Canad. J. Math. Vol. **65** (5), 2013 pp. 1164–1200
http://dx.doi.org/10.4153/CJM-2012-055-0

Partial Differential Hamiltonian Systems

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Abstract. We define partial differential (PD in the following), *i.e.*, field theoretic analogues of Hamiltonian systems on abstract symplectic manifolds and study their main properties, namely, PD Hamilton equations, PD Noether theorem, PD Poisson bracket, *etc.* Unlike the standard multisymplectic approach to Hamiltonian field theory, in our formalism, the geometric structure (kinematics) and the dynamical information on the "phase space" appear as just different components of one single geometric object.

1 Introduction

First order Lagrangian mechanics can be naturally generalized to higher order Lagrangian field theory. Moreover, the latter can be presented in a very elegant and precise algebro-geometric fashion [55]. In particular, it is clear what all the involved geometric structures (higher order jets, Cartan distribution, $\mathscr{C}$-spectral sequence, *etc.*, [4, 55]) are. On the other hand it seems to be quite hard to understand what the most "reasonable, unambiguous, higher order, field theoretic generalization" of Hamiltonian mechanics on abstract symplectic manifolds is. Actually, there exists a universally accepted generalization of the standard mechanical picture

$$\text{Lagrangian mechanics on } TQ \Longrightarrow \text{Hamiltonian mechanics on } T^*Q, \tag{1.1}$$

Q being a smooth manifold, to the picture

$$\text{Lagrangian field theory on } J^1\pi \Longrightarrow \text{Hamiltonian field theory on } \mathscr{M}\pi, \tag{1.2}$$

with π being a fiber bundle, $J^1\pi$ its first jet space and $\mathscr{M}\pi$ its multimomentum space [26] (see also [34], [49] for a recent review, and [30] for an approach *à la* Tulczyjew). Picture (1.2) includes, in particular, a generalization of the Legendre transform. Along this path a structure analogous to the symplectic structure on T^*Q, called the *multisymplectic structure of* $\mathscr{M}\pi$ (see, for instance, [28]), has been discovered. A whole literature exists about properties of such structure, which is generically referred to as *multisymplectic geometry* of $\mathscr{M}\pi$ (see references in [49]). In particular, efforts were made to find multisymplectic analogues of all properties of T^*Q (including, for instance, the Poisson bracket [21, 22, 24, 36, 38]). Now, it is natural to wonder if it is possible to reasonably further generalize in two different directions. The first one is towards a picture

$$\text{Lagrangian field theory on } J^\infty\pi \Longrightarrow \text{higher order Hamiltonian field theory}, \tag{1.3}$$

Received by the editors April 18, 2012; revised September 24, 2012.
Published electronically December 29, 2012.
AMS subject classification: **70S05**, 70S10, 53C80.
Keywords: field theory, fiber bundles, multisymplectic geometry, Hamiltonian systems.

with $J^\infty\pi$ being the ∞-th jet space of π, including a higher order generalization of the Legendre transform. There is no universally accepted answer about picture (1.3) (see, for instance, [1, 2, 10, 39, 40, 52–54] and references therein). Most often they involve the choice of some extra structure other than the natural ones on $J^\infty\pi$. Recently, in [57], we proposed an answer that is free from such ambiguities.

The second direction in which to generalize picture (1.2) can be illustrated as follows. T^*Q is just a very special example of a (pre)symplectic manifold. Actually, Hamiltonian mechanics can (and should, in some cases [29]) be formulated on abstract (pre)symplectic manifolds. Similarly, it is natural to wonder if there exists the concept of abstract multi(pre)symplectic manifolds in such a way that Hamiltonian field theory could be reasonably formulated on them. In the literature there can be found some proposals of would-be abstract multi(pre)symplectic manifolds (see, for instance, [3, 7]). In particular, definitions have been given in such a way to be able to prove multisymplectic analogues of the celebrated Darboux lemma [20, 43]. The recent definitions by Forger and Gomes [20] appear to be the most satisfactory, in that they are "minimal" on one side and duly model in an abstract fashion the relevant geometric properties of $\mathscr{M}\pi$ on the other side. In their work, Forger and Gomes illustrate, in particular, the role played by fiber bundles in the would-be definition of multi(pre)symplectic structure. The next step forward should be to formulate Hamiltonian field theory on multisymplectic bundles.

In this paper we present our own proposal about what should be an abstract, first order, Hamiltonian field theory. We call such a proposal the *theory of partial differential* (*PD* in the following) *Hamiltonian systems* so to

(1) stress that it is a natural generalization of the theory of Hamiltonian systems on abstract symplectic manifolds,
(2) distinguish it from the special case of Hamiltonian field theory on $\mathscr{M}\pi$.

A PD Hamiltonian system encompasses both the kinematics (encoded, in picture (1.2), by the multisymplectic structure in $\mathscr{M}\pi$) and the dynamics (encoded, in picture (1.2), by the so-called *Hamiltonian section* [49]) which appear as just different components of one single geometric object. Namely, the main difference between a PD Hamiltonian system and a multi(pre)symplectic structure (whatever the reader understands by this) is the dynamical content of the former (as opposed to the just kinematical one of the latter). Notice that this idea is already present in literature [37]. However, our formalism differs from the one in [37] in that it is adapted to the fibered structure of the manifold of "field variables".

As already mentioned, standard examples of PD Hamiltonian systems come from Lagrangian field theory. Consider a field theory on a "space-time" M with coordinates $x := (\dots, x^i, \dots)$, defined by a Lagrangian density

$$\mathscr{L} = L(x, u, u')d^n x$$

depending on some field variables $u := (\dots, u^\alpha, \dots)$ and their partial derivatives $u' := (\dots, u^\alpha_i, \dots)$. The Lagrangian density $\mathscr{L}$ determines a Legendre transform[1] $F\mathscr{L}$ from the space J^1 of the (x, u, u')'s to the so-called *multimomentum space* $J^\dagger$ with

[1] See, for instance, [49] for a geometric formulation of the constructions in this paragraph.

coordinates (x, u, p), $p := (\dots, p^i_\alpha, \dots)$ being the *multimomenta*. The Legendre transform $F\mathscr{L}\colon J^1 \to J^\dagger$ is defined as

$$F\mathscr{L}(x, u, u') := (x, u, \partial L/\partial u').$$

Moreover, $\mathscr{L}$ determines in a canonical way the following $(n+1)$-form on J^1:

$$\omega_{\mathscr{L}} := d\tfrac{\partial L}{\partial u^\alpha_i} \wedge du^\alpha \wedge d^{n-1}x_i - dE \wedge d^n x, \quad E := u^\alpha_i \tfrac{\partial L}{\partial u^\alpha_i} - L.$$

The form $\omega_{\mathscr{L}}$ is "constant" along the fibers of $F\mathscr{L}$ and, therefore, determines an $(n+1)$-form ω on the submanifold $C_0 := \operatorname{im} F\mathscr{L} \subset J^\dagger$. The forms $\omega_{\mathscr{L}}$ and ω are standard examples of PD Hamiltonian systems. In a way that will be clear later, they determine PDEs for sections of the bundles $J^1 \to M$ and $C_0 \to M$. When $F\mathscr{L}$ is invertible,

$$\omega = dp^i_\alpha \wedge du^\alpha \wedge d^{n-1}x_i - dH \wedge d^n x, \quad H := E \circ FL^{-1}$$

and the above-mentioned PDEs are the Euler–Lagrange and the de Donder–Weyl equations of the theory, respectively. Using the theory of PD Hamiltonian systems, one can generalize these considerations to more general field theories, in particular those depending on higher derivatives of the fields [57].

The paper is divided into eight sections. In Section 2 we collect our notations and conventions and recall basic differential geometric facts that will be used in the main part of the paper.

In Section 3 we define what we call *affine forms* on fiber bundles. The introduction of affine forms can be motivated as follows. Trajectories in Hamiltonian mechanics are curves whose first derivative at a point is naturally understood as a tangent vector. In their turn, tangent vectors can be inserted into a differential form and, in particular, a symplectic one, and Hamilton equations are written in terms of such an insertion. Trajectories in field theory are sections of a fiber bundle $\alpha\colon P \to M$, whose first derivative at a point is naturally understood as a point in $J^1\alpha$. In their turn, points of $J^1\alpha$ can be inserted into an affine form and, in particular, a PD Hamiltonian system (see Section 5), and PD Hamilton equations are written in terms of such an insertion. Recall now that the natural projection $J^1\alpha \to P$ is an affine bundle whose sections are naturally interpreted as (Ehresmann) connections in α. Thus, connections and affine geometry play a prominent role in the theory of PD Hamiltonian systems. The affine geometry is hidden in standard Hamiltonian mechanics by an *a priori* choice of the parametrization of the time axis (see [31, 32], and references therein, for the role of affine geometry in theoretical mechanics). Similarly, even if the role of connections in field theory has been often recognized (see, for instance, [18, 50]), their affine geometry is sometimes hidden in Hamiltonian field theory on $\mathscr{M}\pi$ by the use of multivectors, or even decomposable ones [47, 48] (which is just a multidimensional analogue of choosing a parameterization of time). Actually, we show in Subsection 3.3 that affine forms can be understood as standard differential forms of a special kind. Nevertheless, we prefer to keep the distinction for foundational reasons.

In Section 4 we discuss standard operations with affine forms. Essentially because of the interpretation of affine forms as standard differential forms we mentioned above, some of these operations (for instance, the insertion of a connection into an affine form [19], or the differential of an affine form) were actually already defined in the literature, or can be understood as standard operations with forms. We stress again that we will keep the distinction. Finally, we discuss relevant affine form cohomologies proving an affine form version of the Poincaré lemma.

In Section 5 we introduce PD (pre)Hamiltonian systems and discuss their geometry and the geometry of the associated PD Hamilton equations with some references to the singular, constrained case (see [11–13, 15] for an account of the constraint algorithm in first order field theory). For completeness, we also relate PD Hamiltonian systems to multi(pre)symplectic structures *à la* Forger [20] and the calculus of variations. Notice that the theory of PD Hamiltonian systems is somehow in between (abstract) multisymplectic field theory and polysymplectic field theory.

In Section 6 we introduce PD Noether symmetries and currents of a PD Hamiltonian system. In view of the dynamical content of the latter we are able to prove a Noether theorem (see also [14]). Moreover, there is a natural Lie bracket (named PD Poisson bracket) among PD Noether currents. As already mentioned, multisymplectic analogues of the Poisson bracket have been previously discussed in the literature [21, 22, 24, 36, 38]. However, we emphasize here the dynamical nature of the PD Poisson bracket (see also [25, 56]). Namely, such a bracket is just part of the Peierls bracket [56] among conservation laws of the underlying Lagrangian theory, and we don't try to extend it to non-conserved currents. Indeed, our opinion is that the existence of a Poisson bracket among non-conserved functions in Hamiltonian mechanics is essentially due to the existence of a preferred Hamiltonian system on any symplectic manifold N, *i.e.*, the one with 0 Hamiltonian, for which every function on N is a conservation law. Finally we discuss the (gauge) reduction of a degenerate (but unconstrained) PD Hamiltonian system.

In Section 7 we propose few examples of PD Hamiltonian systems, including the computation of their PD Noether symmetries and currents or, in one case, their reduction.

We conclude with Section 8 where we briefly discuss the emergence of PD Hamiltonian systems in Mathematical Physics and Geometry, providing further motivations for their introduction.

2 Notations and Conventions

In this section we collect notations and conventions about some general constructions that will be used in the following.

Let N be a manifold. We denote by $C^\infty(N)$ the $\mathbb{R}$-algebra of smooth, $\mathbb{R}$-valued functions on N. A vector field X over N will be always understood as a derivation $X\colon C^\infty(N) \to C^\infty(N)$. We denote by $\mathrm{D}(N)$ the $C^\infty(N)$-module of vector fields over N, by $\Lambda(M) = \bigoplus_k \Lambda^k(N)$ the graded $\mathbb{R}$-algebra of differential forms over N, by $d\colon \Lambda(N) \to \Lambda(N)$ the de Rham differential, and by $H(N) = \bigoplus_k H^k(N)$ the de Rham cohomology. If $F\colon N_1 \to N$ is a smooth map of manifolds, we denote by $F^*\colon \Lambda(N) \to \Lambda(N_1)$ its pull-back. We will everywhere take the wedge product of differential forms, $\wedge$, to be understood, *i.e.*, for $\omega, \omega_1 \in \Lambda(N)$, instead of writing $\omega \wedge \omega_1$, we will simply write $\omega\omega_1$. We assume the reader to be familiar with Frölicher–Nijenhuis calculus on form valued vector fields (insertion $i_Z\omega$ of a form valued vector field Z into a differential form ω, Lie derivative $L_Z\omega$ of a differential form ω along a form valued vector fields Z, Frölicher–Nijenhuis bracket, *etc.*, see, for instance, [44]).

