BULL. AUSTRAL. MATH. SOC. 46A40
VOL. 33 (1986), 177-187.

SYMMETRY GROUPS ON ORDERED BANACH SPACES

SADAYUKI YAMAMURO

A symmetry of an ordered Banach space is an order and norm isomorphism which commutes with its ideal centre. A class of ordered Banach spaces is introduced to show that, for a space in this class, the group of symmetries is trivial if and only if the space is lattice-ordered. When this group becomes larger, the space approaches an antilattice. This phenomenon is also investigated.

1. Preliminaries.

Let B be a real Banach space ordered by a closed and proper positive cone B^+. Throughout this paper, B is always assumed to be archimedean.

The cannonical half-norm N associated with B^+, due to [3], is defined by

$$N(x) = \inf \{||x + y|| : y \in B^+\} \quad \text{for all} \quad x \in B .$$

An element $x \in B$ is said to be orthogonally decomposable if there exist elements y and z of B^+ such that

Reseived 24 May 1985.

$$x = y - z \ , \quad ||y|| = N(x) \text{ and } ||z|| = N(-x) \ .$$

If every element of B is orthogonally decomposable, then B is said to be orthogonally decomposable. (See [17].)

Let $L(B)$ be the Banach space of all continuous linear operators on B with the operator bound norm. The positive cone $L(B)^+$ consists of ϕ in $L(B)$ such that $\phi(B^+) \subset B^+$. B is said to have the Robinson property if

$$||\phi|| = \sup \{||\phi(x)|| : ||x|| \le 1 , \ x \in B^+\}$$

for all $\phi \in L(B)^+$. (See [16].)

An N-automorphism of B is a bijective element ϕ of $L(B)$ such that $N(\phi(x)) = N(x)$ for all $x \in B$. The set of all N-automorphisms of B is deonted by $G(B)$, which is obviously a group. The following two facts have been proved in [17].

(1.1) *When* B *is orthogonally decomposable,* $\phi \in G(B)$ *if and only if* $||\phi(x)|| = ||x||$ *for all* $x \in B^+$ *and* ϕ *is an o. d. isomorphism* (that is, ϕ is a continuous bijection and $\phi(x) = \phi(y) - \phi(z)$ is an orthogonal decomposition of $\phi(x)$ if and only if $x = y - z$ is an orthogonal decomposition of x).

(1.2) *When* B *is orthogonally decomposable and has the Robinson property, then* $\phi \in G(B)$ *if and only if* ϕ *is a bipositive isometry.*

The ideal centre of B is the set $Z(B)$ of all elements T of $L(B)$ such that there exists a number λ, depending on T, such that $-\lambda x \le Tx \le \lambda x$ for all $x \in B^+$. For $T \in Z(B)$, we can define a norm

$$||T||_0 = \inf\{\lambda \ge 0 : -\lambda x \le Tx \le \lambda x \text{ for all } x \in B^+\} \ .$$

A sufficient condition for $||T||_0 = ||T||$ for all $T \in Z(B)$ is that both the norms of B and B^*, the dual of B, are absolutely monotone. (See [15], Lemma 2.3 and [4], Theorem 1.3.1.) If this is the case, $Z(B)$ is an ordered Banach space with an archimedean order and the multiplicative unit. Hence, by [10], it is an abelian real Banach algebra. For the spectrum Ω of $Z(B)$, the Gelfand transform is an isometric, order and algebraic isomorphism onto $C(\Omega)$. Furthermore, by [15], Corollary 1.13, for every $a \in B^+$, the map $T \mapsto Ta$ is a lattice homomorphism of $Z(B)$

onto a sublattice of B. Following [15], we call B regular if $||T||_o = ||T||$ for all $T \in Z(B)$.

An element ϕ of $G(B)$ such that $\phi T = T\phi$ for all $T \in Z(B)$ is called a symmetry. (See [7] and [8].) The set of all symmetries is denoted by $S(B)$, which is obviously a subgroup of $G(B)$.

