition 2.4, S is isomorphic to the spined product of S^* and G with respect to $f: S^* \to G^*$ and $g: G \to G^*$.

$$S \cong S^* \underset{G^*; f, g}{\bowtie} G.$$

The converse is due to Proposition 2.5.

We want to observe a relation between $\Lambda(S)$ and $\Lambda(S')$ in the most general case. Let S and S' be $\mathcal{N}$-semigroups, $h: S \to S'$ a homomorphism of S onto S', that is $S' = Sh$, and $\tilde{h}$ the extension of h to a homomorphism of $Q(S)$ onto $Q(S')$. ($\tilde{h}$ is defined by $(xy)^{-1}\tilde{h} = (xh)(yh)^{-1}$ if $x, y \in S$.) Let N be the kernel of $\tilde{h}$. Then $S \cap N = \emptyset$. Every homomorphism h of S onto S' is determined by a subgroup N of $Q(S)$ such that $S \cap N = \emptyset$ (the author 1973a). It is easy to see that SN is an archimedean subsemigroup of $Q(S)$. Since $S \cap N = \emptyset$ implies $SN \cap N = \emptyset$, SN has no idempotent. Thus SN is an $\mathcal{N}$-semigroup. It is easy to see $Q(S) = Q(SN)$. Let $\tilde{\iota}: Q(S) \to Q(SN)$ denote the identity map of $Q(S)$. Let $\iota_1: S \to Q(S)$, $\iota_1': S' \to Q(S')$ and $\iota_1'': SN \to Q(SN)$ be the inclusion mappings in the natural way. Let $h_1: S \to SN$ be the embedding defined by $x \mapsto xe$ where e is the identity of N. Given h, define $h_2: SN \to S'$ by $(xm)h_2 = xh$ where $x \in S$, $m \in N$. h_2 is well defined and a surjective homomorphism. Elements of $\Lambda(S)$, $\Lambda(S')$ and $\Lambda(SN)$ are denoted

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by λ, λ' and λ'' respectively. However, by Proposition 3.1, every λ of $\Lambda(S)$ is given by $x\lambda = xa$, $x \in S$ where $a \in \mathscr{I}_Q(S)$. The λ associated with a is denoted by λ_a. Similarly λ' associated with $b' \in \mathscr{I}_Q(S')$ and λ'' associated with $c'' \in \mathscr{I}_Q(SN)$ are respectively denoted by $\lambda'_{b'}$, where $b' \in \mathscr{I}_Q(S')$, $\lambda''_{c''}$ where $c'' \in \mathscr{I}_Q(SN)$. Since $S \subseteq SN$, $\mathscr{I}_Q(S) \subseteq \mathscr{I}_Q(SN)$. So let $\bar{h}_1$: $\Lambda(S) \to \Lambda(SN)$ be the inclusion map, that is, $\lambda_a \bar{h}_1 = \lambda''_a$ for $\lambda \in \Lambda(S)$.

Let ι_2: $S \to \Lambda(S)$, ι'_2: $S' \to \Lambda(S')$ and ι''_2: $SN \to \Lambda(\mathrm{SN})$ be the embeddings, that is ι_2, ι'_2 and ι''_2 are the isomorphisms of S, S' and SN onto $\Gamma(S)$, $\Gamma(S')$ and $\Gamma(SN)$ defined by $a\iota_2 = \lambda_a$ $(a \in S)$, $b'\iota'_2 = \lambda'_{b'}$ $(b' \in S')$, $c''\iota''_2 = \lambda''_{c''}$ $(c'' \in SN)$ respectively. Let F: $\Lambda(S) \to L$, F': $\Lambda(S') \to L'$ and F'': $\Lambda(SN) \to L''$ be the greatest semilattice homomorphisms.

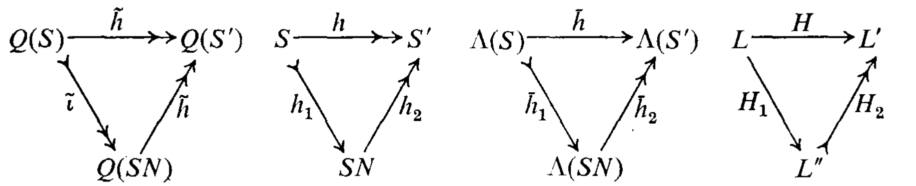
In the diagram below, $\rightarrowtail$ means "one-to-one", $\twoheadrightarrow$ "onto" and $\rightarrowtail\!\!\!\!\twoheadrightarrow$ "one-to-one, onto". We will frequently identify $\mathscr{I}_Q(S)$ with $\Lambda(S)$. Therefore, if $X \subset Q(S)$ $\Lambda(S) \cdot X$ can be considered.

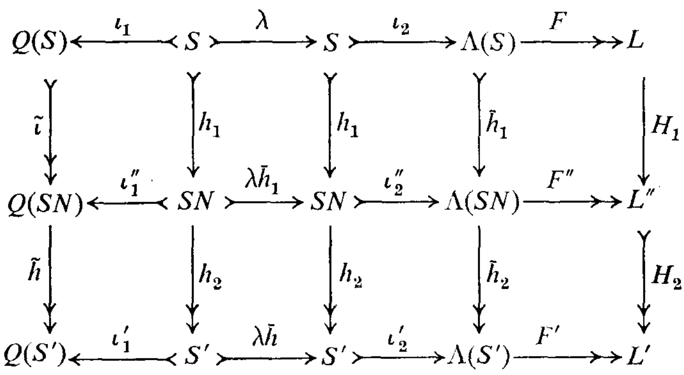
Let A be a semigroup, B a semigroup with idempotent. A homomorphism f of A into B is called trivial if $|Af| = 1$.

