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ON COMMUTATOR LAWS IN GROUPS, 3

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In memoriam Wilhelm Magnus

Abstract

In this final contribution to the investigation of commutator laws in groups, we answer some of the questions left open in the previous two papers. The principal result is the independence of the Jacobi-Witt-Hall type laws from the so-called standard set of laws. The main results of the earlier papers are summarised. An interlude corrects some of the numerous printing errors in the second of our papers.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 20A05, 08A99.

1. Introduction

This paper concludes our study, begun in [1, 2], of the interdependence of commutator laws in groups. To make the present paper self-contained, we repeat notation and definitions, and summarise our earlier results. We work in groups and use the customary group notation, with the minor exception that the product $x \cdot y$ will always be written with the multiplication dot $\cdot$; inversion x^{-1} and the unit element e are as usual. The commutator notation is also as usual:

$$[x, y] := x^{-1} \cdot y^{-1} \cdot x \cdot y.$$

Conjugation is also written as usual:

$$x^y := y^{-1} \cdot x \cdot y, \quad x^{-y} := y \cdot x \cdot y^{-1}.$$

We introduce two further binary operations, one to mimic commutation, denoted by κ, the other to mimic conjugation, and denoted by σ, and both written as right-hand operators. As conjugation is related to commutation by

$$x^y = x \cdot [x, y], \tag{1.1}$$

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we postulate analogously

(1.2) $$xy\sigma = x \cdot xy\kappa.$$

(Thus σ is just a convenient abbreviation.)

The values of $xy\kappa$, as x and y range over the carrier of the underlying group, generate a subgroup called the *kappatator* subgroup of the group.

The second author has presented preliminary reports on the results of this paper to various audiences, but they were all based on what has since turned out to be an invalid model of kappa-groups. The model here presented (at the end of Section 4) has proved singularly elusive, although it is of relatively small order (namely 2^7). It is for this reason that more detail is included below than might otherwise be judged necessary.

2. Interlude

I did not have an opportunity to correct proofs of paper [2] before it was published, and a number of misprints have crept in. In the following (incomplete) list, line numbers counted from the top of the page, not counting the running head, are denoted by $+n$, from the bottom by $-n$. Only misprints affecting the mathematics are listed. BHN.

Page 113, line -5 (in formula (1.2)): For '$x \cdot y\kappa$' read '$x \cdot xy\kappa$'.

Page 114, line +6: For '$\mathscr{G}$' read '$\mathscr{G}'$'.

Page 115, line +1 (in formula (I9′)) the right-hand side should read '$yx\kappa y^{-1}\sigma$'.

Page 115, line +3 (in formula (I11′)) the right-hand side should read '$yx\kappa x^{-1}\sigma$'.

Page 115, line -16: For '(A1)' read '(S1)'.

Page 115, line -5: For '(S3)' read '(S5)'.

Page 117, line +5: Insert the two lines

$$(4.25) \qquad m(x^z, y^z) = m(x, y),$$

then I6 *holds; and conversely,* I6 *holds only if* (4.25) *is satisfied identically. If identically*

Page 117, line -12: For '(4.23)' read '(4.25)'.

Page 121, line +9: For 'that I2' read 'that I1'.

Page 122, line -7 [formula (6.91)]: For '$\cdot y(\kappa$' read '$\cdot yt\kappa$'.

Page 124, line -12 [formula (7.22)]: Delete the last exponent '2'. [This was an error in our manuscript, for which I apologize.]

Page 127, line -3: '$X - 1$' should read 'X^{-1}'.

Page 129, line -12 [reference 3]: For '*quantities*' read '*quandles*'.

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3. The laws

The laws we investigate are the same as in [2], with the addition of one 'quandle' law. We retain the numbering of the laws, except that the prime $'$, denoting the use of the abbreviation σ, is mostly omitted.