The positive cone $(B^*)^+$ of the dual B^* is the set of all $f \in B^*$ such that $f(x) \geq 0$ for all $x \in B^+$. An element f of $(B^*)^+$ is said to be order-continuous if $x = \sup(x_i)$ for an increasing net of positive elements (x_i) implies $f(x) = \sup f(x_i)$. The set of all order-continuous elements of $(B^*)^+$ is denoted by B^{oc}. As in the case of $C[0,1]$, it is possible that B^{oc} can contain only the zero functional. On the other hand, if the norm of B is order-continuous, then $B^{oc} = (B^*)^+$.

An element a of B^+ is said to be (oc)-quasi-interior if $f(a) = 0$ and $f \in B^{oc}$ imply $f = 0$.

Now we set

$$B(G) = \{x \in B : \phi(x) = x \text{ for all } \phi \in G(B)\}$$

and

$$B(S) = \{x \in B : \phi(x) = x \text{ for all } \phi \in S(B)\} .$$

If $B(S)$ contains an (oc)-quasi-interior point, B is said to be H-finite; otherwise, B is called H-infinite.

2. Problems.

When B is a Banach lattice, we have

$$Z(B) = \{T \in L(B) : |Tx| \leq \lambda|x| \text{ for all } x \in B \text{ and some } \lambda\}.$$

It is known ([2], Theorem 3.2) that, when B is σ-complete, $T \in Z(B)$ if and only if T commutes with all band projections.

(2.1) *When B is a σ-complete Banach lattice, we have $S(B) = \{1\}$, where 1 denotes the identity operator.*

Proof. Let $\phi \in S(B)$. Since $Z(B)$ contains all band projections, ϕ commutes with all band projections. Hence, $\phi \in Z(B)$. Now, B is

https://doi.org/10.1017/S000497270000304X Published online by Cambridge University Press

orthogonally decomposable, has the Robinson property and norm, together with the dual norm, is absolutely monotone. Therefore, $||\phi|| = ||\phi||_0 = 1$ and, hence, $0 \leq \phi(x) \leq x$ for all $x \in B^+$. Similarly, $0 \leq \phi^{-1}(x) \leq x$ for all $x \in B^+$. Hence, $\phi = 1$.

Now, let B be a general ordered Banach space. The above fact leads to the following question : is B lattice-ordered if $S(B) = \{1\}$? More generally, we shall consider the following problem.

Problem 1. Is $B(S)$ lattice-ordered?

We shall show that the answer is affirmative for a special class of ordered Banach spaces. This shows that, as B becomes more lattice-like, $S(B)$ will become smaller and $B(S)$ will become larger. An ordered Banach space is called an antilattice if $z = \sup(x,y)$ implies $x \geq y$ or $x \leq y$. (See [11] and [13].) Then, $B(S)$ will be the smallest when B is an antilattice. We consider this problem in the following three forms.

Problem 2. If B is an H-finite antilattice, is $B(S)$ generated by a single (oc)-quasi-interior point?

Problem 3. If B is an H-infinite antilattice, do we have $B(S) = \{0\}$?

Problem 4. If $S(B) = G(B)$, is B an antilattice?

3. Ordered Banach spaces of type (P).

Let B be an ordered Banach space. We suppose that there is a family $\{P_a : a \in B^+\}$ of projections (the idempotent elements of $L(B)$). An orthogonal decomposition $a = b - c$ is called proper if the following two conditions are satisfied:

(1) $P_b(c) = P_c(b) = 0$;

(2) If $\phi(a) = a$ for some $\phi \in S(B)$, then $\phi(b) = b$.