THEOREM 4.3. *Given a homomorphism h of an $\mathscr{N}$-semigroup S onto an $\mathscr{N}$-semigroup S', there exist a unique nontrivial homomorphism $\bar{h}$ of $\Lambda(S)$ into $\Lambda(S')$, unique homomorphism $\bar{h}_2$ of $\Lambda(SN)$ onto $\Lambda(S')$, unique nontrivial homomorphism H and H_1 of L into L' and L'', respectively, and an isomorphism H_2 which is a unique homomorphism of L'' onto L', such that*

(4.3.0) $$\Lambda(S)\bar{h} \subseteq (\Lambda(SN))\bar{h}_2 = \Lambda(S')$$

and for each $\lambda \in \Lambda(S)$, the following diagrams are commutative.





Moreover,

$$(\Psi(S))\bar{h} \subseteq \Psi(S')$$
$$(\mathscr{G}(S))\bar{h} \subseteq \mathscr{G}(\mathrm{S}')$$

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Proof. First we show $\mathscr{I}_Q(S))\tilde{h}\subseteq\mathscr{I}_Q(S')$ where $S'=Sh$. Let $a\in\mathscr{I}_Q(S)$. By definition, $aS\subseteq S$ and then $(a\tilde{h})S'=(a\tilde{h})(Sh)=(aS)\tilde{h}\subseteq S\tilde{h}=Sh=S'$, hence $a\tilde{h}\in\mathscr{I}_Q(S')$. As stated before the theorem, every λ of $\Lambda(S)$ is given by λ_a, $a\in\mathscr{I}_Q(S)$.

Given h, define $\bar{h}$: $\Lambda(S)\to\Lambda(S')$ by

$$\lambda_a\bar{h}=\lambda'_{a\tilde{h}},\quad a\in\mathscr{I}_Q(S).$$

To prove (4.3.0) it is sufficient to show

$$(\mathscr{I}_Q(S))\tilde{h}\subseteq(\mathscr{I}_Q(SN))\tilde{h}=\mathscr{I}_Q(S'), \tag{4.3.0'}$$

but we have shown

$$(\mathscr{I}_Q(S))\tilde{h}\subseteq\mathscr{I}_Q(S'). \tag{4.3.1}$$

Second, we want to show

$$\mathscr{I}_Q(S')\subseteq(\mathscr{I}_Q(SN))\tilde{h}. \tag{4.3.2}$$

Let $b'\in\mathscr{I}_Q(S')$, $b'=b\tilde{h}$ for some $b\in Q(S)$. By definition, $(bx)\tilde{h}=(b\tilde{h})(x\tilde{h})\in S\tilde{h}$ for all $x\in S$. It follows that $bx\in SN$ and $bxm\in SN$ for all $x\in S$, all $m\in N$; then $b\in\mathscr{I}_Q(SN)$, hence $b'\in(\mathscr{I}_Q(SN))\tilde{h}$. We have (4.3.2).

Next, we want to prove

$$(\mathscr{I}_Q(SN))\tilde{h}\subseteq\mathscr{I}_Q(S'). \tag{4.3.3}$$

Recall h_2: $SN\to S'$ is a homomorphism of SN onto S' and so h_2 can be extended to $\tilde{h}_2$: $Q(SN)\to Q(S')$. By (4.3.1) we have $\mathscr{I}_Q(SN))\tilde{h}_2\subseteq\mathscr{I}_Q((SN)h_2)$. However, $Q(SN)=Q(S)$ and we see $\tilde{h}_2=\tilde{h}$ as follows: Let $a\in Q(SN)=Q(S)$,

$$a=(xm)(yn)^{-1}=zu^{-1}$$

where $x,y,z,u\in S$, $m,n\in N$. Then

$$a\tilde{h}_2=(xm)h_2\cdot((yn)h_2)^{-1}=(xh)(yh)^{-1}=(zh)(uh)^{-1}=a\tilde{h}$$

for all $a\in Q(S)$, hence $\tilde{h}_2=\tilde{h}$. Also it is easy to see $(SN)h_2=Sh=S'$. Therefore we have (4.3.3). Combining (4.3.1), (4.3.2) and (4.3.3), we get (4.3.0'). As $\bar{h}$ was defined from h, we define $\bar{h}_2$: $\Lambda(SN)\to\Lambda(S')$ from h_2 as follows:

Since $c\tilde{h}_2=c\tilde{h}$,

$$\lambda_c\bar{h}_2=\lambda'_{c\tilde{h}},\quad c\in\mathscr{I}_Q(SN).$$

We can easily see that $\bar{h}_2$ is onto. Consequently (4.3.0) has been proved.

Next we show the commutativity of the diagrams. First it is obvious that $\bar{\iota}\bar{h}=\bar{h}$. If $x\in S$, $xh_1h_2=(xe)h_2=xh$ where e is the identity of N, hence $h_1h_2=h$. For all $x\in S$ and all $a\in\mathscr{I}_Q(S)$, $\lambda_a\bar{h}_1\bar{h}_2=\lambda''_a\bar{h}_2=\lambda'_{a\tilde{h}}=\lambda_a\bar{h}$, hence $\bar{h}_1\bar{h}_2=\bar{h}$. For $x\in S$, $x\iota_1\bar{\iota}=x\bar{\iota}=x=xe=xh_1=xh_1\iota''_1$, hence $\iota_1\bar{\iota}=h_1\iota''_1$. For $xm\in SN$, $x\in S$, $m\in N$, $(xm)\iota''_1\bar{h}=(xm)\bar{h}=xh=xh\iota'_1=(xm)h_2\iota'_1$, hence $\iota''_1\bar{h}=h_2\iota'_1$. Let $x\in S$, $a\in\mathscr{I}_Q(S)$. Then $x\lambda_ah_1=(x\lambda_a)e=x\lambda_a=(xe)\lambda''_a=(xh_1)(\lambda_a\bar{h}_1)$, hence