The following laws will *always* ('A') be assumed without further mention:

$$xe\kappa = e, \tag{A1}$$

$$xx\kappa = e, \tag{A3}$$

$$xy\kappa \cdot yx\kappa = e. \tag{A4}$$

From A1 and A4 there follows at once the dual of A1:

$$ex\kappa = e. \tag{A2}$$

Some of these laws can be expressed in terms of σ:

$$xe\sigma = x,$$

$$ex\sigma = e,$$

$$xx\sigma = x.$$

The following laws are *sometimes* ('S') of interest, and are therefore here listed:

$$xy\kappa = e, \tag{S1}$$

$$xy\kappa z\kappa = e, \tag{S2}$$

$$xy\kappa y\kappa = e, \tag{S2.1}$$

$$xy\kappa zt\kappa^2 = e, \tag{S3}$$

$$(x \cdot y)z\kappa = xz\kappa \cdot yz\kappa, \tag{S4}$$

$$x(y \cdot z)\kappa = xy\kappa \cdot xz\kappa. \tag{S5}$$

The kappa-group is *kappa-abelian*, *kappa-nilpotent* (of class 2), *kappa-Engel* (of class 2), *kappa-metabelian* if, respectively, S1, S2, S2.1, S3 is satisfied; and if S4 or S5 is satisfied, kappa is, respectively, *left linear* or *right linear*, or *bilinear* if both are satisfied.

The following laws are the *interesting* ('I') laws, whose interrelations were the main theme of the previous two papers:

$$xy\kappa z\kappa = [xy\kappa, z], \tag{I1}$$

which is equivalent to

$$xy\kappa z\sigma = xy\kappa^z,$$

(I2) $$(x \cdot y)z\kappa = xz\kappa y\sigma \cdot yz\kappa,$$

(I3) $$x(y \cdot z)\kappa = xz\kappa \cdot xy\kappa z\sigma,$$

(I4) $$(x \cdot y)z\sigma = xz\sigma \cdot yz\sigma,$$

(I5) $$x(y \cdot z)\sigma = xy\sigma z\sigma,$$

(I6) $$(xy\kappa)^z = x^z y^z \kappa,$$

(I7) $$xy\kappa z\sigma = xz\sigma yz\sigma\kappa,$$

(I8) $$xy^{-1}\kappa = xy\kappa^{-y^{-1}},$$

(I9) $$xy^{-1}\kappa = yx\kappa y^{-1}\sigma,$$

(I10) $$x^{-1}y\kappa = xy\kappa^{-x^{-1}},$$

(I11) $$x^{-1}y\kappa = yx\kappa x^{-1}\sigma,$$

(Q) $$xy\sigma z\sigma = xz\sigma yz\sigma^2.$$

This last law can be considered as a right distributive law for σ. It is here introduced only because it is satisfied in some, but not all, of the models of kappa-groups we list further down.

The last set of laws are the *Jacobi-Witt-Hall* ('J') type laws, of the form

$$w_i(x, y, z) \cdot w_i(y, z, x) \cdot w_i(z, x, y) = e,$$

where we list only the words w_i:

(J1) $$w_1(x, y, z) = xy^{-1}\kappa z\kappa y\sigma,$$

(J2) $$w_2(x, y, z) = yx\kappa zy\sigma\kappa,$$

(J3) $$w_3(x, y, z) = (x \cdot y)z\kappa,$$

(J4) $$w_4(x, y, z) = xz\kappa y\sigma \cdot yz\kappa.$$

4. Preparing the main result

It had been shown in [2] that 7 of the 10 pairs of laws from I1–I5 [for example I1 and I2] imply all these 5 laws, plus I8–I11, and the set of laws consisiting of A1–A4, I1–I5, I8–I11 was there called the *standard set of laws*. We retain this name in what follows. The main result of [2] was that in a kappa-group in which the standard set of laws holds, every further kappatator law is a consequence of these laws and any one of the Jacobi-Witt-Hall type laws, for example J1. It had also been shown there that in a kappa-group that satisfies the standard set of laws, I7 implies J1–J4, and thus should also be considered a Jacobi-Witt-Hall type law. We now show that the standard set of laws by itself does not imply any of the J laws. To do this, we construct a model of a kappa-group that satisfies the standard set of laws, but fails to satisfy one of the J laws.