Let $\varpi\colon W \to N$ be an affine bundle (or, possibly, a vector bundle) and let $F\colon N_1 \to N$ be a smooth map of manifolds. The affine space of smooth sections of ϖ will be denoted by $\Gamma(\varpi)$. For $x \in N$, we sometimes put $\Gamma(\varpi)|_x := \varpi^{-1}(x)$ and, for $\chi \in \Gamma(\varpi)$, we also put $\chi_x := \chi(x)$. The affine bundle on N_1 induced by ϖ via F will be denote by $\varpi|_F\colon W|_F \to N$:

$$\begin{tikzcd} W|_F \arrow[r] \arrow[d, "\varpi|_F"'] & W \arrow[d, "\varpi"] \\ N_1 \arrow[r, "F"] & N. \end{tikzcd}$$

We also denote $\Gamma(\varpi)|_F := \Gamma(\varpi|_F)$. For any section $s \in \Gamma(\varpi)$, there exists a unique section, which, abusing the notation, we denote by $s|_F \in \Gamma(\varpi|_F)$, such that the diagram

$$\begin{tikzcd} W|_F \arrow[r] & W \\ N_1 \arrow[r, "F"] \arrow[u, "s|_F"] & N \arrow[u, "s"'] \end{tikzcd}$$

commutes. Elements in $\Gamma(\varpi)|_F$ are called *sections of ϖ along F*. If F is an embedding $\varpi|_F$, $\Gamma(\varpi)|_F$ and $s|_F$ will be referred to as *the restriction to* N_1 of ϖ, $\Gamma(\varpi)$ and s, respectively. If $\varpi_1\colon W_1 \to N$ is another affine bundle and $A\colon \Gamma(\varpi) \to \Gamma(\varpi_1)$ is an affine map then there exists a unique affine map $A|_F\colon \Gamma(\varpi)|_F \to \Gamma(\varpi_1)|_F$ such that $A|_F(s|_F) = A(s)|_F$ for all $s \in \Gamma(\varpi)$.

Let $\alpha\colon P \to M$ be a fiber bundle. A vector field $X \in \mathrm{D}(P)$ is called *α-projectable* if and only if there exists $\check{X} \in \mathrm{D}(P)$ such that $X \circ \alpha^* = \alpha^* \circ \check{X}$. The vector field $\check{X}$ is called the *α-projection of X*. Vector fields that are α-projectable form a Lie subalgebra in $\mathrm{D}(P)$ denoted by $\mathrm{D}_V(P,\alpha)$ (or simply D_V if this does not lead to confusion). An α-projectable vector field projecting onto the 0 vector field is an *α-vertical vector field*. Vector fields that are α-vertical form an ideal in D_V denoted by $V\mathrm{D}(P,\alpha)$ (or simply $V\mathrm{D}$). Notice that, if α has connected fiber, then D_V is the stabilizer of $V\mathrm{D}$ in $\mathrm{D}(P)$, *i.e.*, $\mathrm{D}_V = \{X \in \mathrm{D}(P) \mid [X, V\mathrm{D}] \subset V\mathrm{D}\}$.

Let $\alpha\colon P \to M$ be as above, $\dim M = n$, $\dim P = m + n$. Denote by $\alpha_1\colon J^1\alpha \to M$ the bundle of 1-jets of local sections of α [4,51], and by $\alpha_{1,0}\colon J^1\alpha \to P$ the canonical projection. For any local section $\sigma\colon U \to P$ of α, $U \subset M$ being an open subset, we denote by $\dot\sigma\colon U \to J^1\alpha$ its 1-st jet prolongation. Any system of α-adapted coordinates $(\ldots, x^i, \ldots, y^a, \ldots)$ on P, x^i being coordinates on M and y^a fiber coordinates on P, gives rise to the system of jet coordinates $(\ldots, x^i, \ldots, u^a, \ldots, y^a_i, \ldots)$ on $J^1\alpha$, $i = 1, \ldots, n$, $a = 1, \ldots, m$. Recall that $\alpha_{1,0}$ is an affine bundle and a section

$\nabla \colon P \to J^1\alpha$ of it is naturally interpreted as a (Ehresmann) connection in α. We assume the reader to be familiar with the geometry of connections (see, for instance, [44]). A connection ∇ is locally represented as

$$\nabla : y^a_i = \nabla^a_i,$$

∇^a_i being local functions on P. The space $\Gamma(\alpha_{1,0})$ of all such sections will be also denoted by $C(P, \alpha)$ (or simply C).

Let $\alpha \colon P \to M$ be as above, $\alpha' \colon P' \to M$ be another fiber bundle and let $G \colon P \to P'$ be a bundle morphism (over the identity $\mathrm{id}_M \colon M \to M$), *i.e.*, a smooth map such that $\alpha' \circ G = \alpha$. First of all, recall that there exists a unique bundle morphism $j_1 G \colon J^1\alpha \to J^1\alpha'$ such that $j_1 G \circ \dot{\sigma} = (G \circ \sigma)^{\cdot}$ for all local sections σ of α. The bundle morphism $j_1 G$ is the *first jet prolongation of* G and the diagram

$$\begin{tikzcd}
J^1\alpha \arrow[rr, "j_1 G"] \arrow[d, "\alpha_{1,0}"'] & & J^1\alpha' \arrow[d, "\alpha'_{1,0}"] \\
P \arrow[rr, "G"] \arrow[dr, "\alpha"'] & & P' \arrow[dl, "\alpha'"] \\
& M &
\end{tikzcd}$$

commutes. Now, a connection $\nabla \in C(P, \alpha)$ and a connection $\nabla' \in C(P', \alpha')$ are said G-*compatible* if and only if $\nabla' \circ G = j_1 G \circ \nabla$.

Let

$$\cdots \longrightarrow K_{l-1} \xrightarrow{\delta_{l-1}} K_l \xrightarrow{\delta_l} K_{l+1} \xrightarrow{\delta_{l+1}} \cdots$$

be a complex. Put $K := \bigoplus_l K_l$ and $\delta := \bigoplus_l \delta_l$. We denote the cohomology space of (K, δ) by $H(K, \delta) := \bigoplus_l H^l(K, \delta)$, where $H^l(K, \delta) := \ker \delta_l / \operatorname{im} \delta_{l-1}$.

Let A be a commutative $\mathbb{R}$-algebra, M, M_1 be A-modules and let A be an affine space modeled over M. We denote by $\mathrm{Aff}_A(A, M_1)$ (resp. $\mathrm{Hom}_A(M, M_1)$) the A-module of affine (resp. A-linear) maps $A \to M_1$ (resp. $M \to M_1$). If $\phi \in \mathrm{Aff}_A(A, M_1)$, its *linear part* $\underline{\phi}$ is an element in $\mathrm{Hom}_A(M, M_1)$.

Let m, r be positive integers and let $A_{a_1 \cdots a_r}$ be elements in a real vector space, for $a_1, \ldots, a_r = 1, \ldots, m$. We denote by $A_{[a_1 \cdots a_r]}$ their skew-symmetrization, *i.e.*,

$$A_{[a_1 \cdots a_r]} := \tfrac{1}{s!} \sum_{\sigma \in S_r} \varepsilon(\sigma) A_{a_{\sigma(1)} \cdots a_{\sigma(r)}},$$

with S_r being the group of permutations of $\{1, \ldots, r\}$ and $\varepsilon(\sigma)$ the sign of $\sigma \in S_r$.

We denote by $\simeq$ (resp. $\approx$) a canonical (resp. non-canonical) isomorphism between algebraic structures and by $\equiv$ an equivalence of notations. For instance, for $\alpha \colon P \to M$ as above, $VD \equiv VD(P, M)$. Finally, we use the Einstein convention for sums over upper-lower pairs of repeated indices.

3 Affine Forms on Fiber Bundles

3.1 Special Forms on Fiber Bundles

Let $\alpha\colon P\to M$ be a fiber bundle, $A:=C^\infty(P)$, $A_0:=C^\infty(M)$, $x^1,\dots,x^n$ be coordinates on M, $\dim M=n$, and let $y^1,\dots,y^m$ be fiber coordinates on P, $\dim P=n+m$. In the following we will often consider the monomorphism of algebras $\alpha^*\colon A_0\to A$, whose image is made of functions on P that are constant along the fibers of α. The space D_V (resp. $V\mathrm{D}$) is made of vector fields X locally of the form $X=X^i\partial_i+Y^a\partial_a$ (resp. $X=Y^a\partial_a$), where $X^i=X^i(x^1,\dots,x^n)$, $\partial_i:=\partial/\partial x^i$, $i=1,\dots,n$, $\partial_a:=\partial/\partial y^a$, $a=1,\dots,m$.

Denote by $\Lambda_1(P,\alpha)=\bigoplus_k\Lambda_1^k(P,\alpha)$ (or simply $\Lambda_1=\bigoplus_k\Lambda_1^k$) the differential (graded) ideal of differential forms on P vanishing when pulled back to fibers of α, *i.e.*, $\omega\in\Lambda_1^k$, $k\geq 0$ if and only if $\omega\in\Lambda^k(P)$ and $i^*_{\alpha^{-1}(x)}(\omega)=0$ for all $x\in M$, $i_{\alpha^{-1}(x)}\colon\alpha^{-1}(x)\to P$ being the embedding of the fiber $\alpha^{-1}(x)$ of α through $x\in M$. Moreover, denote by $\Lambda_p(P,a)=\bigoplus_k\Lambda_p^k(P,\alpha)$ (or simply $\Lambda_p=\bigoplus_k\Lambda_p^k$) the p-th exterior power of Λ_1. For all k and p, Λ_p^k is made of differential k-forms ω such that $(i_{Y_1}\circ\cdots\circ i_{Y_{k-p+1}})\omega=0$ for every $Y_1,\dots,Y_{k-p+1}\in V\mathrm{D}$ or, which is the same, differential k-forms ω locally of the form

$$\omega=\sum_{l\geq 0}\omega_{i_1\cdots i_{p+l}a_1\cdots a_{k-p-l}}dx^{i_1}\cdots dx^{i_{p+l}}dy^{a_1}\cdots dy^{a_{k-p-l}},$$

$\omega_{i_1\cdots i_{p+l}a_1\cdots a_{k-p-l}}$ being local functions on P, $i_1,\dots,i_{p+l}=1,\dots,n$, $a_1,\dots,a_{k-p-l}=1,\dots,m$.

Denote by $V\Lambda(P,\alpha)=\bigoplus_k V\Lambda^k(P,\alpha)$ (or simply $V\Lambda=\bigoplus_k V\Lambda^k$) the quotient differential algebra $\Lambda(P)/\Lambda_1$, with $d^V\colon V\Lambda\to V\Lambda$ its differential and with $p^V\colon\Lambda(P)\ni\omega\longmapsto\omega^V:=\omega+\Lambda_1\in V\Lambda$ the projection onto the quotient. Notice that d^V is A_0-linear. An element ρ^V in $V\Lambda^k$ is locally of the form

$$\rho^V=\rho_{a_1\cdots a_k}d^Vy^{a_1}\cdots d^Vy^{a_k},$$

$\rho_{a_1\cdots a_k}$ being local functions on P, and $d^V\rho^V$ being locally given by

$$d^V\rho^V=\partial_a\rho_{a_1\cdots a_k}d^Vy^ad^Vy^{a_1}\cdots d^Vy^{a_k}=\partial_{[a}\rho_{a_1\cdots a_k]}d^Vy^ad^Vy^{a_1}\cdots d^Vy^{a_k}.$$

Clearly, $V\Lambda^1$ is the dual A-module of $V\mathrm{D}$ and $V\Lambda$ its exterior algebra. In particular, elements in $V\Lambda$ may be interpreted as multilinear, skew-symmetric forms on $V\mathrm{D}$.

Denote by $\overline{\Lambda}(P,\alpha)=\bigoplus_k\overline{\Lambda}^k(P,\alpha):=\bigoplus_k\Lambda_k^k\subset\Lambda(P)$ (or simply $\overline{\Lambda}=\bigoplus_k\overline{\Lambda}^k$) the sub-algebra generated by Λ_1^1. An element $\omega\in\overline{\Lambda}^k$ is locally of the form

$$\omega=\omega_{i_1\cdots i_k}dx^{i_1}\cdots dx^{i_k}.$$

Notice that $\overline{\Lambda}$ is naturally isomorphic to $A\otimes_{A_0}\Lambda(M)$ as an A-algebra.

