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ON A THEOREM OF SYLVESTER AND SCHUR

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In 1892, Sylvester [7] proved that in the set of integers $n, n+1, \ldots, n+k-1$, $n>k>1$, there is a number containing a prime divisor greater than k. This theorem was rediscovered, in 1929, by Schur [6]. More recent results include an elementary proof by Erdös [1] and a proof of the following theorem by Faulkner [2]: Let p_k be the least prime $\geq 2k$; if $n \geq p_k$ then $\binom{n}{k}$ has a prime divisor $\geq p_k$ with the exceptions $\binom{9}{2}$ and $\binom{10}{3}$. In that paper the author uses some deep results of Rosser and Schoenfeld [5] on the distribution of primes. A note by Moser [4] states that a simple extension of Erdös' proof leads to the result that the product of k consecutive integers greater than k is divisible by a prime $\geq \frac{11}{10}k$.

The object of this note is to prove by elementary means the following theorem:

THEOREM. *The product of k consecutive integers $n(n+1)\cdots(n+k-1)$ greater than k contains a prime divisor greater than $\frac{3}{2}k$ with the exceptions* 3.4, 8.9 *and* 6.7.8.9.10.

We may reformulate the theorem as follows: If $n \geq 2k$ then $\binom{n}{k}$ contains a prime divisor greater than $\frac{3}{2}k$ with the above exceptions.

COROLLARY. *For all $k>1$, $n\geq 2k$, $\binom{n}{k}$ has a prime divisor $\geq \frac{7}{5}k$.*

The result of the corollary is suggested in [4].

The first part of the following proof employs methods similar to those used by Erdös in [1]. In [3] we proved by elementary means the following: The product of the prime powers less than or equal to n is less than 3^n for $n>1$, i.e. if $\alpha=\alpha(p,n)$ is such that $p^\alpha \leq n < p^{\alpha+1}$, then $\prod_{p\leq n} p^\alpha < 3^n$. It is this result that enables us to extend Erdös' work.

Since the exponent β_p to which a prime occurs in $\binom{n}{k}$ is

$$\beta_p = \sum_{i=1}^{[\log_p n]} \left(\left[\frac{n}{p^i}\right] - \left[\frac{n-k}{p^i}\right] - \left[\frac{k}{p^i}\right] \right)$$

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it is easy to see that

LEMMA 1. *If* $p^{\beta_p} \parallel \binom{n}{k}$ *then* $p^{\beta_p} \le n$.

Proof of the theorem. (1). Let $\pi(k)$ denote the number of primes $\le k$. Clearly for $k \ge 8$, $\pi(k) \le \frac{1}{2}k$. Thus if $\binom{n}{k}$ has no prime factor greater than $\frac{3}{2}k$, Lemma 1 implies

$$\binom{n}{k} \le n^{1/2 \cdot 3/2k} \le n^{3/4k}.$$

However since

$$\binom{n}{k} = \frac{n}{k} \cdots \frac{n-1}{k-1} \cdots \frac{n-k+1}{1} > \binom{n}{k}^k$$

we must have

$$\left(\frac{n}{k}\right)^k < n^{3/4k}$$

which is false if $k \le n^{1/4}$. Therefore our theorem holds for $8 \le k \le n^{1/4}$.

It is easy to see that $\pi(k) < \frac{1}{3}k$ for $k \ge 37$ and $\pi(k) < \frac{2}{9}k$ for $k \le 300$. In a similar manner as above we then have that the theorem is true for $37 < k \le n^{1/2}$ and $300 \le k \le n^{2/3}$ in these cases respectively.

(2). We now consider the case $k > n^{2/3}$. If $\binom{n}{k}$ contains no prime divisor exceeding $\frac{3}{2}k$ then by Lemma 1

$$\binom{n}{k} < \prod_{p \le 3/2k} p \prod_{p \le n^{1/2}} p \prod_{p \le n^{1/3}} p \cdots. \tag{1}$$

In [3] we proved by elementary methods that

$$3^{n_0} > \prod_{p \le n_0} p \prod_{p \le n_0^{1/2}} p \prod_{p \le n_0^{1/3}} p \cdots. \tag{2}$$

Therefore, since $k > n^{2/3}$ implies $k^{1/l} > n^{1/(2l-1)}$ for $l \ge 2$, we have

$$3^{3/2k} > \prod_{p \le 3/2k} p \prod_{p \le n^{1/3}} p \prod_{p \le n^{1/5}} p \cdots. \tag{3}$$

Now taking $n_0 = n^{1/2}$ in (2), we find

$$3^{n^{1/2}} > \prod_{p \le n^{1/2}} p \prod_{p \le n^{1/4}} p \prod_{p \le n^{1/6}} p \cdots. \tag{4}$$

Combining (1), (3) and (4) we have under the assumption that $\binom{n}{k}$ is not divisible by any prime exceeding $\frac{3}{2}k$ that

$$\binom{n}{k} < 3^{3/2k + n^{1/2}} \tag{5}$$

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It is easy to prove by induction that $\binom{4k}{k} > \left(\frac{4^4}{3^3}\right)\frac{1}{4k}$. Assume that $n \geq 4k$. Then (5) implies

(6) $$3^{3/2k+n^{1/2}} > \left(\frac{4^4}{3^3}\right)^k \frac{1}{4k}.$$

It now follows from (6) that

$$(\tfrac{3}{2}k+n^{1/2}) \log 3 > k(4 \log 4 - 3 \log 3) - \log 4k$$

and under the initial assumption that $k > n^{2/3}$ that

$$n^{1/2} \log 3 > n^{2/3}(8 \log 2 - \tfrac{9}{2} \log 3) - \log n$$

which is false if $n > 240$.

