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Invariant tensor fields of dynamical systems with pinched Lyapunov exponents and rigidity of geodesic flows

RENATO FERES† AND ANATOLY KATOK‡†
Mathematics 253-37, *California Institute of Technology, Pasadena, CA* 91125, *USA*

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Abstract. We consider in this note smooth dynamical systems equipped with smooth invariant affine connections and show that, under a pinching condition on the Lyapunov exponents, certain invariant tensor fields are parallel. We then apply this result to a problem of rigidity of geodesic flows for Riemannian manifolds with negative curvature.

1. *Statement of results*

Let $f: N \to N$ be a diffeomorphism of a smooth (i.e. C^∞) closed manifold N. We will assume that N is equipped with an f-invariant Borel measure μ which is absolutely continuous with respect to Lebesgue measure and positive on nonempty open sets. We also assume that N possesses a nondegenerate, f-invariant bilinear form g and an f-invariant (not necessarily torsion-free) affine connection ∇ with respect to which g is parallel ($\nabla g \equiv 0$). The objects defined above, namely f, N, g, and ∇ are assumed to be smooth.

We say that the Lyapunov exponents of f satisfy the *pinching condition* with respect to μ if for μ-almost every point $x \in N$, $\sup\{|\chi| : \chi$ is a nonzero Lyapunov exponent at $x\} < 2 \cdot \inf\{|\chi| : \chi$ is a nonzero Lyapunov exponent at $x\}$.

Finally, in those cases when 0 is a Lyapunov exponent of f, we will consider tensor fields τ with the following property: $(*)$ τ and $\nabla\tau$ vanish when contracted with vectors or 1-forms associated to the exponent 0.

Most of our work is aimed at establishing the following result:

THEOREM 1. *Let f, g, N, μ, and ∇ be as defined above and let τ be a smooth, f-invariant tensor field on N. If 0 is a Lyapunov exponent of f, we also assume that τ possesses property $(*)$. Then, if the Lyapunov exponents of f satisfy the pinching condition with respect to μ, τ is parallel,* i.e. $\nabla\tau \equiv 0$.

It should be remarked that the exact degree of differentiability required in the proof of Theorem 1 is actually finite, but high, and depends on the type of τ and on the dimension of N, due to the use of Sard's Theorem.

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From the proof of Theorem 1, we extract the following elementary, but nonetheless interesting fact, which is a consequence of Lemma 2. We recall that a tensor field of type (r, s) is a section of the vector bundle $(\bigotimes^r TN)\otimes(\bigotimes^s T^*N)$.

PROPOSITION 1. *Let f be as in Theorem 1. If the Lyapunov exponents of f are nonzero and satisfy the pinching condition, and τ is a continuous, f-invariant tensor field on N of type (r, s), $r+s=3$, then τ must vanish identically. Therefore, under these assumptions a C^1 f-invariant connection on N is torsion-free and unique, as the difference of two such connections is an invariant $(1, 2)$-tensor field.*

An immediate consequence of Theorem 1 and Proposition 1 is:

COROLLARY 1. *Consider f, g, μ, ∇, and N as in Theorem 1, and assume that the exponents of f are nonzero. Then, (N, ∇) is an affine locally symmetric space, not necessarily complete.*

Proof. It is easy to see that the torsion and curvature of an f-invariant connection are f-invariant tensor-fields. By Theorem 1 the curvature tensor of the connection ∇ is parallel; by Proposition 1 ∇ is torsion-free. This is a well-known necessary and sufficient condition for ∇ to be locally symmetric affine connection. □

Let M be a closed C^∞ Riemannian manifold with negative sectional curvature and let $\varphi_t : SM \to SM$ be the geodesic flow on the unit tangent bundle of M. This is an Anosov flow. The foliations of SM into stable and unstable horospheres coincide with the contracting and expanding foliations for φ_t. The geodesic flow has a natural smooth measure, sometimes called Liouville measure.

THEOREM 2. *If at least one of the horospheric foliations on SM is smooth and the Lyapunov exponents of the time-one map φ_1 satisfy the pinching condition with respect to Liouville measure, then the geodesic flow φ_t is smoothly conjugate to the flow on a manifold of constant negative curvature.*

The following result is an immediate corollary of Theorem 2. It represents a stronger and, in fact, sharp version of a result by M. Kanai [**2**].