For any p, the quotient (graded) differential module[2]

$$E_0^{p,\bullet}\equiv E_0^{p,\bullet}(P,\alpha):=\Lambda_p/\Lambda_{p+1}$$

[2] This notation is motivated by the fact that A-modules $E_0^{p,\bullet}$ are columns of the first term of the (cohomological) Leray–Serre spectral sequence of the fiber bundle α (see [46]).

is naturally isomorphic to $V\Lambda \otimes_A \overline{\Lambda}^p$ (or, which is the same, $V\Lambda \otimes_{A_0} \Lambda^p(M)$) via the correspondence

$$E_0^{p,q} \ni \omega + \Lambda_{p+1}^{p+q} \longmapsto \varpi \in V\Lambda^q \otimes_A \overline{\Lambda}^p, \tag{3.1}$$

well defined by putting

$$\varpi(Y_1, \ldots, Y_q) := (i_{Y_q} \circ \cdots \circ i_{Y_1})(\omega) \in \overline{\Lambda}^p,$$

$Y_1, \ldots, Y_q \in V\mathrm{D}$. In the following we denote by $E_0^{p,q}$ the q-th homogeneous piece of $E_0^{p,\bullet}$, $q \in \mathbb{Z}$. According to the above, $V\Lambda \otimes_A \overline{\Lambda}$ (or, which is the same, $V\Lambda \otimes_{A_0} \Lambda(M)$) is the graded object associated with the filtration $\Lambda(P) \supset \Lambda_1 \supset \cdots \supset \Lambda_p \supset \cdots$. As we will see in the next subsection, a connection in α allows one to identify such filtration with its graded object.

Let us now focus on the ideals Λ_{n-1} and Λ_n. Put $d^n x := dx^1 \cdots dx^n$ and $d^{n-1}x_i := i_{\partial_i} d^n x$, so that $dx^j d^{n-1}x_i = \delta_i^j d^n x$, $i, j = 1, \ldots, n$. Then an element $\omega \in \Lambda_{n-1}^{q+n-1}$ (resp. $\omega \in \Lambda_n^{q+n-1}$) is locally in the form

$$\omega = \omega^i_{a_1 \cdots a_q} dy^{a_1} \cdots dy^{a_q} d^{n-1}x_i + \omega_{a_1 \cdots a_{q-1}} dy^{a_1} \cdots dy^{a_{q-1}} d^n x$$

(resp.

$$\omega = \omega_{a_1 \cdots a_{q-1}} dy^{a_1} \cdots dy^{a_{q-1}} d^n x),$$

$\ldots, \omega^i_{a_1 \ldots a_q}, \ldots, \omega_{a_1 \ldots a_{q-1}}, \ldots$ being local functions on P. In particular Λ_{n-1}^{q+n-1} (resp. Λ_n^{q+n-1}) is the module of sections of an $\left[n\binom{q}{m} + \binom{q-1}{m}\right]$ (resp. $\binom{q-1}{m}$)-dimensional vector bundle over P. Below we will provide an alternative description of Λ_{n-1} and Λ_n (Theorem 3.1). In our opinion, such description is more suitable for a better understanding of the role of Λ_{n-1} and Λ_n in first order field theories (see, for instance, [28]).

3.2 Affine Forms

Let $\nabla \in C \equiv C(P, \alpha)$. Recall, preliminarily, that C is an affine space modeled over the A-module $\overline{\Lambda}^1 \otimes_A V\mathrm{D}$, or, which is the same, $\Lambda^1(M) \otimes_{A_0} V\mathrm{D}$. The connection ∇ allows one to split the tangent bundle TP into its vertical part VP and a horizontal part $H_\nabla P$. Denote by $H_\nabla \mathrm{D}(P, \alpha) \subset \mathrm{D}(P)$ (or simply $H_\nabla \mathrm{D} \subset \mathrm{D}(P)$) the submodule of ∇-horizontal vector fields. An element $X \in H_\nabla \mathrm{D}$ is locally in the form $X = X^i \nabla_i$, where $\nabla_i := \partial_i + \nabla_i^a \partial_a$, $i = 1, \ldots, n$. The splitting

$$\mathrm{D}(P) = V\mathrm{D} \oplus H_\nabla \mathrm{D} \tag{3.2}$$

determines a splitting of the de Rham differential $d \colon \Lambda(P) \to \Lambda(P)$ into a horizontal part $d_\nabla \colon \Lambda(P) \to \Lambda(P)$, and a vertical part $d_\nabla^V \colon \Lambda(P) \to \Lambda(P)$, $d = d_\nabla + d_\nabla^V$, where d_∇ (resp. d_∇^V) is the Lie derivative along the horizontal-form valued vector field (resp. the form valued vertical vector field) $H_\nabla \colon A \to \overline{\Lambda}^1(P)$ (resp. $V_\nabla \colon A \to \Lambda^1(P)$) determined by ∇. H_∇ (resp. V_∇) is locally given by $H_\nabla = dx^i \nabla_i$ (resp.

$V_\nabla = (dy^a - \nabla^a_i dx^i)\partial_a)$. Notice that $\left(\Lambda(P), d^V_\nabla, d_\nabla\right)$ is not a bi-complex unless ∇ is flat. Splitting (3.2) also determines an isomorphism $\phi_\nabla : V\Lambda \otimes_A \overline{\Lambda} \to \Lambda(P)$ locally given by

$$\phi_\nabla(d^V y^{a_1} \cdots d^V y^{a_q} \otimes dx^{i_1} \cdots dx^{i_q}) = d^V_\nabla y^{a_1} \cdots d^V_\nabla y^{a_q} dx^{i_1} \cdots dx^{i_q}.$$

In particular, for any q, p, there is an obvious projection $\mathfrak{p}^{q,p}_\nabla : \Lambda(P) \to V\Lambda^q \otimes_A \overline{\Lambda}^p$.

For any $k \geq 0$, put

$$'\Omega^{k+1} := \mathrm{Aff}_A(C, V\Lambda^k \otimes_A \overline{\Lambda}^n).$$

An element $'\vartheta \in {}'\Omega^{k+1}$ is locally given by

$$'\vartheta(\nabla) = ('\vartheta^i_{a,a_1\cdots a_k}\nabla^a_i + {}'\vartheta_{a_1\cdots a_k})d^V y^{a_1} \cdots d^V y^{a_k} \otimes d^n x, \quad \nabla \in C,$$

$\ldots, {}'\vartheta^i_{a,a_1\cdots a_k}, \ldots, {}'\vartheta_{a_1\cdots a_k}, \ldots$ local functions on P. The linear part $'\underline{\vartheta}$ of an element $'\vartheta \in {}'\Omega^{k+1}$ is an element in the A-module

$$\mathrm{Hom}_A(\overline{\Lambda}^1 \otimes_A VD, V\Lambda^k \otimes_A \overline{\Lambda}^n) \simeq \mathrm{Hom}_A(VD, V\Lambda^k \otimes_A \overline{\Lambda}^{n-1}),$$

where we identified $V\Lambda^k \otimes \overline{\Lambda}^{n-1}$ and $\mathrm{Hom}_A(\overline{\Lambda}^1, V\Lambda^k \otimes \overline{\Lambda}^n)$ via the isomorphism

$$V\Lambda^k \otimes \overline{\Lambda}^{n-1} \ni \sigma \otimes \rho \longmapsto \varphi_{\sigma\otimes\rho} \in \mathrm{Hom}_A(\overline{\Lambda}^1, V\Lambda^k \otimes \overline{\Lambda}^n),$$

$\sigma \in V\Lambda^k$, $\rho \in \overline{\Lambda}^{n-1}$, defined by putting

$$\varphi_{\sigma\otimes\rho}(\eta) := (-)^k \sigma \otimes \eta\rho \in V\Lambda^k \otimes \overline{\Lambda}^n, \quad \eta \in \overline{\Lambda}^1.$$

Put $\underline{\Omega}^{k+1} \equiv \underline{\Omega}^{k+1}(P,\alpha) := V\Lambda^{k+1} \otimes_A \overline{\Lambda}^{n-1}$. Similarly as above, $\underline{\Omega}^{k+1}$ can be embedded into $\mathrm{Hom}_A(VD, V\Lambda^k \otimes_A \overline{\Lambda}^{n-1})$ via the correspondence

$$\underline{\Omega}^{k+1} \ni \sigma' \otimes \rho \longmapsto \varphi'_{\sigma'\otimes\rho} \in \mathrm{Hom}_A(VD, V\Lambda^k \otimes_A \overline{\Lambda}^{n-1}), \tag{3.3}$$

$\sigma' \in V\Lambda^k$, $\rho \in \overline{\Lambda}^{n-1}$, defined by putting

$$\varphi'_{\sigma'\otimes\rho}(Y) := i_Y \sigma' \otimes \rho \in V\Lambda^k \otimes_A \overline{\Lambda}^{n-1}, \quad Y \in VD.$$

In the following, embedding (3.3) will be understood.

Put also $\Omega^0 \equiv \Omega^0(P,\alpha) := \overline{\Lambda}^{n-1}$, $\underline{\Omega}^0 \equiv \underline{\Omega}^0(P,\alpha) := \Omega^0$, and for $k \geq 0$,

$$\Omega^{k+1} \equiv \Omega^{k+1}(P,\alpha) := \{\vartheta \in {}'\Omega^{k+1} \mid \underline{\vartheta} \in \underline{\Omega}^{k+1}\},$$

$\Omega \equiv \Omega(P,\alpha) := \bigoplus_{q\geq 0} \Omega^q$ and $\underline{\Omega} \equiv \underline{\Omega}(P,\alpha) := \bigoplus_{q\geq 0} \underline{\Omega}^q$. Elements in Ω^k will be called *affine k-forms* over α, $k \geq 0$. It is easy to show that an element $\vartheta \in {}'\Omega^{k+1}$ is an affine $(k+1)$-form if and only if it is locally given by

$$\vartheta(\nabla) = (\vartheta^i_{aa_1\cdots a_k}\nabla^a_i + \vartheta_{a_1\cdots a_k})d^V y^{a_1} \cdots d^V y^{a_k} \otimes d^n x, \quad \nabla \in C;$$

$\ldots, \vartheta^i_{aa_1\cdots a_k}, \ldots, \vartheta_{a_1\cdots a_k}, \ldots$ being local functions on P such that $\vartheta^i_{aa_1\cdots a_k} = \vartheta^i_{[aa_1\cdots a_k]}$, for $i = 1, \ldots, n$, $a, a_1, \ldots, a_k = 1, \ldots, m$.

According to the above, the linear part $\underline{\vartheta} \in \underline{\Omega}^{k+1}$ of ϑ is implicitly defined by the formula

$$\vartheta(\nabla + \eta \otimes Y)(Y_1, \ldots, Y_k) - \vartheta(\nabla)(Y_1, \ldots, Y_k) = (-)^k \eta \cdot \underline{\vartheta}(Y, Y_1, \ldots, Y_k) \in \overline{\Lambda}^n,$$

$\nabla \in C$, $\eta \in \overline{\Lambda}^{n-1}$, $Y, Y_1, \ldots, Y_k \in VD$, and it is locally given by

$$\underline{\vartheta} = \frac{(-)^k}{k+1}\vartheta^i_{a_1\cdots a_{k+1}} d^V y^{a_1} \cdots d^V y^{a_{k+1}} \otimes d^{n-1}x_i. \tag{3.4}$$

3.3 Affine Forms and Differential Forms

Let $\Omega_0(P,\alpha) = \bigoplus_{q\geq 0} \Omega_0^q(P,\alpha)$ (or, simply, $\Omega_0 \equiv \bigoplus_{q\geq 0} \Omega_0^q$) be the kernel of the projection $\Omega \ni \vartheta \longmapsto \underline{\vartheta} \in \underline{\Omega}$. Clearly, Ω_0^q is canonically isomorphic to $V^{q-1}\Lambda \otimes_A \overline{\Lambda}^n$ for $q > 0$ (and in the following such isomorphism will be understood), while $\Omega_0^0 = 0$. Moreover, Ω^q (resp. Ω_0^q) is the module of sections of an $\left[n\binom{q}{m} + \binom{q-1}{m}\right]$ (resp. $\binom{q-1}{m}$)-dimensional vector bundle over P.