We start with a group construction that is slightly more general than needed, and specialise later. Let $\mathscr{R}$ be a commutative ring with zero 0 and unit element 1. The elements of our group $\mathscr{G}$ are to be triplets

$$g = (\mathbf{x}, \mathbf{y}, z),$$

where $\mathbf{x}$ and $\mathbf{y}$ are 3-dimensional vectors over $\mathscr{R}$ and z is a scalar, that is to say an element of the ring. The vector operations are defined as usual: If

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \quad \text{then } \mathbf{x} + \mathbf{x}' := \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' \end{pmatrix},$$

$$\mathbf{x} \times \mathbf{x}' := \begin{pmatrix} x_2 \cdot x_3' - x_3 \cdot x_2' \\ x_1 \cdot x_3' - x_3 \cdot x_1' \\ x_1 \cdot x_2' - x_2 \cdot x_1' \end{pmatrix},$$

$$\mathbf{x} \cdot \mathbf{x}' := x_1 \cdot x_1' + x_2 \cdot x_2' + x_3 \cdot x_3',$$

the sum, vector product, and scalar product, respectively. If

$$g' = (\mathbf{x}', \mathbf{y}', z'), \qquad g'' = (\mathbf{x}'', \mathbf{y}'', z'')$$

are triplets as above, we define multiplication by

$$g \cdot g' := (\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}', z + z' + \mathbf{x}' \cdot \mathbf{y}). \tag{4.1}$$

Then the neutral element is $e = (\mathbf{0}, \mathbf{0}, 0)$, where $\mathbf{0}$ is the null vector $\mathbf{0} := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The inverse of g is

$$g^{-1} = (-\mathbf{x}, -\mathbf{y}, -z + \mathbf{x} \cdot \mathbf{y}).$$

It is then straightforward to verify the group laws - we omit the verification. It is also straightforward to compute conjugates and commutators:

$$g^{g'} = (\mathbf{x}, \mathbf{y}, z + \mathbf{x}' \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y}'),$$

$$[g, g'] = (\mathbf{0}, \mathbf{0}, \mathbf{x}' \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y}').$$

It follows that the group is nilpotent of class 2.

Next we define kappa by

$$gg'\kappa := (\mathbf{0}, \mathbf{x} \times \mathbf{x}', \mathbf{x}' \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y}' + \phi(\mathbf{x}, \mathbf{x}')), \tag{4.2}$$

where ϕ is a scalar function which will be defined later. For the present we just postulate that

$$\phi(\mathbf{x}, \mathbf{0}) = \phi(\mathbf{x}, \mathbf{x}) = \phi(\mathbf{x}', \mathbf{x}) - \phi(\mathbf{x}, \mathbf{x}') = 0. \tag{4.3}$$

At this stage it is already possible, and easy, to verify that this kappa-group satisfies the laws A1–A4; also I1 is satisfied:

$$gg'\kappa g''\kappa = (\mathbf{0}, \mathbf{0}, \mathbf{x}'' \cdot (\mathbf{x} \times \mathbf{x}')) = [gg'\kappa, g'']. \tag{4.4}$$

Note that the function ϕ does not enter into this, and that our models are kappa-nilpotent of class 3. To establish the rest of the standard set of laws it suffices to establish I2. This requires an appropriate choice of the function ϕ and severe restrictions on the ring $\mathscr{R}$. To motivate the choice of ϕ, we first calculate the two sides of the equation in I2:

$$(g \cdot g')g''\kappa = (\mathbf{0}, (\mathbf{x} + \mathbf{x}') \times \mathbf{x}'', \mathbf{x}'' \cdot (\mathbf{y} + \mathbf{y}') - (\mathbf{x} + \mathbf{x}') \cdot \mathbf{y}'' + \phi(\mathbf{x} + \mathbf{x}', \mathbf{x}'')), \tag{4.5}$$

and, omitting a few steps of the computation,

$$gg''\kappa^{g'} \cdot g'g''\kappa = (\mathbf{0}, (\mathbf{x}+\mathbf{x}') \times \mathbf{x}'', \mathbf{x}'' \cdot (\mathbf{y}+\mathbf{y}') - (\mathbf{x}+\mathbf{x}') \cdot \mathbf{y}'' + \phi(\mathbf{x}, \mathbf{x}'') + \phi(\mathbf{x}', \mathbf{x}'') + \mathbf{x}' \cdot (\mathbf{x} \times \mathbf{x}'')). \tag{4.6}$$

Thus to make these two expressions equal, we have to make ϕ satisfy the law

$$\phi(\mathbf{x} + \mathbf{x}', \mathbf{x}'') = \phi(\mathbf{x}, \mathbf{x}'') + \phi(\mathbf{x}', \mathbf{x}'') + \mathbf{x}' \cdot (\mathbf{x} \times \mathbf{x}''). \tag{4.7}$$