For every $a \in B^+$, we set

$$B_a^+ = \{x \in B^+ : f(x) = 0 \text{ if } f \in B^{oc} \text{ and } f(a) = 0\}.$$

https://doi.org/10.1017/S000497270000304X Published online by Cambridge University Press

An ordered Banach space B is said to be <u>of type</u> (P) if it is regular and it is equipped with a family $\{P_a : a \in B^+\}$ such that the following conditions are satisfied:

$(P1)$ $P_a(B^+) = B_a^+$ <u>and</u> $a \in B_a^+$ <u>for all</u> $a \in B^+$

$(P2)$ <u>If</u> $a \in B(S)$, then $P_a \leq 1$.

$(P3)$ <u>Every element of</u> B <u>admits a proper decomposition</u>.

Before proceding further, we give some examples.

Example 1. Banach lattices which are σ-complete and in which the norms are order-continuous are of type (P). In this case, P_a is defined by

$$P_a(x) = \sup_{n \geq 1} (x \wedge na) ,$$

which is the band projection associated with $\{a\}^{\perp\perp}$. By definition, the norms of Banach lattices are absolutely monotone. Hence B is regular. Since the norm is order-continuous, $B^{oc} = (B^*)^+$, and B_a^+ coincides with the "positive bipolar" considered in [14] where the equality $(P1)$ has been proved. $(P2)$ and $(P3)$ follow immediately from (2.1) and the basic properties of band projections.

Example 2. Let M be a von Neumann algebra and $B = M^h$ be the ordered Banach space of all hermitian elements of M. Then B is of type (P). First note that, this is obviously regular. For each $a \in B^+$, define P_a by $P_a(x) = s(a)xs(a)$, where $s(a)$ is the support of a. Since $B^{oc} = (M_*)^+$, the positive part of the predual M_*, $f(a) = 0$ for $f \in B^{oc}$ implies $f(s(a)) = 0$, and the relation $(P1)$ can be proved directly or by a modification of a result in [5], p. 357. As to $(P2)$, we first note that $T \in Z(B)$ if and only if $T1 \cap M \in M'$ and $Tx = x \cdot T1$ for every $x \in B$. For the proof of this fact, see, for instance, [1]. Since M^h is orthogonally decomposable and has the Robinson property, $G(B)$ is the set of all bipositive isometries on M^h. In other words, $G(B)$ is the set of restrictions, to M^h, of all Jordan *-isomorphisms

https://doi.org/10.1017/S000497270000304X Published online by Cambridge University Press

ϕ of M such that $\phi(1) = 1$. Furthermore, it follows from the above characterization of $Z(B)$ that $\phi \in S(B)$ if and only if ϕ is the restriction of a Jordan $*$-isomorphism of M which is identical on the centre $M \cap M'$. Hence, we have $B(S) = (M \cap M')^h$. Therefore, if $a \in B(S)$, then $s(a)$ is an central projection. Hence, $P_a(x) \leq x$ for all $x \in B^+$. Finally, the condition $(P3)$ is satisfied because the usual orthogonal decomposition $a = a^+ - a^-$, $a^+ a^- = 0$, is a proper decomposition.

Example 3. Let M be a von Neumann algebra on a Hilbert space H and suppose that there is a cyclic and separating vector $\xi_o \in H$ for M. Then, by the Tomita-Takesaki theory, there are a conjugation operator J and a modular operator Δ associated with ξ_o. The real part of H,

$$H^J = \{\xi \in H : J\xi = \xi\},$$

is then an ordered Hilbert space ordered by the "natural cone"

$$H^+ = \overline{\{\Delta^{\frac{1}{4}} x\, \xi_o : x \in M^+\}}.$$

(See [6] and [9].) This is of type (P). Since the norm is absolutely monotone, H^J is regular. We define P_ξ by $P_\xi = p_\xi j(p_\xi)$, where p_ξ is the projection on the subspace $[M'\xi]$ and $j(p_\xi) = Jp_\xi J$. By [9], Theorems 4.5 and 4.6, we have

$$P_\xi(H^+) = \{\eta \in H^+ : (\eta, \rho) = 0 \text{ if } \rho \in H^+ \text{ and } (\rho, \xi) = 0\}.$$