Theorem 3.1 *There are canonical isomorphisms of A-modules*

$$\iota_{0,q}\colon \Lambda_n^{q+n-1} \longrightarrow \Omega_0^q,$$
$$\iota_q\colon \Lambda_{n-1}^{q+n-1} \longrightarrow \Omega^q,$$
$$\underline{\iota}_q\colon E_0^{n-1,q} \longrightarrow \underline{\Omega}^q,$$

$q \geq 0$, *such that diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda_n & \longrightarrow & \Lambda_{n-1} & \longrightarrow & E_0^{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \underline{\iota}} & & \\ 0 & \longrightarrow & \Omega_0 & \longrightarrow & \Omega & \longrightarrow & \underline{\Omega} & \longrightarrow & 0 \end{array} \tag{3.5}$$

commutes, where $\iota_0 := \bigoplus_q \iota_{0,q}$, $\iota := \bigoplus_q \iota_q$ *and* $\underline{\iota} := \bigoplus_q \underline{\iota}_q$.

Proof Let $q > 0$. First of all, denote by $\underline{\iota}_q\colon E_0^{n-1,q} \to \underline{\Omega}^q$ the already-mentioned natural isomorphism (3.1) and notice that for any $\omega \in \Lambda_n^{q+n-1}$ and $Y_1,\dots,Y_{q-1} \in VD$, $(i_{Y_1} \circ \cdots \circ i_{Y_{q-1}})(\omega) \in \overline{\Lambda}^n$. Therefore, an element $\iota_{0,q}(\omega) \in \Omega_0^q$ is well defined by putting $\iota_{0,q}(\omega)(Y_1,\dots,Y_{q-1}) := (i_{Y_1} \circ \cdots \circ i_{Y_{q-1}})(\omega) \in \overline{\Lambda}^n$, $Y_1,\dots,Y_{q-1} \in VD$. Moreover, the correspondence $\Lambda_n^{q+n-1} \ni \omega \longmapsto \iota_{0,q}(\omega) \in \Omega_0^q$ is an isomorphism of A-modules. Indeed, let $\omega \in \Lambda_n^{q+n-1}$ and $(i_{Y_1}\circ\cdots\circ i_{Y_{q-1}})(\omega) = 0$ for all $Y_1,\dots,Y_{q-1} \in VD$, then $\omega \in \Lambda_{n+1}^{q+n-1} = 0$, so that $\iota_{0,q}$ is injective. Moreover, Λ_n^{q+n-1} and Ω_0^q are locally free A-modules of the same local dimension. We conclude that

$$\iota_0 := \bigoplus_q \iota_{0,q}\colon \Lambda_n \longrightarrow \Omega_0$$

is a canonical isomorphism of A-modules as well, sending Λ_n^{q+n-1} into Ω_0^q, $q \geq 0$. Finally, if $\omega \in \Lambda_n^{q+n-1}$ is locally given by

$$\omega = \omega_{a_1\cdots a_{q-1}} dy^{a_1} \cdots dy^{a_{q-1}} d^n x,$$

then $\iota_0(\omega) \in \Omega_0$ is locally given by $\iota_0(\omega) = \omega_{a_1\cdots a_{q-1}} d^V y^{a_1} \cdots d^V y^{a_{q-1}} \otimes d^n x$.

Now, for $\omega \in \Lambda_{n-1}^{q+n-1}$ and $\nabla \in C$ put

$$\iota_q(\omega)(\nabla) := \mathfrak{p}_\nabla^{q-1,n}(\omega).$$

If ω is locally given by

$$\omega = \omega^i_{a_1\cdots a_q} dy^{a_1}\cdots dy^{a_q} d^{n-1}x_i + \omega_{a_1\cdots a_{q-1}} dy^{a_1}\cdots dy^{a_{q-1}} d^n x,$$

then

$$\begin{aligned}\omega &= \omega^i_{a_1\cdots a_q}(d^V_\nabla + d_\nabla)(y^{a_1})\cdots(d^V_\nabla + d_\nabla)(y^{a_q})d^{n-1}x_i\\ &\quad + \omega_{a_1\cdots a_{q-1}} d^V_\nabla y^{a_1}\cdots d^V_\nabla y^{a_{q-1}} d^n x\\ &= \omega_{a_1\cdots a_{q-1}} d^V_\nabla y^{a_1}\cdots d^V_\nabla y^{a_{q-1}} d^n x + \omega^i_{a_1\cdots a_q} d^V_\nabla y^{a_1}\cdots d^V_\nabla y^{a_q} d^{n-1}x_i\\ &\quad + \sum_s (-)^{p-s}\omega^i_{a_1\cdots a_q}\nabla^{a_s}_j d^V_\nabla y^{a_1}\cdots \widehat{d^V_\nabla y^{a_s}}\cdots d^V_\nabla y^{a_q} dx^j d^{n-1}x_i\\ &= \omega^{q,n-1} + [q(-)^{q-1}\omega^i_{aa_1\cdots a_{q-1}}\nabla^a_i + \omega_{a_1\cdots a_{q-1}}] d^V_\nabla y^{a_1}\cdots d^V_\nabla y^{a_{q-1}} d^n x\end{aligned}$$

where a cap "$\widehat{\ }$" denotes omission of the factor below it, and $\omega^{q,n-1} \in \Lambda(P)$ is a suitable form such that $\mathfrak{p}^{q-1,n}_\nabla(\omega^{q,n-1}) = 0$. Therefore, locally

$$\begin{aligned}\iota_q(\omega)(\nabla) &= \mathfrak{p}^{q-1,n}_\nabla(\omega)\\ &= [q(-)^{q-1}\omega^i_{aa_1\ldots a_{q-1}}\nabla^a_i + \omega_{a_1\ldots a_{q-1}}] d^V y^{a_1}\cdots d^V y^{a_{q-1}} \otimes d^n x.\end{aligned} \tag{3.6}$$

This shows simultaneously that $\iota_q(\omega)$ is affine, that it is in Ω^q, and that ι_q is injective. Since Λ^{p+n}_{n-1} and Ω^q are locally free A-modules of the same local dimension, then the correspondence $\iota_q\colon \Lambda^{q+n-1}_{n-1} \ni \omega \longmapsto \iota_q(\omega) \in \Omega^q$ is an isomorphism. Commutativity of diagram (3.5) immediately follows from local formulas (3.4) and (3.6). ■

Notice that isomorphism ι generalizes considerably the well-known isomorphism $\Lambda^n_{n-1} \simeq \mathrm{Aff}_A(C, \overline{\Lambda}^n)$ [28].

Finally, let $\pi\colon E \to M$ be a fiber bundle and let q^A be fiber coordinates on E. Notice that $\Omega^1(E,\pi)$ (resp. $\underline{\Omega}^1(E,\pi)$) is the $C^\infty(E)$-module of sections of a vector bundle $\mu_0\pi\colon \mathscr{M}\pi \to E$ (resp. $\tau^\dagger_0\pi\colon J^\dagger\pi \to E$). Recall that there is a distinguished element Θ in $\Omega^1(\mathscr{M}\pi, \mu\pi)$ (resp. $\underline{\Theta} \in \underline{\Omega}^1(J^\dagger\pi, \tau^\dagger\pi)$), with $\mu\pi := \pi\circ\mu_0\pi$ (resp. $\tau^\dagger\pi := \pi\circ\tau^\dagger_0\pi$), the tautological one [28], which in standard coordinates

$$\ldots, x^i, \ldots, q^A, \ldots, p^i_A, \ldots, p$$

on $\mathscr{M}\pi$ (resp. $\ldots, x^i, \ldots, q^A, \ldots, p^i_A, \ldots$ on $J^\dagger\alpha$) is given by

$$\Theta = p^i_A dq^A d^{n-1}x_i - p d^n x \quad (\text{resp. } \underline{\Theta} = p^i_A d^V q^A \otimes d^{n-1}x_i).$$

4 Affine Form Calculus

4.1 Natural Operations with Affine Forms

In this section we derive the main formulas of calculus on affine forms. Such formulas will be useful in generalizing proofs from the context of Hamiltonian systems to the context of PD Hamiltonian systems (see Section 6).

Let $\alpha\colon P \to M$ be as in the previous section. Isomorphism ι (resp. ι_0, $\underline{\iota}$) can be used to "transfer structures" from Λ_{n-1} (resp. Λ_n, E_0^{n-1}) to Ω (resp. Ω_0, $\underline{\Omega}$) and back. As an instance, notice that Ω has a natural structure of $\Lambda(P)$-module given by

$$\lambda\vartheta := \iota(\lambda\omega),$$

$\lambda \in \Lambda(P)$, $\vartheta = \iota(\omega) \in \Omega$, $\omega \in \Lambda_{n-1}$. Moreover, Ω is generated by Ω^0 as a $\Lambda(P)$-module. Similarly, Λ_n (resp. E_0^{n-1}) has a structure of $V\Lambda$-module given by

$$\lambda^V\omega_0 := \iota_0^{-1}(\lambda^V\rho^V \otimes \nu) \quad (\text{resp. } \lambda^V\underline{\omega} := \iota_0^{-1}(\lambda^V\rho^V \otimes \sigma)),$$

$\lambda^V \in V\Lambda$, $\omega_0 = \iota_0^{-1}(\rho^V \otimes \nu)$ (resp. $\underline{\omega} = \underline{\iota}^{-1}(\rho^V \otimes \sigma)$), $\rho^V \in V\Lambda$, $\nu \in \overline{\Lambda}^n$ (resp. $\sigma \in \overline{\Lambda}^{n-1}$), so that $\rho^V \otimes \nu \in V\Lambda \otimes_A \overline{\Lambda}^n = \Omega_0$ (resp. $\rho^V \otimes \sigma \in V\Lambda \otimes_A \overline{\Lambda}^{n-1} = \underline{\Omega}$). Clearly, Λ_n (resp. E_0^{n-1}) is generated by $\overline{\Lambda}^n$ (resp. $\overline{\Lambda}^{n-1}$) as a $V\Lambda$-module. Finally, the presented structures are compatible in the sense that for $\omega_0 \in \Lambda_n$, $\omega \in \Lambda_{n-1}$ and $\lambda \in \Lambda(P)$, we have

$$\lambda^V\omega_0 = \lambda\omega_0 \quad \text{and} \quad \underline{\lambda\omega} = \lambda^V\underline{\omega}.$$

As a last instance of how to use isomorphisms in (3.5) to transfer a structure from one space to the other we define the insertion of a connection $\nabla \in C$ into a differential form $\omega \in \Lambda_n$ as

$$i_\nabla\omega := \iota_0^{-1}\big(\vartheta(\nabla)\big) = (\iota_0^{-1} \circ \mathfrak{p}_\nabla^{q-1,n})(\omega) \in \Lambda_n,$$

$\vartheta = \iota(\omega) \in \Omega$. Notice that this insertion of a connection in an element $\omega \in \Lambda_n$ has been already discussed in [19]. In the following we will always understand isomorphisms ι, ι_0, $\underline{\iota}$.

Notice that $\underline{\Omega}$ inherits many operations from Ω. Indeed, let

$$\nabla \in C, \quad Z \in \overline{\Lambda}^1 \otimes_A V\mathrm{D} \subset \Lambda(P) \otimes_A \mathrm{D}(P), \quad Y \in V\mathrm{D}, \quad X \in \mathrm{D}_V, \quad q \geq 0.$$

Then the following hold:

(i) $i_Z(\Omega) \subset \Omega_0$ and $i_Z(\Omega_0) = 0$, so that an operator, which, abusing the notation, we again denote by $i_Z\colon \underline{\Omega} \to \Omega_0$, is well defined via the formula

$$i_Z\underline{\omega} := i_Z\omega \in \Omega_0,$$

$\omega \in \Omega$. Moreover, it is easy to show that

$$i_Z\underline{\omega} = i_{\nabla+Z}\omega - i_\nabla\omega.$$

Finally, for $Z = \eta \otimes Y_1$, and $\underline{\omega} = \rho^V \otimes \sigma$, $\eta \in \overline{\Lambda}^1$, $Y_1 \in V\mathrm{D}$, $\rho^V \in V\Lambda^q$ and $\sigma \in \overline{\Lambda}^{n-1}$, we have

$$i_Z\underline{\omega} = (-)^{q-1} i_{Y_1}\rho^V \otimes \eta\sigma.$$

(ii) $i_Y(\Omega) \subset \Omega$ (resp. $L_X(\Omega) \subset \Omega$) and $i_Y(\Omega_0) \subset \Omega_0$ (resp. $L_X(\Omega_0) \subset \Omega_0$) so that the quotient map, which, abusing the notation, we again denote by $i_Y : \underline{\Omega} \to \underline{\Omega}$ (resp. $L_X : \underline{\Omega} \to \underline{\Omega}$), is well defined via the formula

$$i_Y \underline{\omega} := \underline{i_Y \omega} \in \underline{\Omega} \quad (\text{resp. } L_X \underline{\omega} = \underline{L_X \omega} \in \underline{\Omega}).$$

Finally, for $\underline{\omega} = \rho^V \otimes \sigma$, $\rho^V \in V\Lambda^q$ and $\sigma \in \overline{\Lambda}^{n-1}$, we have

$$i_Y \underline{\omega} = i_Y \rho^V \otimes \sigma.$$

(iii) $d_\nabla(\Omega) \subset \Omega_0$ and $d_\nabla(\Omega_0) = 0$ so that an operator, which, abusing the notation, we again denote by $d_\nabla : \underline{\Omega} \to \Omega_0$, is well defined via the formula $d_\nabla \underline{\omega} := d_\nabla \omega \in \Omega_0$, $\omega \in \Omega$.