$$\lambda_ah_1=h_1(\lambda_a\bar{h}_1).$$

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Let $a \in \mathscr{I}_Q(S)$. If $x \in S$ and $m \in N$,

$$(xm)(\lambda_a \bar{h}_1) h_2 = (xm)\lambda''_a h_2 = (xma) h_2 = ((xa)m) h_2 = (xa) h$$
$$= (xa)\bar{h} = (x\bar{h})(a\bar{h}) = (xh)\lambda'_{a\bar{h}} = (xm) h_2 \cdot (\lambda_a \bar{h}).$$

Thus $(\lambda_a \bar{h}_1) h_2 = h_2 \cdot (\lambda_a \bar{h})$. For $a \in S \subseteq SN$,

$$a\iota_2 \bar{h}_1 = \lambda_a \bar{h}_1 = \lambda_a = a\iota''_2 = (ae)\iota''_2 = ah_1 \iota''_2$$

whence $\iota_2 \bar{h}_1 = h_1 \iota''_2$. Let $a \in S$, $n \in N$,

$$(an)\iota''_2 \bar{h}_2 = \lambda''_{an} \bar{h}_2 = \lambda'_{(an)\bar{h}_2} = \lambda'_{(an)\bar{h}} = \lambda'_{a\bar{h}} = \lambda'_{ah} = (ah)\iota'_2 = (an) h_2 \iota'_2.$$

Hence $\iota''_2 \bar{h}_2 = h_2 \cdot \iota'_2$. Consider the greatest semilattice decompositions:

$$\Lambda(S) = \bigcup_{\alpha \in L} \Lambda_\alpha(S), \qquad \Lambda(SN) = \bigcup_{\beta \in L''} \Lambda_\beta(SN), \qquad \Lambda(S') = \bigcup_{\gamma \in L'} \Lambda_\gamma(S'),$$

where $\Lambda_\alpha(S)$, $\Lambda_\beta(SN)$ and $\Lambda_\gamma(S')$ are archimedean components, in other words, if $\lambda \in \Lambda(S)$, $\lambda \varepsilon \Lambda_{\lambda F}(S)$; if $\lambda'' \in \Lambda(SN)$, $\lambda'' \in \Lambda_{\lambda'' F''}(SN)$; if $\lambda' \in \Lambda(S')$, $\lambda' \in \Lambda'_{\lambda' F'}(S')$. Recall $\bar{h}_1$ is the inclusion. Since $\Lambda_\alpha(S)$ is archimedean for each $\alpha \in L$, $(\Lambda_\alpha(S))\bar{h} \subseteq \Lambda_\beta(SN)$ for some $\beta \in L''$. Define $H_1 : L \to L''$ by $\alpha H_1 = \beta$, that is, $(\Lambda_\alpha(S))\bar{h}_1 \subseteq \Lambda_{\alpha H_1}(SN)$. For $\alpha_1, \alpha_2 \in L$, we have $(\Lambda_{\alpha_1\alpha_2}(S))\bar{h}_1 \subseteq \Lambda_{(\alpha_1\alpha_2)H_1}(SN)$ and $(\Lambda_{\alpha_1}(S) \cdot \Lambda_{\alpha_2}(S))\bar{h}_1 = (\Lambda_{\alpha_1}(S))\bar{h} \cdot (\Lambda_{\alpha_2}(S))\bar{h} \subseteq \Lambda_{\alpha_1 H_1}(SN) \cdot \Lambda_{\alpha_2 H_1}(SN) \subseteq \Lambda_{(\alpha_1 H_1)(\alpha_2 H_1)}(SN)$,
hence $\Lambda_{(\alpha_1\alpha_2)H_1}(SN) = \Lambda_{(\alpha_1 H_1)(\alpha_2 H_1)}(SN)$ which implies $(\alpha_1 \alpha_2) H_1 = (\alpha_1 H_1)(\alpha_2 H_1)$. Thus H_1 is a homomorphism of L into L''. Let $\lambda \in \Lambda(S)$. Then $\lambda \in \Lambda_{\lambda F}(S)$ and $(\Lambda_{\lambda F}(S))\bar{h} \subseteq \Lambda_{\lambda F H_1}(SN)$ but $\lambda \bar{h}_1 = \lambda$ and $\lambda \bar{h}_1 = \lambda \in \Lambda_{\lambda F''}(SN)$, whence $FH_1 = \bar{h}_1 F''$. Now $\bar{h}_2 F' : \Lambda(SN) \twoheadrightarrow L'$ is a semilattice homomorphism of $\Lambda(SN)$. Since F'': $\Lambda(SN) \to L''$ is greatest, there is a homomorphism $H_2 : L'' \twoheadrightarrow L'$ such that $F'' H_2 = \bar{h}_2 F'$. Of course H is defined by $H = H_1 H_2$. (The commutativity of the remaining parts immediately follows: for example, $\iota_2 \bar{h} = h\iota'_2$ is a consequence of $h = h_1 h_2$, $\iota_2 \bar{h}_1 = h_1 \iota''_2$ and $\iota''_2 \bar{h}_2 = h_2 \iota'_2$.) Thus the commutativity has been proved.