Therefore, we have the equality $(P1)$ if the norm is order-continuous. To prove this, suppose that $\eta = \sup(\eta_i)$ for an increasing net $(\eta_i) \subset H^+$. Then, $-\eta \leq \eta_i \leq \eta$. The element η can be assumed to be cyclic and separating, because otherwise we can take $\eta + (1 - p_\eta)j(1 - p_\eta)\xi_o$ instead of η. Since $\eta \in H^+$, H^+ is equal to the closure of $\{\Delta_\eta^{\frac{1}{4}} x\, \eta : x \in M^+\}$ and there is an order isomorphism Φ of M^h onto the set

$$\{\rho \in H^J : -\lambda\eta \leq \rho \leq \lambda\eta \text{ for some } \lambda\}$$

https://doi.org/10.1017/S000497270000304X Published online by Cambridge University Press

defined by $\Phi(x) = \Delta_\eta^{1/4} x \eta$. ([9] , Theorem 2.7.) Since (η_i) is contained in this set, there are $x_i \in M^h$ and $x \in M^h$ such that $\Phi(x_i) = \eta_i$, $\Phi(x) = \eta$ and $x = \sup(x_i)$. Furthermore, $||x_i\xi - x\xi|| \to 0$ for every $\xi \in H$. Then, since $||J(x_i - x\xi)|| \to 0$,

$$||\eta_i - \eta||^2 = (\Delta_\eta^{1/4}(x_i - x)\eta , \Delta_\eta^{1/4}(x_i - x)\eta)$$

$$= (\Delta_\eta^{1/2}(x_i - x)\eta , (x_i - x)\eta)$$

$$= (J(x_i - x)\eta , (x_i - x)\eta)$$

$$\le ||J(x_i - x)\eta||\cdot||(x_i - x)\eta|| \to 0 .$$

To prove *(P2)*, we start with a result in [7] that $Z(H^J) = (M \cap M')^h$. On the other hand, $G(H^J)$ is obviously the set of all unitary operators $u \in L(H)$ such that $u(H^+) = H^+$. Hence, $S(H^+)$ is the set of all unitary operators u in $R(M,M')$, the von Neumann algebra generated by M and M' , such that $u(H^+) = H^+$. Now, suppose that $\xi \in H^J(S)$. Then, $uj(u)\xi = \xi$ for all unitary element u of M, because $uj(u) \in S(H^J)$. Since $j(u)\xi = u^*\xi$, we have $[M'\xi] = [M\xi]$, which means that p_ξ is a central projection. Therefore, $p_\xi \le 1$, and, hence, $P_\xi \le 1$. The condition *(P3)* is satisfied because the usual orthogonal decomposition $\xi = \xi^+ - \xi^-$, $(\xi^+ , \xi^-) = 0$, is a proper decomposition.

4. Problems 1, 2 and 3.

We start with a lemma.

(4.1) *Suppose that B is an ordered Banach space which is regular and B^+ is generating. Then, if B is an antilattice and $0 \le P = P^2 \le 1$, then $P = 0$ or $P = 1$.*

Proof. Since $P \in Z(B)$ and $1 - P \in Z(B)$, for each $a \in B^+$ we have that Pa and $(1 - P)a$ belong to a lattice-ordered subset of B , because B is regular. Then, since B is an antilattice, Pa and

https://doi.org/10.1017/S000497270000304X Published online by Cambridge University Press

$(1 - P)a$ must be comparable, that is, $Pa \leq (1 - P)a$ or $Pa \geq (1 - P)a$. It then follows that, for every $a \in B^+$, we have either $Pa = 0$ or $Pa = a$. Suppose that there are nonzero elements a and b of B^+ such that $Pa = 0$ and $Pb = b$. Then, $P(a + b) = b$ and $a + b \neq b$, which is a contradiction. Hence, since B^+ is generating, we have either $P = 0$ or $P = 1$.