Remark 4.1 Notice that the insertion $i_\nabla \omega$, being affine in ∇, is actually point wise, *i.e.*, if $\nabla' \in C$ is such that $\nabla'_y = \nabla_y \in C|_y = \alpha_{1,0}^{-1}(y)$ for some $y \in P$, then $(i_{\nabla'}\omega)_y = (i_\nabla \omega)_y$. Therefore, the insertion $i_c \omega_y$ of an element $c \in \alpha_{1,0}^{-1}(y)$, $y \in P$, into ω_y is well defined. Similar considerations apply to both the above-defined insertions i_Z and i_Y. Finally, for all $y \in P$, the projection $\Omega \to \underline{\Omega}$ also determines a well-defined linear map $\Omega|_y \ni \omega_y \longmapsto \underline{\omega}_y \in \underline{\Omega}|_y$ whose kernel is $\Omega_0|_y$.

In the following we will denote by $\delta : \Omega \to \Omega$ (resp. $\delta_0 : \Omega_0 \to \Omega_0$) the restricted de Rham differential, *i.e.*, for $\omega \in \Omega$ (resp. $\omega_0 \in \Omega_0$), $\delta\omega := d\omega \in \Omega$ (resp. $\delta_0\omega_0 := d\omega_0 \in \Omega_0$) and with $\underline{\delta} : \underline{\Omega} \to \underline{\Omega}$ the quotient differential. Then, for $\omega_0 = \rho^V \otimes \alpha^*(\nu_0)$ (resp. $\omega = \rho^V \otimes \alpha^*(\sigma_0)$), $\rho^V \in V\Lambda$, $\nu_0 \in \Lambda^n(M)$ (resp. $\sigma_0 \in \Lambda^{n-1}(M)$), we have

$$\delta_0 \omega_0 = d^V \rho^V \otimes \alpha^*(\nu_0) \quad (\text{resp. } \underline{\delta}\underline{\omega} = d^V \rho^V \otimes \alpha^*(\sigma_0)).$$

In other words δ_0 (resp. $\underline{\delta}$) is isomorphic to the differential $d^V \otimes \mathrm{id} : V\Lambda \otimes_{A_0} \Lambda^n(M) \to V\Lambda \otimes_{A_0} \Lambda^n(M)$ (resp. $d^V \otimes \mathrm{id} : V\Lambda \otimes_{A_0} \Lambda^{n-1}(M) \to V\Lambda \otimes_{A_0} \Lambda^{n-1}(M)$).

All the above-mentioned formulas can be proved by straightforward computations.

Now, let ∇, Y and X be as above. Denote by $[\![\cdot,\cdot]\!]$ the Frölicher–Nijenhuis bracket in $\Lambda(P) \otimes_A \mathrm{D}(P)$. It is easy to see that $[\![H_\nabla, X]\!] \in \overline{\Lambda}^1 \otimes_A V\mathrm{D} \subset \Lambda(P) \otimes_A \mathrm{D}(P)$.

Theorem 4.2 *Let $\omega \in \Omega$, then*

$$\begin{aligned} &[i_\nabla, \delta]\omega := (i_\nabla \circ \delta - \delta_0 \circ i_\nabla)\omega = d_\nabla \omega \in \Omega_0, \\ &[i_\nabla, i_Y]\omega := (i_\nabla \circ i_Y - i_Y \circ i_\nabla)\omega = 0 \in \Omega_0, \\ &[i_\nabla, L_X]\omega := (i_\nabla \circ L_X - L_X \circ i_\nabla)\omega = i_{[\![H_\nabla, X]\!]}\omega \in \Omega_0. \end{aligned} \tag{4.1}$$

Proof First we prove that $i_\nabla : \Omega \to \Omega_0$ satisfies the "Leibniz rule"

$$i_\nabla(\lambda\omega) = \lambda \cdot i_\nabla \omega + i_{H_\nabla}\lambda \cdot \omega, \tag{4.2}$$

$\lambda \in \Lambda(P)$, $\omega \in \Omega$. For $\rho \in \Lambda(P)$, denote $\rho_\nabla^{\bullet,p} := \sum_q \mathfrak{p}_\nabla^{q,p}(\rho)$, so that $\rho = \sum_p \rho_\nabla^{\bullet,p}$. Notice that for $\omega \in \Omega$ and $\lambda \in \Lambda(P)$, we have $\omega = \omega_\nabla^{\bullet,n} + \omega_\nabla^{\bullet,n-1}$ so that

$$i_\nabla(\lambda\omega) = \mathfrak{p}_\nabla^{\bullet,n}(\lambda\omega) = \lambda_\nabla^{l,0}\omega_\nabla^{\bullet,n} + \lambda_\nabla^{\bullet,1}\omega_\nabla^{\bullet,n-1} = \lambda \cdot i_\nabla\omega + \lambda_\nabla^{\bullet,1} \cdot \omega_\nabla^{\bullet,n-1}.$$

Moreover, $i_{H_\nabla}\lambda = \sum_p i_{H_\nabla}\lambda_\nabla^{\bullet,p} = \sum_p p\lambda_\nabla^{\bullet,p}$, which in turn implies $\lambda_\nabla^{\bullet,p} = i_{H_\nabla}\lambda - \sum_{p>1} p\lambda_\nabla^{\bullet,p}$. Therefore

$$\begin{aligned} i_\nabla(\lambda\omega) &= \lambda \cdot i_\nabla\omega + \lambda_\nabla^{\bullet,1} \cdot \omega_\nabla^{\bullet,n-1} \\ &= \lambda \cdot i_\nabla\omega + i_{H_\nabla}\lambda \cdot \omega_\nabla^{\bullet,n-1} - \sum_{p>1} p\lambda_\nabla^{\bullet,p}\omega_\nabla^{\bullet,n-1} \\ &= \lambda \cdot i_\nabla\omega + i_{H_\nabla}\lambda \cdot \omega. \end{aligned}$$

In view of (4.2), the above defined operators $[i_\nabla, \delta]$, $[i_\nabla, i_Y]$, $[i_\nabla, L_X]\colon \Omega \to \Omega_0$, satisfy analogous "Leibniz rules":

$$\begin{aligned} [i_\nabla, \delta](\lambda\omega) &= d_\nabla\lambda \cdot \omega + (-)^l\lambda \cdot [i_\nabla, \delta](\omega), \\ [i_\nabla, i_Y](\lambda\omega) &= \lambda \cdot [i_\nabla, i_Y](\omega), \\ [i_\nabla, L_X](\lambda\omega) &= i_{[\![H_\nabla, X]\!]}\lambda \cdot \omega + \lambda \cdot [i_\nabla, L_X](\omega). \end{aligned} \tag{4.3}$$

Since Ω is generated by $\overline{\Lambda}^{n-1}$ as a $\Lambda(P)$-module, in view of (4.3), it is enough to prove (4.1) for $\omega \in \overline{\Lambda}^{n-1}$. In this case

$$\begin{aligned} [i_\nabla, \delta]\omega = i_\nabla d\omega &= (d\omega)_\nabla^{\bullet,n} = d_\nabla\omega, \\ [i_\nabla, i_Y]\omega &= 0, \\ [i_\nabla, L_X]\omega = i_\nabla L_X\omega &= (L_X\omega)_\nabla^{\bullet,n} = 0 = i_{[\![H_\nabla, X]\!]}\omega. \end{aligned}$$ ■

We now discuss the interaction between affine forms and bundle morphisms. Let $\alpha'\colon P' \to M$ be another fiber bundle and let $G\colon P \to P'$ be a bundle morphism. Clearly, G preserves the ideals Λ_p, $p \geq 0$, *i.e.*, $G^*\left(\Lambda_p(P', \alpha')\right) \subset \Lambda_p(P, \alpha)$. In particular,

$$G^*\left(\Omega(P', \alpha')\right) \subset \Omega(P, \alpha) \quad \text{and} \quad G^*\left(\Omega_0(P', \alpha')\right) \subset \Omega_0(P, \alpha).$$

We conclude that the quotient map which, abusing the notation, we again denote by $G^*\colon \underline{\Omega}(P', \alpha') \to \underline{\Omega}(P, \alpha)$, is well defined. Now, consider G-compatible connections $\nabla \in C(P, \alpha)$ and $\nabla' \in C(P', \alpha')$. It is easy to show that

$$G^* \circ i_{\nabla'} = i_\nabla \circ G^*\colon \Omega(P', \alpha') \longrightarrow \Omega_0(P, \alpha).$$

4.2 Cohomology

Remark 4.3 (See [46]) In the following we denote by $\mathcal{F}$ the abstract fiber of α. Notice that, for any $q \geq 0$, $VH^q \equiv VH^q(P,\alpha) := H^q(V\Lambda, d^V)$ is the A_0-module of sections of a (pro-finite) vector bundle $\alpha^q \colon P^q \to M$ over M whose abstract fiber is $H^q(\mathcal{F})$. Moreover, α^q is endowed with a canonical flat connection ∇^q (∇^q is a smooth analogue of Gauss–Manin connection in algebraic geometry). Correspondingly, there is a de Rham-like complex

$$\cdots \longrightarrow \Lambda^{p-1} \otimes_{A_0} VH^q \xrightarrow{d_1^{p-1,q}} \Lambda^p \otimes_{A_0} VH^q \xrightarrow{d_1^{p,q}} \cdots ,$$

whose cohomology we denote by[3]

$$E_2^{\bullet,q} := \bigoplus_p E_2^{p,q}, \quad E_2^{p,q} := H^p\left(\Lambda(M) \otimes_{A_0} V\Lambda^q, d_1^{\bullet,q}\right), \quad q \geq 0.$$

It can be proved that, if α is trivial or M is simply connected, then there is a (generically non-canonical), isomorphism

$$E_2^{p,q} \approx H^p(M) \otimes H^q(\mathcal{F}), \quad p,q \geq 0.$$

Finally, notice also that, for any $q \geq 0$,

$$H^q(\Omega_0, \delta_0) \simeq \Lambda^n(M) \otimes_{A_0} VH^q,$$
$$H^q(\underline{\Omega}, \underline{\delta}) \simeq \Lambda^{n-1}(M) \otimes_{A_0} VH^q.$$

Proposition 4.4 *Let $\alpha \colon P \to M$ be a fiber bundle. Then, for any $q \geq 0$, there exists a short exact sequence of vector spaces*

$$0 \longrightarrow \operatorname{coker} d_1^{n,q-1} \longrightarrow H^q(\Omega, \delta) \longrightarrow \ker d_1^{n-1,q} \longrightarrow 0.$$

In particular, $H^q(\Omega,\delta) \approx \operatorname{coker} d_1^{n,q-1} \oplus \ker d_1^{n-1,q} = E_2^{n,q-1} \oplus \ker d_1^{n-1,q}$.

Proof Consider the short exact sequence of complexes

$$0 \longrightarrow \Omega_0 \longrightarrow \Omega \longrightarrow \underline{\Omega} \longrightarrow 0,$$

and the associated long sequence in cohomology

$$\cdots \longrightarrow H^{q-1}(\underline{\Omega}, \underline{\delta}) \xrightarrow{\partial} H^q(\Omega_0, \delta_0) \longrightarrow H^q(\Omega, \delta) \longrightarrow H^q(\underline{\Omega}, \underline{\delta}) \xrightarrow{\partial} \cdots . \tag{4.4}$$

We already commented in the above remark that for any q, $H^q(\Omega_0, \delta_0)$ is identified with $\Lambda^n(M) \otimes_{A_0} VH^q$ and $H^q(\underline{\Omega}, \underline{\delta})$ is identified with $\Lambda^{n-1}(M) \otimes_{A_0} VH^q$. Similarly, it is easy to show that the connecting operator

$$\partial \colon H^{q-1}(\underline{\Omega}, \underline{\delta}) \longrightarrow H^q(\Omega_0, \delta_0)$$

[3] Similarly as above, these notations are motivated by the fact that the differentials $d_1^{\bullet,q}$ (resp. the vector spaces $E_2^{\bullet,q}$) are the ones in the first term (resp. are rows of the second term) of the (cohomological) Leray–Serre spectral sequence of the fiber bundle α [46].

is identified with the de Rham-like differential

$$d_1^{n-1,q}\colon \Lambda^{n-1}(M)\otimes_{A_0} VH^q \longrightarrow \Lambda^n(M)\otimes_{A_0} VH^q.$$

The thesis then follows from exactness of (4.4). ■

Corollary 4.5 *If $\mathcal{F}$ is connected, then $H^0(\Omega,\delta)\simeq \ker d_M^{n-1}$,*

$$d_M^{n-1}\colon \Lambda^{n-1}(M)\longrightarrow \Lambda^n(M)$$

being the last de Rham differential of M.