We have to show H_2 is an isomorphism. Recall $\bar{h}_2 = \bar{h}$, $\Lambda(SN)\bar{h}_2 = \Lambda(S')$, $(\mathscr{I}_Q(SN)\bar{h} = \mathscr{I}_Q(S')$ and $N \subseteq \mathscr{G}(SN) \subset \Lambda(SN)$. We see that if $\lambda \in \Lambda_\beta(SN)$, then $\lambda N \subseteq \Lambda_\beta(SN)$, thus $\Lambda_\beta(SN)$ is a union of N-cosets for each $\beta \in L''$. Let $\bar{h}_{2,\beta} = \bar{h}_\lambda | \Lambda_\beta(SN)$. Then $\bar{h}_{2,\beta}$ maps $\Lambda_\beta(SN)$ onto some $\Lambda_{\beta'}(S')$, and each $\Lambda_\gamma(S')$, $\gamma \in L''$, is obtained as an image of $\Lambda_\beta(SN)$. Hence H_2: $L'' \twoheadrightarrow L'$ is one-to-one.

It remains to prove the uniqueness of $\bar{h}_2$, $\bar{h}$, H_1, H_2 and H. Suppose $\bar{\bar{h}}_2$: $\Lambda(SN) \twoheadrightarrow \Lambda(S')$, $\bar{\bar{h}}$: $\Lambda(S) \to \Lambda(S')$, satisfy:

$$\iota''_2 \bar{\bar{h}}_2 = h_2 \iota'_2, \qquad h_1 \bar{\bar{h}}_2 = \bar{\bar{h}}.$$

For all $a \in SN$, $\lambda''_a \bar{\bar{h}}_2 = a\iota''_2 \bar{\bar{h}}_2 = ah_2 \iota'_2 = a\iota''_2 \bar{h}_2 = \lambda''_a \bar{h}_2$. Let $c \in \mathscr{I}_Q(SN)$, $c = ab^{-1}$, $a, b \in SN$. Since $\mathscr{I}_Q(SN) \cong \Lambda(SN)$ under $z \mapsto \lambda''_z$, $cb = a$ implies $\lambda''_c \lambda''_b = \lambda''_a$. Take the homomorphic images under $\bar{\bar{h}}_2$ and $\bar{h}_2$, then

$$(\lambda''_c \bar{\bar{h}}_2)(\lambda''_b \bar{\bar{h}}_2) = \lambda''_a \bar{\bar{h}}_2, \qquad (\lambda''_c \bar{h}_2)(\lambda''_b \bar{h}_2) = \lambda''_a \bar{h}_2,$$

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but $\lambda''_b\bar{\bar{h}}_2 = \lambda'_{b\tilde{h}_2} = \lambda'_{b\tilde{h}}$, $\lambda''_b\bar{h}_2 = \lambda'_{b\tilde{h}_2} = \lambda'_{b\tilde{h}}$, $\lambda''_a\bar{\bar{h}}_2 = \lambda'_{a\tilde{h}} = \lambda''_a\bar{h}_2$. Since $\Lambda(S')$ is cancellative, we have $\lambda''_c\bar{\bar{h}}_2 = \lambda''_c\bar{h}_2$ for all $c\in\mathscr{I}_Q(SN)$, that is, $\bar{\bar{h}}_2 = \bar{h}_2$. Accordingly $\bar{\bar{h}} = h_1\bar{\bar{h}}_2 = h_1\bar{h}_2 = \bar{h}$. Suppose $H'_1: L \twoheadrightarrow L''$, $H'_2: L'' \twoheadrightarrow L'$ and $H': L \twoheadrightarrow L'$ satisfy $H'_1H'_2 = H'$, $FH'_1 = \bar{h}_1F''$, $F''H'_2 = \bar{h}_2F'$. Since F and F'' are onto, $FH'_1 = FH_1$ implies $H'_1 = H_1$; $F''H'_2 = F''H_2$ implies $H'_2 = H_2$; hence $H = H'$. The uniqueness has been shown.

Since $\Psi(S)$ and $\Psi(S')$ are the archimedean ideal components of $\Lambda(S)$ and $\Lambda(S')$ respectively, $(\Psi(S))\bar{h}\subseteq\Psi(S')$. Since $\mathscr{G}(S)$ and $\mathscr{G}(S')$ are the greatest subgroups of $\Lambda(S)$ and $\Lambda(S')$, respectively, $(\mathscr{G}(S))\bar{h}\subseteq\mathscr{G}(S')$. The nontrivality of the homomorphisms immediately follows from this fact. The proof of the theorem has been completed.

Corollary 4.4. *The following are equivalent*:

(4.4.1) $N\subseteq\mathscr{G}(S)$.

(4.4.2) $N\subseteq\Lambda(S)$.

(4.4.3) $\Lambda(S) = \Lambda(SN)$.

(4.4.4) $\bar{h}_1$ *is onto.*

Corollary 4.5. $\bar{h}$ *is onto if and only if* $\Lambda(S)N = \Lambda(SN)$.

Proof. Assume $\bar{h}$ is onto. Then $(\mathscr{I}_Q(S))\tilde{h} = \mathscr{I}_Q(S')$, hence $(\mathscr{I}_Q(S))\tilde{h} = (\mathscr{I}_Q(SN))\tilde{h}$ by (4.3.0′). It follows that $\mathscr{I}_Q(S)N = \mathscr{I}_Q(SN)N$. But, since $N\subseteq\mathscr{I}_Q(SN)$, we have $\mathscr{I}_Q(S)N = \mathscr{I}_Q(SN)$. Conversely if $\mathscr{I}_Q(S)N = \mathscr{I}_Q(SN)$, then

$$(\mathscr{I}_Q(S))\tilde{h} = (\mathscr{I}_Q(SN))\tilde{h} = \mathscr{I}_Q(S'),$$

hence $\bar{h}$ is onto.