We now give the answers to the first three problems when B is of type (P).

(4.2) *Let B be an ordered Banach space of type (P).*

(1). $B(S)$ is lattice-ordered.

(2). If B is an H-finite antilattice, $B(S)$ is generated by an (oc)-quasi-interior point.

(3). If B is an H-infinite antilattice, $B(S) = \{0\}$.

Proof. (1). Let $a \in B(S)$ and $a = b - c$ be a proper decomposition. By $(P3)$, $b \in B(S)$. Therefore, $P_b \leq 1$ by $(P2)$. Furthermore, $(P1)$ implies $P_b \geq 0$. Therefore, if $x \geq a$ and $x \geq 0$, we have $b = P_b a \leq P_b x \leq x$. This means $b = \sup(a,0)$.

(2). Since $B(S)$ is a sublattice of an antilattice, it is totally ordered. Hence it is generated by a single element. Since B is H-finite, the element must be an (oc)-quasi-interior point.

(3). Suppose that $a \in B(S)$ and $a \neq 0$. By $(P3)$, we can assume that $a \in B^+$. By $(P1)$ and $(P2)$, we have $0 \leq P_a \leq 1$. Hence by (4.1), we have $P_a = 0$ or $P_a = 1$. However, by $(P1)$, $P_a = 0$ implies $a = 0$, a contradiction. Hence, $P_a = 1$, or, equivalently, $B_a^+ = B^+$ by $(P1)$. Hence, a is an (oc)-quasi-interior point. This contradicts the assumption.

An immediate consequence of (4.2) (1) is the following fact.

(4.3) *When B is an ordered Banach space of type (P), $S(B) = \{1\}$ implies that B is lattice-ordered.*

https://doi.org/10.1017/S000497270000304X Published online by Cambridge University Press

5. Problem 4.

The answer to this problem is in the negative. We shall give a negative example when B is a finite-dimensional von Neumann algebra. We recall that every finite-dimensional von Neumann algebra M is a direct sum

$$M = M_{k_1} \oplus M_{k_2} \oplus \dots \oplus M_{k_m} ,$$

where M_{k_n} is the algebra of all $k_n \times k_n$ matrices. The set $\{k_1, k_2, \dots k_m\}$ characterizes the structure of M. We shall show that $G(M^h) = S(M^h)$ if and only if the numbers in this set are different.

(5.1) *Let M be a finite-dimensional von Neumann algebra. Then, $G(M^h) = S(M^h)$ if and only if M is a direct sum of factors which are not mutually Jordan *-isomorphic.*

Proof. There are factors $M_n (n = 1, 2, \dots m)$ such that M is a direct sum : $M = M_1 \oplus M_2 \oplus \dots \oplus M_m$. Since the centre of M is the direct sum of centres of M_n , the central projections of M are linear combinations of the following projections:

$$e_1 = 1 \oplus 0 \oplus \dots \oplus , \; e_2 = 0 \oplus 1 \oplus \dots \oplus 0 ,$$

$$e_m = 0 \oplus \dots \oplus 0 \oplus 1 .$$

Now, suppose, for instance, that M_1 and M_2 are Jordan *-isomorphic and ψ is the isomorphism. Then, the map defined by

$$\phi(x_1 \oplus x_2 \oplus \dots \oplus x_m) = (\psi^{-1}(x_2) \oplus \psi(x_1) \oplus x_3 \oplus \dots \oplus x_m)$$

is a Jordan *-isomorphism of M and $\phi(e_1) = e_2$. Therefore, $G(M^h) \neq S(M^h)$. To prove the converse, let $\phi \in G(M^h)$. Since ϕ preserves the minimal projections, ϕ maps the set $\{e_1, e_2, \dots, e_m\}$ into itself. If ϕ is not identical on the centre of M, we may suppose that $\phi(e_1) = e_2$. Then, a map $\psi : M_1 \to M_2$ is defined by the following relation:

$$\phi(x \oplus 0 \oplus \ldots \oplus 0) = \phi((x \oplus 0 \oplus \ldots \oplus 0)e_1)$$
$$= \phi(x \oplus 0 \oplus \ldots \oplus 0)e_2$$
$$= 0 \oplus \psi(x) \oplus 0 \oplus \ldots \oplus 0 .$$

This ψ is a Jordan *-isomorphism of M_1 onto M_2 .