Proof If $\mathcal{F}$ is connected, then $VH^0\simeq A_0$ and $d_1^{n-1,0}$ is identified with d_M^{n-1}. ■

Corollary 4.6 *Let $q\geq 0$ and $\omega\in\Omega^q$ be δ-closed,* i.e., *$\delta\omega=0$. Then,*

(1) *if $q=0$, ω is locally of the form $\alpha^*(\eta)$ for some $\eta\in\Lambda^{n-1}(M)$,*
(2) *if $q>0$, then ω is locally δ-exact,* i.e., *ω is locally of the form $\delta\theta$, θ being a local element in Ω^{q-1}.*

Proof If $\mathcal{F}$ is contractible, then $VH^q=0$, and therefore $H^q(\Omega,\delta)=0$, for all $q>0$. ■

Let $\omega\in\Omega$ and $\theta\in\Omega$ be such that $\omega=\delta\theta$. Then θ will be called a *potential* of ω.

5 PD Hamiltonian Systems

5.1 PD Hamiltonian Systems and PD Hamilton Equations

In this section we introduce what we think should be understood as the partial differential, *i.e.*, field theoretic analogue of a Hamiltonian (mechanical) system on an abstract symplectic manifold.

Let $\alpha\colon P\to M$ be as in the previous section and let $\omega\in\Omega^2(P,\alpha)$ be such that $\delta\omega=0$. Put

$$\ker\omega := \{Y\in VD \mid i_Y\omega=0\},\quad \ker\underline{\omega} := \{Y\in VD \mid i_Y\underline{\omega}=0\},$$

$$\operatorname{Ker}\omega := \{\nabla\in C \mid i_\nabla\omega=0\},\quad \operatorname{Ker}\underline{\omega} := \{Z\in VD\otimes_A \overline{\Lambda}^1 \mid i_Z\underline{\omega}=0\}.$$

Since ω is closed, both $\ker\omega$ and $\ker\underline{\omega}$ are modules of smooth sections of involutive α-vertical distributions D^ω and $\underline{D}^\omega$ on P, where, for $y\in P$,

$$D^\omega_y := \{\xi\in V_yP \mid i_\xi\omega_y=0\},\quad \underline{D}^\omega_y := \{\xi\in V_yP \mid i_\xi\underline{\omega}_y=0\}.$$

Similarly, $\operatorname{Ker}\underline{\omega}$ is a sub-module in $VD\otimes_A\overline{\Lambda}^1$. As a minimal regularity requirement, assume that $\underline{D}^\omega$ has got constant rank $\underline{r}$. Then, it is easy to check that, as a consequence, $\operatorname{Ker}\underline{\omega}$ is the module of sections of a smooth vector bundle $\varpi\colon W\to P$. For $y\in P$, denote $r(y)=\dim D^\omega_y$. In general, $r(y)$ will change from point to point $y\in P$. However, we are proving in brief that $r(y)$ cannot change that much. First of all, since obviously $D^\omega\subset\underline{D}^\omega$, we have $r(y)\leq\underline{r}$ for all $y\in P$. Now, for $y\in P$, denote

$$\operatorname{Ker}\omega_y := \{c\in\alpha_{1,0}^{-1}(y) \mid i_c\omega_y=0\}.$$

Then $\operatorname{Ker}\omega_y$ is either empty or an affine space modeled over $\varpi^{-1}(y)$ (see also Theorem 4 of [20]).

Proposition 5.1 *For any $y \in P$, $\underline{r} - r(y) \leq 1$.*

Proof Let $y \in P$ and suppose $r(y) < \underline{r}$. If $\xi \in \underline{D}^{\omega}_{y}$ then (see Remark 4.1) $\underline{i_{\xi}\omega_y} = i_{\xi}\underline{\omega}_y = 0$, so that $i_{\xi}\omega_y \in \Omega^1_0|_y = \overline{\Lambda}^n|_y$. Then consider the map $\gamma_y \colon \underline{D}^{\omega}_{y} \ni \xi \longmapsto \gamma_y(\xi) := i_{\xi}\omega_y \in \overline{\Lambda}^n|_y$. Since $r(y) < \underline{r}$, γ_y is surjective and the sequence of vector spaces $0 \longrightarrow D^{\omega}_{y} \to \underline{D}^{\omega}_{y} \xrightarrow{\gamma_y} \overline{\Lambda}^n|_y \to 0$ is exact. Since $\overline{\Lambda}^n|_y$ is 1-dimensional, it follows that $\underline{r} - r(y) = 1$. ■

The following proposition characterizes the case $r(y) = \underline{r}$.

Proposition 5.2 *Let ω be as above. Then $r(y) = \underline{r}$ if and only if* $\operatorname{Ker} \omega_y \neq \varnothing$.

Proof The result is nothing more than an application of the Rouché–Capelli theorem. We here propose a dual proof. Let $\xi \in V_yP$ be given by $\xi = \xi^a \partial_a|_y$. Then $\xi \in \underline{D}^{\omega}_{y}$ if and only if

$$\omega^i_{ab}(y)\xi^a = 0, \quad a = 1, \ldots, m, \quad i = 1, \ldots, n. \tag{5.1}$$

Similarly, $\xi \in D^{\omega}_{y}$ if and only if they are satisfied both (5.1) and

$$\omega_a(y)\xi^a = 0. \tag{5.2}$$

Therefore, $D^{\omega}_{y} = \underline{D}^{\omega}_{y}$ if and only if equation (5.2) linearly depends on equations (5.1), *i.e.*, if and only if there are real numbers h^b_i, $b = 1, \ldots, m, i = 1, \ldots, n$, such that

$$\omega_a(y) = \omega^i_{ab}(y)h^b_i, \quad a = 1, \ldots, m.$$

Now, let $c \in \alpha^{-1}_{1,0}(y)$ be given by $y^a_i(c) = -\frac{1}{2}h^a_i$. Then $i_c\omega_y$ is given by

$$\begin{aligned} i_c\omega_y &= \left(-2\omega^i_{ba}(y)y^a_i(c) + \omega_a(y)\right) dy^a d^n x|_y \\ &= -\left(\omega^i_{ab}(y)h^b_i - \omega_a(y)\right) dy^a d^n x|_y \\ &= 0. \end{aligned}$$ ■

Definition 5.3 A *PD prehamiltonian system* on the fiber bundle $\alpha \colon P \to M$ is a δ-closed element $\omega \in \Omega^2(P, \alpha)$. A *PD Hamiltonian system on α* is a PD prehamiltonian system ω such that $\ker \underline{\omega} = 0$ (and, therefore, $\ker \omega = 0$ as well).

Let $\theta \in \Omega^1$ be locally given by $\theta = \theta^i_a dy^a d^{n-1}x_i - H d^n x, \ldots, \theta^i_a, \ldots, H$ being local functions on P. Then $\delta\theta$ is locally given by

$$\delta\theta = \partial_{[a}\theta^i_{b]} dy^a dy^b d^{n-1}x_i - (\partial_a H + \partial_i \theta^i_a) dy^a d^n x.$$

Similarly, let $\omega \in \Omega^2$ and $Y \in VD$ be locally given by $\omega = \omega^i_{ab} dy^a dy^b d^{n-1}x_i + \omega_a dy^a d^n x$ and $Y = Y^a \partial_a$, respectively. Then $\delta\omega$, $i_Y\omega$ and $i_Y\underline{\omega}$ are locally given by

$$\begin{aligned} \delta\omega &= \partial_{[a}\omega^i_{bc]} dy^a dy^b dy^c d^{n-1}x_i + (\partial_i \omega^i_{ab} + \partial_{[a}\omega_{b]}) dy^a dy^b d^n x, \\ i_Y\omega &= 2\omega^i_{ab} Y^a dy^b d^{n-1}x_i + \omega_a Y^a d^n x, \\ i_Y\underline{\omega} &= 2\omega^i_{ab} Y^a d^V y^b \otimes d^{n-1}x_i, \end{aligned}$$

so that ω is a PD prehamiltonian system if and only if

$$\partial_{[a}\omega^i_{bc]} = 0, \quad \partial_i\omega^i_{ab} + \partial_{[a}\omega_{b]} = 0, \tag{5.3}$$

or, which is the same (see Corollary 4.6),

$$\omega^i_{ab} = \partial_{[a}\theta^i_{b]}, \quad \omega_a = -\partial_a H - \partial_i\theta^i_a, \tag{5.4}$$

for some $\dots, \theta^i_a, \dots, H$ local functions on P. Moreover, ω is a PD Hamiltonian system if and only if

$$\omega^i_{ab}Y^a = 0 \Longrightarrow Y^a = 0. \tag{5.5}$$

In its turn (5.5) implies $\omega_a = \omega^i_{ab}f^b_i$ for some $\dots, f^b_i, \dots$ local functions on P (see the proof of Proposition 5.2).

Let ω be a PD prehamiltonian system on α, and let $\sigma\colon U \to P$ be a local section of α, $U \subset M$ being an open subset. The first jet prolongation $\dot\sigma\colon U \to J^1\alpha$ of σ may be interpreted as a "connection in α along σ", *i.e.*, a section of the restricted bundle $\alpha_{1,0}|_\sigma\colon J^1\alpha|_\sigma \to M$. Moreover, elements in $\Omega|_\sigma$ may be interpreted as affine maps from $C|_\sigma$ to $\Omega_0|_\sigma \simeq V\Lambda|_\sigma \otimes_{A_0} \Lambda^n(M)$ whose linear part is in $\underline{\Omega}|_\sigma \simeq V\Lambda|_\sigma \otimes_{A_0} \Lambda^{n-1}(M)$. Namely, an element $\Diamond \in C|_\sigma$ can "be inserted" into an element $\rho|_\sigma \in \Omega|_\sigma$, $\rho \in \Omega$, giving an element $i_\Diamond\rho|_\sigma \in \Omega_0|_\sigma$. Thus, we can search for local sections σ of α such that

$$i_{\dot\sigma}\omega|_\sigma = 0. \tag{5.6}$$

Definition 5.4 Equations (5.6) are called the *PD Hamilton equations* (of the PD prehamiltonian system ω).

If ω is locally given by $\omega = \omega^i_{ab}dy^a dy^b d^{n-1}x_i + \omega_a dy^a d^n x$, then the associated PD Hamilton equations are locally given by

$$2\omega^i_{ab}\partial_i y^a - \omega_b = 0. \tag{5.7}$$

Conversely, a system of PDEs in the form (5.7) is a PD Hamilton equation for some PD prehamiltonian (resp. PD Hamiltonian) system if and only if coefficients ω^i_{ab}, ω_b satisfy (5.3) (or, which is the same, (5.4)) (resp. (5.3) and (5.5)). Notice that, in view of (5.7), a general PD prehamiltonian system ω encodes "kinematical information", which can be identified with $\underline{\omega}$, and "dynamical information", which can be identified with the specific choice of ω in the class of those PD Hamiltonian systems with linear part $\underline{\omega}$ (see the comment at the end of Section 3.3, Remark 5.10 and Example 5.11).

Searching for solutions of PD Hamilton equations of a PD prehamiltonian system ω, we could proceed in two steps:

(i) search for a connection $\nabla \in \operatorname{Ker}\omega$,
(ii) search for n-dimensional integral submanifolds of the horizontal distribution $H_\nabla P$.

However, a solution to the first step of the above-mentioned procedure exists if and only if $\ker\omega = \ker\underline{\omega}$, which is not always the case. Therefore, in general, we are led to weaken (i) and search for connections ∇' in a subbundle $P' \subset P$ such that $i_{\nabla'}\omega|_{P'} = 0$. As shown in the next proposition, there is always an "algorithmic" way to find a maximal subbundle $\check{\alpha}\colon \check{P} \to M$ of α such that the affine equation $i_{\check{\nabla}}\omega|_{\check{P}} = 0$, $\check{\nabla} \in C(\check{P}, \check{\alpha})$ admits at least one solution. We will refer to the above-mentioned "algorithm" as the PD constraint algorithm (see also [11–13, 15, 29]).