THEOREM 3. *If at least one of the horospheric foliations on SM is smooth and the sectional curvature k is $\frac{1}{4}$-pinched,* i.e. $-4a^2 < k \le -a^2$, *then the geodesic flow on M is smoothly conjugate to the geodesic flow on a manifold of constant negative curvature.*

Remark. The condition $-4a^2 < k \le -a^2$ cannot be improved. In fact, let M be a closed manifold covered by the complex hyperbolic space or by any other non-compact Riemannian symmetric space of rank 1 with nonconstant curvature. In that case the horospheric foliations are smooth and the sectional negative curvature k satisfies $-4a^2 \le k \le -a^2$. However, the covariant derivative ∇R of the curvature tensor R for the Kanai connection (see § 3) does not vanish.

We believe that the assertion of Theorems 2 or 3 implies that the metric on M has constant negative curvature. This fact is known for surfaces. The following corollaries provide some evidence in favor of this conjecture.

COROLLARY 2. *M is homotopy equivalent to a compact manifold of constant negative curvature.*

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Proof. We can assume that dim $M \geq 3$. Since the unit tangent bundle SM is diffeomorphic to the unit tangent bundle SM_0 for a manifold M_0 of constant negative curvature, $\pi_1(M) = \pi_1(M_0)$ and M is homotopy equivalent to M_0 since they are both $K(\pi, 1)$ spaces. □

Remark. It is not difficult to construct a smooth homotopy equivalence between M and M_0 explicitly using the map conjugating the geodesic flows and the concept of center of gravity.

COROLLARY 3. *All positive Lyapunov exponents for φ_t with respect to any invariant measure are equal.*

Proof. This property holds for metrics of constant negative curvature and is invariant under a smooth flow conjugacy. □

COROLLARY 4. *The topological entropy for φ_t is equal to the metric entropy with respect to the Liouville measure and the latter is the measure of maximal entropy.*

Proof. The same as Corollary 3. □

2. *Proof of Theorem* 1

Let f, N, and μ be as before. Consider the subset Λ of N defined as follows: for each $x \in \Lambda$ there exist finitely many numbers (Lyapunov exponents) $\chi_1(x) < \cdots < \chi_{k(x)}(x)$ and a decomposition $T_xN = E_1 \oplus \cdots \oplus E_{k(x)}(x)$ such that

$$\lim_{n \to \pm\infty} \frac{\log \|(Df^n)_x v\|}{n} = \chi_i(x)$$

for every $v \in E_i(x)$, $v \neq 0$, $i = 1, \ldots, k(x)$. Here, $\|\cdot\|$ is any continuous norm on TN. It is a consequence of Oseledec Multiplicative Ergodic Theorem (cf. e.g. [5] Theorem 10.1) that Λ is a set of full measure.

Let τ be a continuous, f-invariant tensor field on N. One can assume without loss of generality that τ is a tensor field of type $(0, r)$; in fact, if τ is of type $(1, m)$, we can use the isomorphism $TN \simeq T^*N$ defined via the bilinear form g, $v \to g(v, \cdot)$, to obtain a tensor field of type $(0, 1+m)$. Notice that this correspondence between $(1, m)$ and $(0, 1+m)$-tensors preserves f-invariance and sends parallel tensors into parallel tensors, as g is itself f-invariant and parallel.

LEMMA 1. *Let $x \in \Lambda$ and suppose $v_i \in E_{l_i}$, $i = 1, \ldots, r$, are vectors at x for which $\tau(v_1, \ldots, v_r) \neq 0$. Then $\sum_{i=1}^r \chi_{l_i} = 0$.*

Proof. $0 \neq \tau_x(v_1, \ldots, v_r) = (f^{n*}\tau)_x(v_1, \ldots, v_r) = \tau_{f^n(x)}((Df^n)_x v_1, \ldots, (Df^n)_x v_r)$. But $\|\tau_x\|$ is bounded for all $x \in N$, so that

$$0 < |\tau_x(v_1, \ldots, v_r)| \leq c\|(Df^n)_x v_1\| \cdots \|(Df^n)_x v_r\|$$

for some constant $c > 0$ and

$$\frac{\log |\tau_x(v_1, \ldots, v_r)|}{|n|} \leq \frac{\log c}{|n|} + \frac{n}{|n|} \sum_{i=1}^r \frac{\log \|(Df^n)_x v_i\|}{n}.$$