Remark. $G(M^h) = S(M^h)$ if and only if every bijective *o. d.* homomorphism on H is a normal operator. This, and the related problems on *o. d.* homomorphisms, will be discussed in a subsequent paper.

References

[1] E. Alfsen, *Compact Convex Sets and Boundary Integrals*, (Springer-Verlag, Berlin-Heidelberg-New York, 1971.)

[2] W. Arendt, "Spectral properties of Lamperti operators", *Indiana University Math. J.* 32 (1983), 199-215.

[3] W. Arendt, P. Chernoff and T. Kato, "A generalization of dissipativity and positive semigroups", *J. Operator Theory* 8 (1982), 167-180.

[4] C.J.K. Batty and D.W. Robinson, "Positive one-parameter semigroups on ordered Banach spaces", *Acta Math. Applicandae* 2 (1984), 221-296.

[5] O. Bratteli, T. Digernes and D.W. Robinson, "Positive semigroups on ordered Banach spaces", *J. Operator Theory* 9 (1983), 371-400.

[6] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, (Springer-Verlag, Berlin-Heidelberg-New York, 1979).

[7] W. Bos, "A classification for selfdual cones in Hilbert space", *Arch. Math.* 30 (1978), 75-82.

[8] Cho-ho Chu and J.D.M. Wright, "Une théorie des types pour une classe d'espaces de Banach ordonnés", *C. R. Acad. Sc. Paris, Serie A* 281 (1975), 633-636.

[9] A. Connes, "Charactérisation des espaces vectoriels ordonnes sous-jacents aux algèbres de von Neumann", *Ann. Inst. Fourier, Grenoble* 24 (1974), 121-155.

[10] R.V. Kadison, *A representation theory for commutative topological algebras*, (Memoirs of Amer. Math. Soc., No. 7, 1951).

[11] R. V. Kadison, "Order properties of bounded selfadjoint operators", *Proc. Amer. Math. Soc.* 2 (1951), 505-510.

[12] R. V. Kadison, "Isometries of operator algebras", *Ann. Math.* 54 (1951), 325-338.

[13] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, (North Holland, Amsterdam, 1971).

[14] D. W. Robinson and S. Yamamuro, "Hereditary cones, order ideals and half-norms", *Pacific J. Math.* 110 (1984), 335-343.

[15] W. Wils, "The ideal centre of partially ordered vector spaces", *Acta Math.* 127 (1971), 41-77.

[16] S. Yamamuro, "On linear operators on ordered Banach spaces", *Bull. Austral. Math. Soc.* 27 (1983), 285-305.

[17] S. Yamamuro, "On orthogonally decomposable ordered Banach spaces", *Bull. Aust. Math. Soc.* 30 (1984), 357-380.

[18] S. Yamamuro, "Absolute values in orthogonally decomposable spaces", *Bull. Austral. Math. Soc.* 31 (1985), 215-233.

Department of Mathematics
Institute of Advanced Studies
Australian National University
Canberra, A.C.T.
Australia.

Added in proof. 14 January, 1986. The fact, on p.179, that $T \in Z(B)$, for a Banach lattice B , if and only if T commutes with all band projections was given first by W.A.J. Luxemburg in his lecture at the University of Arkansas in 1979.

https://doi.org/10.1017/S000497270000304X Published online by Cambridge University Press