Proposition 5.5 *Let ω be as above and* $\mathrm{Ker}\,\omega = \varnothing$ (i.e., $D^{\omega}_{y} \neq \underline{D}^{\omega}_{y}$ *for some* $y \in P$*). Under suitable regularity conditions on ω (to be specified in the proof), there exists a (maximal) subbundle $\check{P} \subset P$ such that $i_{\check{\nabla}}\omega|_{\check{P}} = 0$ for some $\check{\nabla} \in C(\check{P}, \check{\alpha})$.*

Proof For $s = 1, 2, \ldots$, define recursively

$$P_{(s)} := \{y \in P_{(s-1)} \mid \mathrm{Ker}\,\omega_y \cap (\alpha_{(s-1)})_1^{-1}(y) \neq \varnothing\} \subset P,$$
$$\alpha_{(s)} := \alpha|_{P_{(s)}} \colon P_s \longrightarrow M,$$

where $P_{(0)} := P$, $\alpha_{(0)} := \alpha$ (in particular $P_1 = \{y \in P \mid r(y) = \underline{r}\}$). We assume that $\alpha_{(s)}\colon P_{(s)} \to M$ is a smooth (closed) subbundle for all s (regularity conditions). Then, for dimensional reasons, there exists $\bar{s}$ such that $P_{(s)} = P_{(\bar{s})}$ for all $s \geq \bar{s}$. Put $\check{P} := P_{(\bar{s})}$. ■

The subbundle $\check{\alpha} := \alpha|_{\check{P}}\colon \check{P} \to M$ will be called the *constraint subbundle*. Notice that $\check{P}$ can be empty (for instance when $r(y) = \underline{r} - 1$ for all $y \in P$) and, in this case, PD Hamilton equations do not possess solutions.

Corollary 5.6 *Let ω be a PD prehamiltonian system on α and let σ be a solution of the PD Hamilton equations. Then* $\mathrm{im}\,\sigma \subset \check{P}$.

Proof By induction on s, $\mathrm{im}\,\sigma \subset P_{(s)}$ for all $s = 1, 2, \ldots$. ■

The converse of the above corollary is, *a priori*, only true for $n = 1$. Namely, we may wonder if for any $y \in \check{P}$ there is a solution σ of PD Hamilton equations such that $y \in \mathrm{im}\,\sigma$. We know that there is a connection $\check{\nabla}$ in $\check{P}$ which is "a solution of PD Hamilton equations up to first order", *i.e.*, $i_{\check{\nabla}}\check{\omega}|_{\check{P}} = 0$. n-dimensional integral manifolds of the horizontal distribution $H_{\check{\nabla}}\check{P}$ determined on $\check{P}$ by $\check{\nabla}$ are clearly images of solutions of PD Hamilton equations. If $n = 1$, $\check{\nabla}$ is trivially flat and Frobenius theorem guarantees that for any $y \in \check{P}$ there is a solution "through y". The same is *a priori* untrue for $n > 2$. Integrability conditions on $H_{\check{\nabla}}\check{P}$ will be discussed elsewhere.

5.2 PD Hamiltonian Systems and Multisymplectic Geometry à la Forger

Forger and Gomes have recently proposed a definition of multipresymplectic structure on a fiber bundle [20]. Their work aims to define such a structure so that

(1) the differential $d\Theta$ of the tautological n-form Θ on the affine adjoint bundle of the first jet bundle (see the end of Section 3.3) is multisymplectic;
(2) every multipresymplectic structure is locally isomorphic to the pull-back of Θ along a fibration (Darboux lemma).

Since, in our opinion,

- [20] is the best motivated and established work about fundamentals of multisymplectic geometry,
- abstract fiber bundles play in [20] a similar role as in this paper,

we analyze in this subsection the relationship between PD prehamiltonian systems and multipresymplectic structures *à la* Forger, referring to [20] for the main definition. Here we just mention two of the main results of [20] (which can eventually be understood as definitions of polypresymplectic structure and multipresymplectic structure on a fiber bundle, respectively).

Theorem 5.7 (Forger and Gomes I) *Let $\alpha\colon P \to M$ be a fiber bundle, x^i be local coordinates on M, $i = 1, \dots, n = \dim M$, and $\underline{\omega} \in \underline{\Omega}^2$. The form $\underline{\omega}$ is a polypresymplectic structure on α if and only if around every point of P, there are local fiber coordinates q^A, p^i_A, $z^1 \dots, z^s$, for $A = 1, \dots, m$, $i = 1, \dots, n$ (so that $\dim P = n + m + mn + s$) such that $\underline{\omega}$ is locally given by*

$$\omega = d^V p^i_A d^V q^A \otimes d^{n-1}x_i.$$

Theorem 5.8 (Forger and Gomes II) *Let $\alpha\colon P \to M$ be a fiber bundle, x^i be local coordinates on M, $i = 1, \dots, n = \dim M$, and $\omega \in \Omega^2$. The form ω is a multipresymplectic structure on α if and only if around every point of P, there are local fiber coordinates q^A, p^i_A, p, $z^1 \dots, z^r$, for $A = 1, \dots, m$, $i = 1, \dots, n$ (so that $\dim P = (n+1)(m+1)+r$) such that ω is locally given by*

$$\omega = dp^i_A dq^A d^{n-1}x_i - dp d^n x.$$

Proposition 5.9 *Let ω be a PD prehamiltonian system on α. The following two conditions are equivalent:*

(i) *$\underline{\omega}$ is a polypresymplectic structure and $r(y) = \underline{r} - 1$ for all $y \in P$;*
(ii) *ω is a multipresymplectic structure.*

Proof Recall that, in view of Proposition 4.4, ω is locally δ-exact. Suppose $\underline{\omega}$ is polypresymplectic and $r(y) = \underline{r} - 1$ for all $y \in P$. Then $\underline{r} > 0$ and, in view of Theorem 5.7, (around every point in P) there are α-adapted local coordinates

$$x^i, q^A, p^i_A, z^0, z^1, \dots, z^{\underline{r}-1},$$

such that, locally, $\underline{\omega} = d^V p^i_A d^V q^A \otimes d^{n-1}x_i$ (in particular, $\ker \underline{\omega}$ is locally spanned by the $\partial/\partial z^\alpha$). Therefore, $\underline{\omega} = \underline{\delta\theta}_0$, where $\underline{\theta}_0 := p^i_A d^V q^A \otimes d^{n-1}x_i$ is a local element of $\underline{\Omega}^1$. A general (local) potential of ω is then $\theta' \in \Omega^1$ such that $\underline{\theta}' = \underline{\theta}_0 + d^V \nu$, ν being a local element in $\underline{\Omega}^0 = \overline{\Lambda}^{n-1}$. The (local) potential $\theta := \theta' - \delta\nu$ is locally in the form $\theta = p^i_A dq^A d^{n-1}x_i - p d^n x$, where p is a local function on P. Therefore ω is locally given by

$$\omega = dp^i_A dq^A d^{n-1}x_i - dp d^n x.$$

The module $\ker \omega$ is locally spanned by those local elements $Y^\alpha \frac{\partial}{\partial z^\alpha}$ in $\ker \underline{\omega}$ such that $Y^\alpha \frac{\partial h}{\partial z^\alpha} = 0$. Since $\ker \omega \neq \ker \underline{\omega}$, then $\frac{\partial p}{\partial z^\alpha} dz^\alpha \neq 0$. Let, for instance, be $\frac{\partial p}{\partial z^0} \neq 0$.

Then x^i, q^A, p^i_A, p, $z^1, \ldots, z^{r-1}$ is a new local coordinate system on P. In view of Theorem 5.8, ω is then multipresymplectic.

On the other hand, let ω be multipresymplectic. Then $\underline{\omega}$ is polypresymplectic. Moreover, (around every point in P) there are α-adapted local coordinates

$$x^i, q^A, p^i_A, p, z^1, \ldots, z^r \tag{5.8}$$

such that, locally, $\omega = dp^i_A dq^A d^{n-1}x_i - dp d^n x$ and $\underline{\omega} = d^V p^i_A d^V q^A \otimes d^{n-1}x_i$. This shows that for all $y \in P$,

$$D^\omega_y = \left\langle \ldots, \frac{\partial}{\partial z^\alpha}\Big|_y, \ldots \right\rangle \neq \underline{D}^\omega_y = \left\langle \ldots, \frac{\partial}{\partial z^\alpha}\Big|_y, \ldots, \frac{\partial}{\partial p}\Big|_y \right\rangle. \qquad \blacksquare$$

Remark 5.10 Let ω be a PD prehamiltonian system. First of all, notice that, if ω is a multipresymplectic structure then, in view of Proposition 5.9, PD Hamilton equations of ω do not possess solutions. In this sense, multipresymplectic structures do not contain any dynamical information.

Now, the proof of Proposition 5.9 also shows that if $\underline{\omega}$ is a polypresymplectic structure and $\ker \omega = \ker \underline{\omega}$, then ω is locally in the form

$$\omega = dp^i_A dq^A d^{n-1}x_i - dH d^n x$$

where $\frac{\partial H}{\partial z^\alpha} = 0$, $\alpha = 1, 2, \ldots$, *i.e.*, H is constant along the leaves of the distribution $D^\omega = \underline{D}^\omega$.

Example 5.11 Let $\omega \in \Omega^2$ be a multisymplectic structure on α. In this case $\ker \omega = 0$, while $\underline{D}^\omega$ is a 1-dimensional (involutive) distribution. Leaves of $\underline{D}^\omega$ are submanifolds in the fibers of α. Denote by $\underline{P}$ the set of leaves of $\underline{D}^\omega$. There is an obvious projection $\underline{\alpha}\colon \underline{P} \to M$. Suppose that $\underline{\alpha}\colon \underline{P} \to M$ is a smooth fiber bundle and $\mathfrak{p}\colon P \to \underline{P}$ a smooth submersion (which is always true locally). There is a distinguished class of (local) PD Hamiltonian systems on $\underline{\alpha}$. Indeed, let $U \subset \underline{P}$ be an open subbundle and let $\mathscr{H}\colon U \to P$ be a local section of $\mathfrak{p}$. Then $\omega' := \mathscr{H}^*(\omega) \in \Omega^2(U, \underline{\alpha})$ is a PD Hamiltonian system. In particular, if we choose coordinates on P as in (5.8) (here $r = 0$), then x^i, q^A, p^i_A are coordinates on $\underline{P}$, $\mathscr{H}$ is given by

$$\mathscr{H}^*(p) = H,$$

H being a local function on $\underline{P}$, and ω' is locally given by

$$\omega' = dp^i_A dq^A d^{n-1}x_i - dH d^n x,$$

in particular $\theta' := p^i_A dq^A d^{n-1}x_i - H d^n x$ is a local potential of ω'. Finally, PD Hamilton equations of ω' read

$$q^A_i = \frac{\partial H}{\partial p^i_A},$$

$$p^i_{A,i} = -\frac{\partial H}{\partial q^A},$$

which are de Donder–Weyl equations (see, for instance, [26]).

5.3 PD Hamiltonian Systems and Variational Calculus

We show in this subsection that PD Hamilton equations are locally variational. First of all, an element $\theta \in \Omega^1$ may be understood as a (fiber-wise affine) horizontal n-form over $J^1\alpha$, *i.e.*, as an element $\mathscr{L}^\theta \in \overline{\Lambda}^n(J^1\alpha, \alpha_1)$ via

$$\mathscr{L}^\theta_c := i_c\theta_y, \quad c \in J^1\alpha, \quad y = \alpha_{1,0}(c) \in P.$$

In its turn $\mathscr{L}^\theta$ is a 1-st order Lagrangian density in the fiber bundle α determining an action functional which we denote by $S^\theta = \int \mathscr{L}^\theta$. If θ is locally given by $\theta = \theta^i_a dy^a d^{n-1}x_i - H d^n x$, with θ^i_a, H being local functions on P, then $\mathscr{L}^\theta$ is locally given by $\mathscr{L}^\theta = L^\theta d^n x$, where L^θ is the local function on $J^1\alpha$ given by

$$L^\theta = (\theta^i_b y^b_i - H).$$

In particular, if $\theta = \delta\nu$ for some $\nu \in \Omega^0 = \overline{\Lambda}^{n-1}$ locally given by $\nu = \nu^i d^{n-1}x_i$, then

$$L^\theta = (\partial_i + y^a_i \partial_a)\nu^i, \tag{5.9}$$

i.e., L^θ is a total divergence.