Let A be a commutative semigroup. We define a quasi-order $\|$ on A as follows:

$$x\|y \text{ if and only if } xz = y^m \text{ for some } z\in A \text{ and some } m\in Z_+.$$

Let B be a subsemigroup of A. B is called $\|$-cofinal in A if, for every $a\in A$, there are $b, c\in B$ such that $b\|a$ and $a\|c$.

Corollary 4.6. *The following are equivalent*:

(4.6.1) $\Lambda(S)$ *is* $\|$*-cofinal in* $\Lambda(SN)$.

(4.6.2) H_1 *is onto.*

(4.6.3) H *is onto.*

Corollary 4.7. *The following are equivalent*:

(4.7.1) H_1 *is one-to-one.*

(4.7.2) H *is one-to-one.*

(4.7.3) *For* $\lambda,\mu\in\Lambda(S)$, $\lambda^m\in\mu\cdot\Lambda(SN)$ *and* $\mu^n\in\lambda\cdot\Lambda(SN)$ *for some* $m,n\in Z_+$ *imply* $\lambda^k\in\mu\cdot\Lambda(S)$ *and* $\mu^l\in\lambda\cdot\Lambda(S)$ *for some* $k, l\in Z_+$.

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COROLLARY 4.8. *If $N \subseteq \Lambda(S)$, then H is an isomorphism of L onto L'.*

Returning to Theorem 4.2, the $\tilde{h}$ has

$$\ker \tilde{h} = \{(0, \alpha) : \alpha \in K\} = \mathscr{G}(S).$$

By Corollary 4.8, $\Lambda(S)$ and $\Lambda(S^*)$ have the isomorphic greatest semilattice homomorphic images. For example, S is mid-free if and only if S^* is mid-free.

We give another kind of corollary to Theorem 4.3.

COROLLARY 4.9. *If S is an $\mathscr{N}$-semigroup and S is an ideal of a commutative semigroup T, then a homomorphism h of S onto an $\mathscr{N}$-semigroup S' can be extended to a homomorphism h' of T onto a semigroup T' where S' is an ideal of T'.*

This corollary can be proved by using $h \cdot \lambda'_{a\tilde{h}} = \lambda_a \cdot h$.

EXAMPLE 4.10. Let $S = \{(x,y) : x, y \in Z_+, x+y \geqslant 3\}$ with usual addition. Then $Q(S) = Z \oplus Z$, $\Lambda(S) \cong \mathscr{I}_Q(S) = Z^0_+ \oplus Z^0_+$. Let $N = \{(2z, -2z) : z \in Z\}$. Then

$$S + N = \{(x,y) : x, y \in Z, x+y \geqslant 3\}$$

and

$$\Lambda(S+N) \cong \mathscr{I}_Q(S+N) = \{(x,y) : x, y \in Z, x+y \geqslant 0\},$$

$$\mathscr{I}_Q(S) + N = N \cup \{(x,y) : x, y \in Z, x+y > 0\} \subsetneqq \mathscr{I}_Q(S+N).$$

Let $Q(S') = Q(S)/N = Z_2 \oplus Z$ where Z_2 is a cyclic group of order 2; let $\tilde{h}: Q(S) \twoheadrightarrow Q(S')$, $h = \tilde{h} | S$. Then

$$S' = Sh = Z_2 \oplus \{3, 4, 5, \ldots\}, \quad \Lambda(S') \cong \mathscr{I}_Q(S') = Z_2 \oplus Z^0_+,$$
$$(\mathscr{I}_Q(S))\tilde{h} = (Z_2 \oplus Z_+) \cup \{\mathbf{0}\}$$

where $\mathbf{0}$ is the identity. On the other hand, $(\mathscr{I}_Q(S+N))\tilde{h} = Z_2 \oplus Z^0_+$. Let $\overline{(x,y)}$ denote the element of $Q(S')$ corresponding to $(x,y) \in Q(S)$. We see

$$\overline{(1,-1)} \in (\mathscr{I}_Q(S+N))\tilde{h} \quad \text{but} \quad \notin (\mathscr{I}_Q(S))\tilde{h}.$$

Hence $(\mathscr{I}_Q(S))\tilde{h} \subsetneqq \mathscr{I}_Q(S')$. Now L is isomorphic to the lattice of subsets of a two-element set; $L' \cong L''$ and it is isomorphic to the chain of two elements. $\tilde{h}$ is not onto but H is onto.

5. Steady or endless $\mathscr{N}$-semigroups

Steady $\mathscr{N}$-semigroups have a simple property relative to direct products.

PROPOSITION 5.1.

(5.1.1) *Let $S_1, \ldots, S_n$ be $\mathscr{N}$-semigroups. The direct product $S_1 \times \ldots \times S_n$ is a steady $\mathscr{N}$-semigroup if and only if S_i is steady for each $i = 1, \ldots, n$.*

(5.1.2) *Let S be an $\mathscr{N}$-semigroup and G an abelian group. $S \times G$ is a steady $\mathscr{N}$-semigroup if and only if S is steady.*

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PROOF. To prove (5.1.1), use the restatement of (3.5.1). The proof of (5.1.2) is technically included in the proof of (5.1.1) since a group trivially satisfies (3.5.1).

Obviously steadiness is not preserved by $\mathcal{N}$-subsemigroups. If S is a steady $\mathcal{N}$-semigroup, the particular homomorphic image S^* given in Theorem 4.2 is steadys. However, a homomorphic image of a steady $\mathcal{N}$-semigroup need not be steady.