Passing to the limit as $n \to +\infty$, and $n \to -\infty$, we obtain $0 \leq \pm\sum_{i=1}^r \chi_{l_i}$, so that $\sum_{i=1}^r \chi_{l_i} = 0$. □

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LEMMA 2. *If χ is a Lyapunov exponent of f, then $-\chi$ is also a Lyapunov exponent. Furthermore, if the exponents of f satisfy the pinching condition, then there cannot be nonzero exponents χ_1, χ_2, χ_3 such that $\chi_1+\chi_2+\chi_3=0$.*

Proof. The first assertion is an immediate consequence of Lemma 1 and the existence of the invariant and nondegenerate form g. As for the second assertion, we may assume without loss of generality that $\chi_1>0$, $\chi_2<0$, $\chi_3<0$. If this is not the case, we can use the first assertion and a convenient permutation of the indices to have the exponents in that form. But then, $\chi_1=|\chi_2|+|\chi_3|\geq 2\cdot \min\{|\chi_2|, |\chi_3|\}$, which violates the pinching condition. □

Suppose that f satisfies the conditions of Theorem 1. Let τ be a smooth f-invariant tensor field of type $(0, r)$ and assume that $\nabla\tau$ is not identically zero. We will later show that this last assumption leads to a contradiction. Define $\mathscr{A}'=\{x\in N: \tau_x\neq 0$ and $(\nabla\tau)_x\neq 0\}$. $\mathscr{A}'$ is an open, nonempty subset of N, hence its measure is positive. Define $\mathscr{A}=\mathscr{A}'\cap\Lambda$. This set also has positive measure. Denote by V the direct sum bundle ρ: $V=\bigoplus^r TN\to N$. It will be convenient to view τ as a smooth real valued function on the manifold V. We will use the notation $\omega_x(v_0, v)=(\nabla_{v_0}\tau)_x(v)$, where $v_0\in T_xN$ and $v=(v_1,\ldots,v_r)\in V_x=\rho^{-1}(x)$. For each $x\in\mathscr{A}$, denote by Ξ_x the set of all $v=(v_1,\ldots,v_r)\in V_x$ for which: (i) $\omega_x(\cdot, v)\neq 0$, (ii) $v_i\in E_{l_i}$ for some l_i, $i=1,\ldots,r$.

LEMMA 3. *Under the assumptions of Theorem* 1, *suppose $x\in\mathscr{A}$ and $v\in\Xi_x$. Then $\tau(v)=0$ and there is $v_0\in T_xN$ such that $(D\tau)_vX\neq 0$ for all $X\in T_vV$ projecting onto v_0*, i.e. *with $(D\rho)_vX=v_0$. In particular $(D\tau)_vX\neq 0$ for each v in Ξ_x.*

Proof. We can choose $v_0\in T_xN$ such that $\omega_x(v_0, v)\neq 0$ and $v_0\in E_{l_0}$ for some l_0. Suppose $X\in T_vV$ projects onto v_0. *Let* $\tilde v=(\tilde v_1,\ldots,\tilde v_r)$ be a smooth local section of V and $\gamma(t)$ a differentiable curve in N such that $\tilde v(x)=v$, $\gamma(0)=x$, $\gamma'(0)=v_0$ and $(\tilde v\circ\gamma)'(0)=X\in T_vV$. Then,

$$(D\tau)_vX=\frac{d}{dt}\bigg|_{t=0}\tau(\tilde v\circ\gamma)=(\nabla_{v_0}\tau)_x(v)+\sum_{i=1}^{r}\tau_x(v_1,\ldots,\nabla_{v_0}\tilde v_i,\ldots,v_r).$$