We now assume $3k \leq n < 4k$. Inductively we can show $\binom{3k}{k} > \left(\frac{3^3}{2^2}\right)^k \frac{1}{3k}$, then as above we have

$$3^{3/2k+n^{1/2}} > \binom{3k}{k} > \left(\frac{3^3}{2^2}\right)^k \frac{1}{3k}$$

which implies

$$(\tfrac{3}{2}k+n^{1/2}) \log 3 > k(3 \log 3 - 2 \log 2) - \log 3k.$$

But since $n < 4k$, we have

$$2k^{1/2} \log 3 > k(\tfrac{3}{2} \log 3 - 2 \log 2) - \log 3k,$$

which is false for $k > 120$ and our theorem holds for $n \geq 480$.

It now only remains to check the cases where $2k \leq n < 3k$, $k > n^{2/3}$. We first prove the following.

LEMMA 2. *There is a prime between* $3n$ *and* $4n$ *for* $n > 1$.

Proof. Assume the contrary. Consider the binomial coefficient $\binom{4n}{n}$. It is easy to see that no prime p, such that $2n < p \leq 3n$ divides $\binom{4n}{n}$. Thus our assumption is that no prime between $2n$ and $4n$ occurs in $\binom{4n}{n}$.

If α_p is the exponent of p in $\binom{4n}{n}$ then

$$\alpha_p = \sum_{i=1}^{[\log_p 4n]} \left(\left[\frac{4n}{p^i}\right] - \left[\frac{3n}{p^i}\right] - \left[\frac{n}{p^i}\right]\right).$$

Since each term appearing in this sum is either 0 or 1 for any p, if $\alpha_p \geq 2$ then $p \leq (4n)^{1/2}$. It now follows that under our assumption

(7) $$\binom{4n}{n} < \prod_{p^\alpha \leq 2n} p \prod_{p \leq (4n)^{1/2}} p$$

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since if $p^{\alpha_p} \leq 2n < p^{\alpha_p+1}$ then $4n < p^{\alpha_p+2}$. On the other hand we can prove by induction that $\binom{4n}{n} > \left(\frac{4^4}{3^3}\right)\frac{n_1}{4n}$. By (2) and (7) we then have

$$\left(\frac{4^4}{3^3}\right)^n \frac{1}{4n} < 3^{2n+(4n)1/2}$$

which is false for $n \geq 2200$, and a straight-forward check of a table of primes for $1 \leq n < 2200$ concludes the proof of Lemma 2.

If we now consider the case $2k \leq n < 3k$, $k > n^{2/3}$, our conclusion holds for $k > 4$ by Lemma 2 since there is a prime between $[\frac{3}{4}n]$ and n, and $[\frac{3}{4}n] \geq \frac{3}{2}k$.

Thus our theorem holds for $k \geq 8$ with a finite number of exceptions which may be checked by a table of primes.

(3) Consider the case $k=5$, we want to show that $n(n-1)\cdots(n-4)$ where $n-4>5$ is divisable by a prime ≥ 11. Assume the contrary and consider the binomial coefficient $\binom{n}{5}$. By Lemma 1 we have

$$\frac{n(n-1)\cdots(n-4)}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \binom{n}{5} < n^{\pi((3/2)5)} = n^4$$

which is certainly false for say $n \geq 129$. A check of tables of primes for $n \leq 129$ reveals one exception to our theorem i.e. 6.7.8.9.10 has no prime divisor >7. We may treat the case $k=4$ in the same manner and no exceptions occur.

The cases $k=6$ and $k=7$ now follows from the case $k=5$ since $\frac{3}{2}\cdot 6 < \frac{3}{2}\cdot 7 < 11$ and the product of any five consecutive numbers greater than 6 contains a prime divisor ≥ 11.

For $k=3$, consider the integers n, $n+1$, $n+2$, $n>3$. If $n \equiv 0$ (3), then either n or $n+1$ is divisable by a prime greater than 3 since $(n, n+1)=1$ and $n>3$. The case $n+2\equiv 0$ (3) is identical. If $n+1\equiv 0$ (3) the only time whether neither n or $n+2$ is divisable by a prime greater than 3 is when n and $n+2$ are powers of 2 i.e. when $n=2$. Therefore our theorem holds for $k=3$.

When $k=2$, by the same approach we only have the exceptions 3.4 and 8.9, since the only solutions to $2^\alpha - 3^\beta = \pm 1$ are $\alpha=2$, $\beta=1$ and $\alpha=3$, $\beta=2$. The case $k=1$ is trivially true.

The exception $\binom{10}{5}$ proves the corollary to the theorem i.e. that $\frac{7}{5}$ is the "best possible" constant c such that $\binom{n}{k}$ is divisable by a prime $\geq ck$ for $n \geq 2k$.

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