EXAMPLE 5.2. We give an example of a steady $\mathcal{N}$-semigroup $S = ((G; \varphi))$ while $\mathcal{R}(\varphi)$ is not steady.

Let $S = ((Z; \varphi))$ where φ is defined by

$$\varphi(m) = \tfrac{1}{3}(2m+3) \quad \text{if } m \in Z_+^0,$$
$$\varphi(-n) = \tfrac{1}{3}(n+3) \quad \text{if } n \in Z_+.$$

Then we see that φ is a defining function on Z and S is steady since φ satisfies (3.7.2). However, $\mathcal{R}(\varphi) = \{\frac{1}{3}(x+3)\colon x \in Z_+^0\}$ and $\mathcal{R}(\varphi)$ is not steady since $\frac{2}{3} \notin \mathcal{R}(\varphi)$ and (3.5.1.2) is not fulfilled. Accordingly "$S \times G$" in (5.1.2) cannot be replaced be "a subdirect product of S and G".

A semigroup of positive real numbers under addition is called a *cone* if it consists of the positive elements of a group of real numbers under addition.

Let S be a positive real number semigroup under addition. As is well known, S is a cone if and only if it is naturally totally ordered, that is, for any distinct elements, $x, y \in S$, either $x \mid y$ or $y \mid x$.

PROPOSITION 5.3. *If S is a cone, then S is an endless $\mathcal{N}$-semigroup. However, the converse is not true in general.*

PROOF. It is obvious that S is an $\mathcal{N}$-semigroup. Suppose S contains the smallest element a with respect to the usual order. Then every element x of S has the form $x = ia$ where $i \in Z_+$, hence $S \cong Z_+$. By using (3.4), we see Z_+ is endless. Assume S contains no smallest element. Also we can assume that S contains the particular number 1. This does not lose generality. Then $P = \{p \in S\colon p \leqslant 1\}$ is the set of prime elements of S with respect to 1. For every $p \in P$, there is a $q \in P$ such that $p = q + r$ for some $r \in P$. This shows that S is endless. The converse is not true since the following counter-example shows:

Let $R(R_+)$ be the semigroup of (positive) rational numbers under addition. Let $S = \{x + y\sqrt{2}\colon x, y \in R_+,\ 1/\sqrt{2} < y/x < \sqrt{3}\}$. S is an endless $\mathcal{N}$-semigroup but it is not a cone since

$$S \subsetneqq \{x + y\sqrt{2}\colon x, y \in R,\ x + y\sqrt{2} > 0\}.$$

We assume S is a power-joined $\mathcal{N}$-semigroup, hence $S = (G; I)$ or $((G; \varphi))$ where G is torsion by Proposition 2.3. The notation $S = (G; I) = ((G; \varphi))$ means that φ satisfies (2.2.4).

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THEOREM 5.4 (Hall, 1972). *If S is a power-joined $\mathcal{N}$-semigroup then S is mid-free.*

PROOF. Suppose $\Lambda(S)$ has a middle component, say, $\Lambda_\alpha(S)$, whose elements have the form $[0,\alpha]$ by Proposition 3.3. Since $I(\alpha,\alpha^n)=1$ for all $n\in Z_+$, we have $[0,\alpha]^n=[0,\varepsilon]\in\Lambda_\alpha(S)$, but $[0,\varepsilon]\in\mathscr{G}(S)$. This is a contradiction.

In Theorem 4.2, let us assume S is a power-joined steady $\mathcal{N}$-semigroup. By Theorem 5.4, S is mid-free. From Theorem 4.3 and Corollaries 4.4, 4.5, 4.8 it follows that $\Psi(S^*)=(\Psi(S))\bar{h}=(\Gamma(S))\bar{h}=\Gamma(S^*)$ and $\mathscr{G}(S^*)=(\mathscr{G}(S))\bar{h}=\{\varepsilon^*\}$. Therefore S^* is endless. Thus we have

THEOREM 5.5. *S is a power-joined steady $\mathcal{N}$-semigroup if and only if S is isomorphic to a spined product of a power-joined endless $\mathcal{N}$-semigroup and a torsion abelian group.*

We end this section by constructing more complicated endless $\mathcal{N}$-semigroups.

THEOREM 5.6. *Let $\{S_\xi=((G_\xi;\varphi_\xi)):\xi\in\Xi\}$ be a family of (power-joined) semigroups S_ξ where $|\Xi|$ is infinite. Then there is an endless (power-joined) $\mathcal{N}$-semigroup $S=((G;\varphi))$ such that*

(5.6.1) *G is the direct sum: $G=\sum_{\xi\in\Xi}G_\xi$.*

(5.6.2) *S_ξ can be embedded into S for all $\xi\in\Xi$.*

PROOF. Let $G=\sum_{\xi\in\Xi}G$. The operation in G is additively denoted, the identity being denoted by 0. If $\alpha\in G$ and if α is not the identity, α is uniquely expressed as

$$\alpha=\alpha_{\xi_1}+\ldots+\alpha_{\xi_k},$$

where $\alpha_{\xi_i}\in G_{\xi_i}$ and α_{ξ_i} is not the identity $(i=1,\ldots,k)$. We want to define $\varphi: G\to\mathbf{R}_+$ such that $\varphi|G_\xi=\varphi_\xi$ for each $\xi\in\Xi$. Now define φ by

$$\varphi(\alpha)=\begin{cases}\varphi_{\xi_1}(\alpha_{\xi_1})+\ldots+\varphi_{\xi_k}(\alpha_{\xi_k}) & \text{if } \alpha\neq 0 \text{ and } \alpha=\alpha_{\xi_1},+\ldots+a_{\xi_k},\\ 1 & \text{if } \alpha=0.\end{cases}$$