We claim that the terms $\tau_x(v_1,\ldots,\nabla_{v_0}\tilde v_i,\ldots,v_r)$, $i=1,\ldots,r$, vanish. Suppose not. Decompose $(\nabla_{v_0}\tilde v_i)_x=\sum_{j=1}^{k(x)}u_{ij}$, $u_{ij}\in E_j(x)$. There will be indices i and j for which $\tau_x(v_1,\ldots,u_{ij},\ldots,v_r)\neq 0$. According to Lemma 1, the Lyapunov exponents must satisfy the following relations: (a) $\sum_{k=0}^{r}\chi_{l_k}=0$, since $\omega_x(v_0, v)\neq 0$ and (b) $\chi_j+\sum_{k=1,k\neq i}^{r}\chi_{l_k}=0$, since $\tau_x(v,\ldots,u_{ij},\ldots,v_r)\neq 0$. Subtracting (a) from (b) we obtain $\chi_j-\chi_{l_0}-\chi_{l_i}=0$. Since these exponents are not zero, this equation violates the pinching condition, as we have shown in Lemma 2. Hence we must have $\tau_x(v_1,\ldots,\nabla_{v_0}\tilde v_1,\ldots,v_r)=0$ for each i, and $(D\tau)_vX=\omega_x(v_0, v)\neq 0$, as claimed. Finally, if we had $\tau_x(v)\neq 0$, then $\sum_{k=1}^{r}\chi_{l_k}=0$ so that $\chi_{l_0}=0$ by (a). Since this violates property (*), we must have $\tau_x(v)=0$. □

COROLLARY 5. *All the conditions as in Lemma 2. Define $\mathscr{N}=\{v\in V: \tau(v)=0\}$. Then, for all $x\in\mathscr{A}$ and $v\in\Xi_x$, we can find a neighborhood U of v in V such that $\mathscr{N}\cap U$ is a smooth manifold embedded in V of codimension* 1 *containing v. Furthermore, v is a critical point of the projection $\rho|_{\mathscr{N}\cap U}:\mathscr{N}\cap U\to N$.*

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Proof. The first assertion is an immediate consequence of Lemma 3 and the Implicit Function Theorem. The second one follows from Lemma 3 since if v were not a critical point of the projection, there would be for every $v_0 \in T_x N$ an $X \in T_v V$ with $(D\rho)_v X = v_0$ and $(D\tau)_v X = 0$. □

Proof of Theorem 1. Define $\mathcal{N}' = \{v \in \mathcal{N}: \exists$ neighborhood U of v in V such that $\mathcal{N} \cap U$ is a smooth manifold embedded in $V\}$. $\mathcal{N}'$ is a (nonempty) smooth manifold. The restriction of ρ to $\mathcal{N}'$ is a smooth map which, according to Corollary 5 has a set of positive measure of critical values, since each $x \in \mathcal{A}$ is such a value. But this is impossible by Sard's Theorem. Therefore, we must have $\nabla \tau = 0$. □

Remark. There is an alternative approach to the proof of Theorem 1 which is based on the study of singularities of the zero-set $\{v \in V_x, x \in \mathcal{N}: \tau_x(v) = 0\}$. This method allows to avoid the use of Sard's Theorem and consequently requires lower differentiability of the stable and unstable foliations. In fact, it is sufficient to assume just enough differentiability in order to properly define the invariant tensor fields involved in the argument. The necessary results about the structure of singularities are rather difficult and can be extracted from [**6**].

3. *Geodesic flows*

In order to discuss Theorem 2, we need to recall some notions which were introduced in [**2**] by M. Kanai. For more details concerning the following material, see [**2**], [**3**], [**1**].

Let M be a C^∞ closed Riemannian manifold of negative curvature. Denote by $\tilde{M}$ its universal covering space and by $S\tilde{M}$ the unit tangent bundle of $\tilde{M}$. As is shown in [2], $S\tilde{M}$ is fibered over a manifold P, $S\tilde{M} \xrightarrow{\pi} P = S\tilde{M}/\mathbb{R}$, with the fibres being the orbits of the $\mathbb{R}$-action of the geodesic flow on $S\tilde{M}$. It is well known that the tangent bundles of SM and $S\tilde{M}$ possess a flow-invariant splitting, the Anosov splitting, into the subbundles of exponentially contracting vectors E^-, exponentially expanding vectors E^+, and the direction E° spanned by the geodesic spray $\dot{\varphi}$: $T(S\tilde{M}) = E^- \oplus E^\circ \oplus E^+$. This splitting projects to $TP = F^- \oplus F^+$. We assume that the Anosov splitting is smooth, so that the splitting of TP has the same property. P also possesses a symplectic form Ω defined as the push forward via π of the exterior derivative of the contact form θ of $S\tilde{M}$ (the contact form can be defined by the property that it vanishes on E^- and E^+ and $\theta(\dot{\varphi}) = 1$). F^- and F^+ are Lagrangian distributions with respect to Ω. TP has an involution $c: TP \to TP$, $c|_{F^\pm} = \pm \mathrm{Id}_{F^\pm}$ and we can define a nondegenerate bilinear symmetric form $g(\xi, \eta) = \Omega(\xi, c\eta)$, for $\xi, \eta \in T_pP$. We will call the Levi-Civita connection ∇ associated to g the *Kanai connection* of P. Notice that $\nabla\Omega \equiv 0$.