The only way to satisfy this is to put

$$\phi(\mathbf{x}, \mathbf{x}') = x_1 \cdot x_2 \cdot x_3' + x_1 \cdot x_2' \cdot x_3 + x_1' \cdot x_2 \cdot x_3 + x_1' \cdot x_2' \cdot x_3 + x_1' \cdot x_2 \cdot x_3' + x_1 \cdot x_2' \cdot x_3', \tag{4.8}$$

and simultaneously restrict $\mathscr{R}$ to have characteristic 2, so that there is no distinction between positive and negative signs. It then becomes easy to verify that ϕ satisfies (4.3) and (4.7), and thus I2 is satisfied, and hence the whole standard set of laws holds [see the result quoted from [2] above].

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It only remains to show that the Jacobi-Witt-Hall laws are not satisfied; by the other result of [2] quoted earlier, it suffices to show that one of them fails, and J3 is the most convenient one to check, because $(g \cdot g')g''\kappa$ has already been calculated. We introduce the unit vectors

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and define the group elements

$$a = (\mathbf{i}, \mathbf{0}, 0), \quad a' = (\mathbf{j}, \mathbf{0}, 0), \quad a'' = (\mathbf{k}, \mathbf{0}, 0).$$

Then [see (4.5)]

$$(a \cdot a')a''\kappa = (a' \cdot a'')a\kappa = (a'' \cdot a)a'\kappa = (\mathbf{0}, \mathbf{0}, 1),$$

and thus

$$(a \cdot a')a''\kappa \cdot (a' \cdot a'')a\kappa \cdot (a'' \cdot a)a'\kappa = (\mathbf{0}, \mathbf{0}, 1),$$

as the characteristic of our ring is 2. This shows that J3 is not satisfied in our model.

The natural choice for our ring $\mathscr{R}$ is the Galois field of order 2, when the underlying group becomes the central cube, of order 128, of a dihedral group of order 8. This model will be denoted by M6.

5. Conclusion

The model M6 just constructed allows us to formulate our main result:

THEOREM 5.1. *The standard set of laws does not imply any of the Jacobi-Witt-Hall type laws.*

It is easy to verify that this model is not kappa-abelian, nor kappa-nilpotent of class 2, but that it is kappa-Engel of class 2 and kappa-metabelian; also that it is neither left nor right kappa-linear. A short calculation, which we omit, shows that the model does not satisfy I6. A rather more involved calculation, which we nevertheless also omit, shows that the model does not satisfy the quandle law Q.

We now have most of the information summarised in the following table, which sets out the *profiles* of the models described in [1] and [2], as well as the new model M6.

This table summarises the profiles of all models of [1], [2], and the present paper. The entries are the boolean values: a '1' means the model heading the column satisfies the law listed at the beginning of the row, a '0' that it does not. The entries δ and ϵ can be chosen to take either the value 1 or 0. The last column is new, as is also the

TABLE 1

Law	Model :	M1	M2	M3	M4.$\delta\epsilon$	M5.$\delta\epsilon$	M6
S1		0	0	0	0	0	0
S2		0	0	0	0	1	0
S2.1		0	0	0	0	1	1
S3		1	1	0	0	1	1
S4		1	0	1	0	0	0
S5		1	0	1	0	0	0
I1		0	0	0	0	0	1
I2		0	1	0	0	0	1
I3		0	1	0	0	0	1
I4		1	0	1	0	0	1
I5		0	1	0	0	0	1
I6		1	0	1	δ	δ	0
I7		0	1	0	0	δ	0
I8		1	0	0	ϵ	ϵ	1
I9		0	0	0	ϵ	ϵ	1
I10		1	0	0	ϵ	ϵ	1
I11		0	0	0	ϵ	ϵ	1
Q		1	1	0	0	δ^*	0
J1		0	1	0	0	0	0
J2		1	1	0	0	0	0
J3		1	1	1	0	0	0
J4		1	1	1	0	1	0

row labelled 'Q' [that is, the quandle law]. The entry 'δ^*' in that row means that an apparently slightly stronger condition than the one that makes δ in that column equal to 1 (namely Condition (5.12) in [2]) will also make this entry equal to 1; but we have not investigated whether this condition is in fact stronger.

References

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