Then it is easy to see that φ is a defining function on G. Moreover, if $\alpha\neq 0$ and $\alpha=\alpha_{\xi_1}+\ldots+\alpha_{\xi_k}$, choose $\beta\in G$, $\beta\neq 0$, such that

$$\beta=\beta_{\eta_1}+\ldots+\beta_{\eta_l} \quad\text{and}\quad \eta_i\neq\xi \quad \text{for all } i=1,\ldots,l \text{ and } j=1,\ldots,k.$$

This is possible since $|\Xi|$ is infinite. Then obviously $\alpha+\beta\neq 0$ and

$$\varphi(\alpha)+\varphi(\beta)=\varphi(\alpha+\beta).$$

Therefore $S=((G;\varphi))$ is endless. Obviously each S_ξ can be embedded into S, and it is evident that S is power-joined if and only if each G_ξ is torsion.

The $S=((G;\varphi))$ which has been obtained from $\{S_\xi=((G_\xi;\varphi_\xi)):\xi\in\Xi\}$ is called the *direct union* of $\{S_\xi=((G_\xi;\varphi_\xi)):\xi\in\Xi\}$, and φ is denoted by $\varphi=\sum_{\xi\in\Xi}\varphi_\xi$.

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COROLLARY 5.7. *Every (power-joined) $\mathcal{N}$-semigroup can be embedded into an endless (power-joined) $\mathcal{N}$-semigroup.*

EXAMPLE 5.8. Let Z_4 be the cyclic group of order 4: $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ where $\bar{4} = \bar{0}$. Define $\varphi: Z_4 \to \mathbf{R}_+$ by

$$\varphi(\bar{m}) = \tfrac{1}{2}(m+2)$$

where $\bar{m} = \bar{0}, \bar{1}, \bar{2}, \bar{3}$. Then φ is a defining function on Z_4. Let S be the direct union of $\{((G_i;\ \varphi_i)) : i \in Z_+\}$ where

$$G_i = Z_4 \quad \text{for all } i \in Z_+,$$
$$\varphi_i = \varphi \quad \text{for all } i \in Z_+.$$

Let $\bar{\varphi} = \sum_{i=1}^{\infty} \varphi_i$. Then $\mathscr{R}(\bar{\varphi}) = \{\frac{1}{2}(m+2) : m \in Z_+^0\}$, and $S = ((\sum_{i=1}^{\infty} G_i;\ \bar{\varphi}))$ is power-joined, endless, but $\mathscr{R}(\bar{\varphi})$ is not steady.

Thus endlessness (steadiness) is not preserved under homomorphisms even if the $\mathcal{N}$-semigroup is power-joined. Furthermore, we see that the above example S is never homomorphic onto a steady positive real number semigroup. It is obvious that endlessness is not preserved under subsemigroups. In particular, note that endlessness is not preserved under direct products. For example, Z_+ is endless but $Z_+ \oplus Z_+$ is not endless since the prime elements of $Z_+ \oplus Z_+$ with respect to the element $(1,1)$ have the form $(1,y)$ or $(x,1)$, $x,y \geqslant 1$. The product of any two prime elements cannot be a prime element with respect to $(1,1)$.

6. Admissibility of groups for endlessness

What abelian group G can be a structure group of some endless $\mathcal{N}$-semigroup?

DEFINITION 6.1. Let G be an abelian group. We say G *admits* φ or G is *admissible* if there is a defining function φ on G such that, for each $\alpha \neq \varepsilon$, $\alpha \in G$, $\varphi(\alpha) + \varphi(\beta) = \varphi(\alpha\beta)$ for some $\beta \in G$. In particular, if a β is obtained such that $\alpha\beta \neq \varepsilon$, then we say that *$G$ strongly admits φ* or *G is strongly admissible.*

THEOREM 6.2. *Let G be a finite abelian group.*
(6.2.1) *G is admissible if and only if G is cyclic.*
(6.2.2) *G is never strongly admissible.*

PROOF. (6.2.1) Assume G admits φ. Then $S = ((G;\ \varphi))$ is finitely generated and endless. Since S is steady, $S \cong Z_+ \times G$ by Theorem 7 of the author (1973). Then $\varphi(\xi) = 1$ for all $\xi \in G$, hence S is endless if and only if $|G| = 1$, that is, $S \cong Z_+$. Since every structure group of Z_+ is cyclic, G is cyclic. Conversely, assume that G is a finite cyclic group, say $G = \{\varepsilon, \alpha, \ldots, \alpha^{n-1}\}$, $\alpha^n = \varepsilon$. Define φ by

$$\varphi(\varepsilon) = 1,$$
$$\varphi(\alpha^i) = \frac{i}{n} \quad (i = 1, \ldots, n-1).$$

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Then, if $1 \leqslant i \leqslant n-2$, $\varphi(\alpha^i)+\varphi(\alpha^{n-i-1}) = \varphi(\alpha^{n-1})$ and $\varphi(\alpha^{n-1})+\varphi(\alpha) = \varphi(\varepsilon)$. Hence G is admissible.

(6.2.2) Suppose G strongly admits φ. By (6.2.1) $S = ((G;\varphi)) \cong Z_+$. Choose any element n as a standard element. Then $\{1,2,\ldots,n\}$ is the set of prime elements with respect to n, and the $\mathscr{I}$-function is

$$I_n(i,j) = \begin{cases} 1 & i+j > n, \\ 0 & i+j \leqslant n, \end{cases}$$

hence the defining function φ_n is given by

$$\varphi_n(i) = \frac{1}{n}\left(\sum_{j=1}^{n} I_n(i,j)\right) = \frac{i}{n}$$

because a unique defining function belongs to I_n. Z_n admits φ_n but for the element $n-1$, only the element 1 has the property that $\varphi_n(n-1)+\varphi_n(1) = \varphi_n(n)$. Thus, for any structure group Z_n of S, Z_n is not strongly admissible. Accordingly G is never strongly admissible.