Next we define a connection on SM or $S\tilde{M}$ which is a lift of the Kanai connection. For a vector field ξ on P, let ξ^* be the unique horizontal lift of ξ to $S\tilde{M}$, i.e. ξ^* is a section of $E^- \oplus E^+$ such that $\pi_* \xi^* = \xi$. Now, define a connection ∇' on $S\tilde{M}$ in the following way:

(i) $\nabla'_{\xi^*} \eta^* = (\nabla_\xi \eta)^*$, where ξ and η are vector fields on P,

(ii) $\nabla'_{\dot{\varphi}} \eta^* = \nabla'_{\eta^*} \dot{\varphi} = \nabla'_{\dot{\varphi}} \dot{\varphi} = 0$,

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and extend it to arbitrary vector fields using the general properties of an affine connection. It is not difficult to verify that the connection so defined is invariant under the action of the fundamental group of M and satisfies the properties: $\nabla'\theta \equiv 0$, $\nabla' d\theta \equiv 0$, and if τ is a $(0, r)$-tensor field on P, then $\pi^*\nabla\tau = \nabla'\pi^*\tau$. We note that ∇' is not torsion-free.

Denote by R the curvature tensor of ∇ of P. In **[2]**, it is shown how to deduce the conjugacy of the geodesic flows of Theorem 2 from the assumption that (P, ∇) is an affine locally symmetric space. More precisely, we have

THEOREM (M. Kanai **[2]**.) *If the Anosov splitting of SM is smooth and the curvature tensor for the Kanai connection on P is parallel* ($\nabla R \equiv 0$) *then the geodesic flow of M is smoothly conjugate to the flow for a metric of constant negative curvature.*

Denote by $\check{R}$ the $(0, 4)$-tensor field on P associated to R via Ω, that is, $\check{R}(\xi_1, \ldots, \xi_4) = \Omega(R(\xi_1, \xi_2)\xi_3, \xi_4)$. Since Ω is parallel with respect to ∇, it follows that $\nabla\check{R} = 0$ iff $\nabla R = 0$.

Proof of Theorem 2. Let $J: SM \to SM$ be the 'flip' map which sends tangent vector $v \in T_xM$, $x \in M$ to the vector $-v$. This map interchanges stable and unstable horospheres. Thus if one of the foliations is smooth so is the other. According to the above discussion, we need to show that $\nabla\check{R} \equiv 0$. As we noted before, $\nabla'\pi^*\check{R} = \pi^*\nabla\check{R}$, so that $\pi^*\check{R}$ possesses property $(*)$ and $\nabla\check{R} = 0$ if $\nabla'\pi^*\check{R} = 0$. But Theorem 1, applied to $f = \varphi_1$, $N = SM$, μ Liouville measure on SM, and the bilinear symmetric form $g = \pi^*g + \theta \otimes \theta$ (the latter g, as defined in the previous paragraph, i.e. $g(\xi, \eta) = \Omega(\xi, c\eta)$), yields $\nabla'\pi^*\check{R} = 0$. □

Proof of Theorem 3. Notice that the condition $-(Ca)^2 < k \leq -a^2$ implies that at each point $x \in SM$ where Lyapunov exponents for the map φ_1 are defined $\sup\{|\chi|: \chi$ is a non-zero Lyapunov exponent of $x\} < C \inf\{|\chi|: \chi$ is a non-zero Lyapunov exponent at $x\}$ (see **[4]**, Theorem 3.2.17). Hence Theorem 3 follows from Theorem 2. □

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