LEMMA 6.3. *If an abelian group G_1 strongly admits φ_1 and if φ_2 is any defining function on G_2, then there is a φ such that $G = G_1 \times G_2$ strongly admits φ, $\varphi \mid G_i = \varphi_i$ and $S_i = ((G_i;\varphi_i))$ can be embedded into $S = ((G;\varphi))$ $(i = 1,2)$.*

PROOF. Let $G = G_1 \times G_2 = \{(\alpha,\beta)\colon \alpha \in G_1, \beta \in G_2\}$. Define φ by

$$\varphi(\alpha,\beta) = \begin{cases} \varphi_1(\alpha)+\varphi_2(\beta) & \text{if } \alpha \neq \varepsilon \text{ and } \beta \neq \varepsilon, \\ \varphi_1(\alpha) & \text{if } \beta = \varepsilon, \\ \varphi_2(\beta) & \text{if } \alpha = \varepsilon, \\ 1 & \text{if } \alpha = \beta = \varepsilon. \end{cases}$$

It is routine but easy to verify $\varphi(\alpha,\beta)+\varphi(\gamma,\delta)-\varphi(\alpha\gamma,\beta\delta) \in Z_+^0$. Let $(\alpha,\beta) \neq (\varepsilon,\varepsilon)$. For (α,ε) with $\alpha \neq \varepsilon$, choose (ε,δ) with $\delta \neq \varepsilon$; then $\varphi(\alpha,\varepsilon)+\varphi(\varepsilon,\delta) = \varphi(\alpha,\delta)$ where $(\alpha,\delta) \neq (\varepsilon,\varepsilon)$. Similarly treat for (ε,β) with $\beta \neq \varepsilon$. If $\alpha \neq \varepsilon$ and $\beta \neq \varepsilon$, we can choose $\gamma \in G_1$ such that $\alpha\gamma \neq \varepsilon$ and $\varphi_1(\alpha)+\varphi_1(\gamma) = \varphi_1(\alpha\gamma)$. Then

$$\varphi(\alpha,\beta)+\varphi(\gamma,\varepsilon) = \varphi_1(\alpha)+\varphi_2(\beta)+\varphi_1(\gamma) = \varphi_1(\alpha\gamma)+\varphi_2(\beta) = \varphi(\alpha\gamma,\beta).$$

In any case $\varphi \mid G_i = \varphi_i$ $(i = 1,2)$ and $S_i = ((G_i;\varphi_i))$ can be embedded into $S = ((G;\varphi))$.

Does there exist a torsion abelian group G and a defining function φ such that G strongly admits φ? G is necessarily infinite because of (6.2.2).

EXAMPLE 6.4. Let p be a positive prime number, and let $S = \{x/p^m\colon x, m \in Z_+\}$. The semigroup S under addition is obviously a cone. By Proposition 5.3, S is endless. The structure group of S with respect to any element of the form p^l $(l \in Z)$ is isomorphic to a quasi-cyclic group $G(p^\infty) = \bigcup_{n=1}^{\infty} C(p^n)$, the union of the ascending

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chain of cyclic groups $C(p^n)$ of order p^n in the natural way. After identifying the isomorphic images we denote it by $G(p^\infty)=\{\tilde{x}/p^m: m\in Z_+, 1\leqslant x\leqslant p^m\}$ whose operation is defined by the usual addition mod 1. (Here the element "1" is an idempotent.) Then the defining function ψ: $G(p^\infty)\to S$ is given by

$$\psi\left(\frac{\tilde{x}}{p^m}\right)=\frac{x}{p^m}.$$

Then if $1\leqslant x\leqslant p^m-1$,

$$\psi\left(\frac{\tilde{x}}{p^m}\right)+\psi\left(\frac{\tilde{1}}{p^{m+1}}\right)=\psi\left(\frac{\widetilde{px+1}}{p^{m+1}}\right).$$

Since $p+1\leqslant px+1\leqslant p^{m+1}-p+1<p^{m+1}$, $\widetilde{px+1}/p^{m+1}$ is not the identity. Thus $G(p^\infty)$ strongly admits ψ.

THEOREM 6.5. *Let G be a torsion abelian group. G is admissible if and only if G is either an infinite torsion group or a finite cyclic group.*

PROOF. Every torsion abelian group G is the direct sum of primary groups (for example, see Rotman, 1965). Thus G is a direct sum

$$G=\sum_{\xi\in\Xi}G_\xi,$$

where G_ξ is either a cyclic group of prime power or a quasi-cyclic group $G(p^\infty)$. If G is finite it has been done in Theorem 6.2. Assume G is infinite. Either $|\Xi|=\infty$ or $|\Xi|<\infty$ and at least one of G_ξ is infinite, hence $G_\xi=G(p^\infty)$. In case $|\Xi|=\infty$, Theorem 5.6 can be applied; in case $G_\xi=G(p^\infty)$, apply Lemma 6.3 and Example 6.4. Thus the theorem has been proved.

PROBLEMS.

1. Is the ψ defined in Example 6.4 the only defining function which is strongly admitted by $G(p^\infty)$?
2. Is every infinite abelian group admissible?
3. Every power-joined $\mathcal{N}$-semigroup S is a subdirect product of a torsion abelian group G and a positive rational semigroup P under addition. Find a necessary and sufficient condition on the subdirect product of G and P in order that S be steady (or endless).

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