Proposition 5.12 *Let $\omega \in \Omega^2$ be a δ-exact PD prehamiltonian system. Then the PD Hamilton equations of ω coincide with Euler–Lagrange equations associated with the action $S^\theta := \int \mathscr{L}^\theta$, where $\theta \in \Omega^1$ is the opposite of any potential of ω,* i.e., *$-\delta\theta = \omega$. Moreover, if $H^1(\Omega, \delta) = 0$ then S^θ is independent of the choice of θ and does only depend on ω.*

Proof The first part of the proposition can be proved in local coordinates. Indeed, we compute variational derivatives of L^θ,

$$\begin{aligned}\frac{\delta}{\delta y^b} L^\theta &:= \partial_b L^\theta - (\partial_i + y^a_i \partial_a)\frac{\partial}{\partial y^b_i} L^\theta \\ &= y^a_i(\partial_b \theta^i_a - \partial_a \theta^i_b) - \partial_a H - \partial_i \theta^i_a \\ &= -2\omega^i_{ab} y^a_i + \omega_b,\end{aligned}$$

where we used (5.4). To prove the second part of the proposition, use (5.9) to conclude that, for $\nu \in \Omega^0$, $\delta L^{\delta\nu}/\delta y^a = 0$. ∎

Remark 5.13 Condition $H^1(\Omega, \delta) = 0$ depends on the topology of the fiber bundle α. It is satisfied, for instance, if $H^n(M) = 0$ and $H^1(\mathcal{F}) = 0$, $\mathcal{F}$ being, as above, the abstract fiber of α. Indeed, if $H^1(\mathcal{F}) = 0$ then $H^1(\underline{\Omega}, \underline{\delta}) = 0$ so that, the first part of the exact sequence (4.4) reads

$$0 \longrightarrow H^0(\Omega, \delta) \longrightarrow \Lambda^{n-1}(M) \xrightarrow{d_M^{n-1}} \Lambda^n(M) \longrightarrow H^1(\Omega, \delta) \longrightarrow 0$$

and $H^1(\Omega, \delta) \simeq H^n(M) = 0$.

6 PD Noether Symmetries and Currents

6.1 PD Noether Theorem and PD Poisson Bracket

The multisymplectic analogues of Hamiltonian vector fields and Poisson bracket in symplectic geometry have been extensively investigated [21, 22, 24, 25, 36, 38]. We here propose the natural definitions for general PD Hamiltonian systems. Notice that, even if they look formally identical to (or possibly less general than) the ones proposed in [21, 22, 24, 38], our definitions actually have a dynamical content, not only a kinematical one (see Remark 5.10), so that, for instance, we can prove a PD version of the (Hamiltonian) Noether theorem. That is why, *e.g.*, we prefer to speak about PD Noether symmetries rather than Hamiltonian (multi)vector fields [23].

Let ω be a PD prehamiltonian system on the bundle $\alpha\colon P \to M$. In the following we assume α to have connected fiber.

Definition 6.1 Let $Y \in VD$ and $f \in \Omega^0$. If $i_Y\omega = \delta f$, then Y and f are said to be a *PD Noether symmetry* and a *PD Noether current* of ω (relative to each other), respectively.

Denote by $\mathscr{S}(\omega)$ and $\mathscr{C}(\omega)$ the sets of PD Noether symmetries and PD Noether currents of ω, respectively. A PD Noether symmetry Y (relative to a PD Noether current f) is a symmetry of ω in the sense that

$$L_Y\omega = i_Y\delta\omega + \delta i_Y\omega = \delta\delta f = 0.$$

The next proposition clarifies in what sense a PD Noether current is a conserved current for ω.

Proposition 6.2 (PD Noether theorem) *Let $Y \in \mathscr{S}(\omega)$ and $f \in \mathscr{C}(\omega)$ be a PD Noether symmetry and a PD Noether current of ω relative to each other. Then $\sigma^*(f) \in \Lambda^{n-1}(M)$ is a closed form for every solution σ of PD Hamilton equations.*

Proof First of all, let $\varrho \in \Omega^1$ and let τ be a (local) section of α. It is easy to show (for instance, using local coordinates) that $\tau^*(\varrho) = i_{\dot\tau}\varrho|_\tau \in \Lambda^n(M)$. Then

$$\begin{aligned} d\sigma^*(f) &= \sigma^*(df) = \sigma^*(\delta f) = i_{\dot\sigma}\delta f|_\sigma \\ &= i_{\dot\sigma} i_Y\omega|_\sigma = i_{Y|_\sigma} i_{\dot\sigma}\omega|_\sigma = 0. \end{aligned}$$ ∎

We are now in the position to introduce a Lie bracket among PD Noether currents.

Proposition 6.3 *Let $Y_1, Y_2 \in \mathscr{S}(\omega)$ be PD Noether symmetries relative to the PD Noether currents $f_1, f_2 \in \mathscr{C}(\omega)$, respectively. Then $[Y_1, Y_2] \in \mathscr{S}(\omega)$ and $f := L_{Y_1} f_2 \in \mathscr{C}(\omega)$, and they are relative to each other. Moreover, f is independent of the choice of Y_1 among the PD Noether symmetries relative to the PD Noether current f_1.*

Proof Compute

$$\begin{aligned} \delta L_{Y_1} f_2 &= L_{Y_1}\delta f_2 = L_{Y_1} i_{Y_2}\omega \\ &= i_{[Y_1,Y_2]}\omega + i_{Y_2} L_{Y_1}\omega = i_{[Y_1,Y_2]}\omega. \end{aligned}$$

Now, let $V \in \ker \omega$. Then $L_V f_2 = i_V \delta f_2 = i_V i_{Y_2} \omega = 0$. This proves the second part of the proposition. ■

Let Y_1, Y_2, f_1, f_2 be as in the above proposition.

Proposition 6.4 *The $\mathbb{R}$-bilinear map*

$$\mathscr{C}(\omega) \times \mathscr{C}(\omega) \ni (f_1, f_2) \longmapsto \{f_1, f_2\} := L_{Y_1} f_2 \in H(\omega),$$

Y_1 being a PD Noether symmetry relative to f_1, is a Lie bracket.

Proof Let $Y_2 \in \mathscr{S}(\omega)$ be a PD Noether symmetry relative to $f_2 \in \mathscr{C}(\omega)$. Skew-symmetry of $\{\cdot, \cdot\}$ immediately follows from

$$\{f_1, f_2\} = L_{Y_1} f_2 = i_{Y_1} \delta f_2 + \delta i_{Y_1} f_2 = i_{Y_1} i_{Y_2} \omega.$$

Now we check the Leibniz rule. Let $Y_3 \in \mathscr{S}(\omega)$ and $f_3 \in \mathscr{C}(\omega)$ be another pair of PD Noether symmetry, PD Noether current relative to each other. Then

$$\begin{aligned}\left\{ f_1, \{f_2, f_3\} \right\} &= L_{Y_1} \{f_2, f_3\} = L_{Y_1} L_{Y_2} f_3 \\ &= L_{[Y_1, Y_2]} f_3 + L_{Y_2} L_{Y_1} f_3 = \left\{ \{f_1, f_2\}, f_3 \right\} + \left\{ f_2, \{f_1, f_3\} \right\}.\end{aligned}$$ ■

PD Noether symmetries and PD Noether currents of a PD Hamiltonian system constitute very small Lie subalgebras of the Lie algebras of higher symmetries and conservation laws of PD Hamilton equations, for which fully satisfactory definitions have been given and many infinite-jet based computational techniques have been developed [4]. Nevertheless, it is worthwhile to give Definition 6.1 and to carefully analyze it, independently of infinite jets, in view of the possibility of developing a "(multi)symplectic theory" of higher symmetries and conservation laws (see, for instance, [56]). In Section 7 we propose some specific examples.

Finally, notice that, in general, neither is a PD Noether current uniquely determined by the relative PD Noether symmetry nor vice versa (unless $\ker \omega = 0$). However, "non-trivial PD Noether symmetries" are in one-to-one correspondence with "non-trivial PD Noether currents" in the following sense. Clearly, $\ker \omega \subset \mathscr{S}(\omega)$ and $H^0(\Omega, \delta) \subset \mathscr{C}(\omega)$. We will call elements in $\ker \omega$ *gauge PD Noether symmetries* (see below) and elements in $H^0(\Omega, \delta)$ (*i.e.*, closed $(n-1)$-forms on M, see Corollary 4.5) *trivial PD Noether currents*.

Remark 6.5 It is easy to see that $\ker \omega$ and $H^0(\Omega, \delta)$ are ideals in the Lie algebras $\mathscr{S}(\omega)$ and $\mathscr{C}(\omega)$, respectively. Let $\overline{\mathscr{S}}(\omega) := \mathscr{S}(\omega)/\ker \omega$ and let $\overline{\mathscr{C}}(\omega) := \mathscr{C}(\omega)/H^0(\Omega, \delta)$ be the quotient Lie algebras. Then the map

$$\overline{\mathscr{S}}(\omega) \ni Y + \ker \omega \longmapsto f + H^0(\Omega, \delta) \in \overline{\mathscr{C}}(\omega),$$

where $Y \in \mathscr{S}(\omega)$ and $f \in \mathscr{C}(\omega)$ are relative to each other, is a well-defined isomorphism of Lie algebras. It is natural to call elements in $\overline{\mathscr{S}}(\omega)$ and $\overline{\mathscr{C}}(\omega)$ *non-trivial PD Noether symmetries* and *non-trivial PD Noether currents*, respectively. Indeed, elements in $\ker \omega$ are trivial symmetries in that they are infinitesimal gauge transformations (see next subsection), and elements in $H^0(\Omega, \delta)$ are trivial conserved currents in that they are conserved currents for every PD prehamiltonian system ω, independently of ω.

6.2 Gauge Reduction of PD Hamiltonian Systems

From a physical point of view, elements in $\ker \omega$ are infinitesimal gauge transformations and therefore should be factored out via a reduction of the system. In this section we assume $\ker \omega = \ker \underline{\omega}$ or, what is the same, $\operatorname{Ker} \omega \neq \varnothing$. As a further regularity condition we assume that the leaves of $D^\omega = \underline{D}^\omega$ form a smooth fiber bundle $\widetilde{P}$ over M, whose projection we denote by $\widetilde{\alpha} \colon \widetilde{P} \to M$, in such a way that the canonical projection $\mathfrak{p} \colon P \to \widetilde{P}$ is a smooth bundle. The last condition is always fulfilled at least locally. Notice, also, that, by construction, $\mathfrak{p}$ has connected fibers.

Theorem 6.6 *There exists a unique PD Hamiltonian system $\widetilde{\omega}$ in $\widetilde{\alpha}$ such that*

(i) $\omega = \mathfrak{p}^*(\widetilde{\omega})$,
(ii) $\ker \widetilde{\omega} = \ker \underline{\widetilde{\omega}} = 0$, *and*
(iii) *a local section σ of α is a solution of the PD Hamilton equation of ω if and only if $\mathfrak{p} \circ \sigma$ (which is a local section of $\widetilde{\alpha}$) is a solution of PD Hamilton equations of $\widetilde{\omega}$.*

Proof Let $\widetilde{\nabla} \in C(\widetilde{P}, \widetilde{\alpha})$. There exists a (non-unique) connection $\nabla \in C(P, \alpha)$ such that ∇ and $\widetilde{\nabla}$ are $\mathfrak{p}$-compatible. To prove this, choose a connection $\square$ in $\mathfrak{p}$ and lift the planes of $\widetilde{\nabla}$ to P by means of $\square$. It is easy to show that the so-obtained distribution on P defines a connection ∇ in α with the required property. Similarly, every vector field $\widetilde{X} \in V\mathrm{D}(\widetilde{P}, \widetilde{\alpha})$ can be lifted to a (non-unique) $\mathfrak{p}$-projectable vector field $X \in V\mathrm{D}(P, \alpha)$ such that $\widetilde{X}$ is its projection. Then $X \in \mathrm{D}_V(P, \mathfrak{p})$. Consider $\eta := \omega(\nabla)(X) \in \Omega^0(P, \alpha)$. We will prove that $L_Y \eta = 0$ for any $Y \in V\mathrm{D}(P, \mathfrak{p})$. Indeed, let $Y \in V\mathrm{D}(P, \mathfrak{p})$. Then $[Y, X] \in V\mathrm{D}(P, \mathfrak{p})$. Similarly

$$[\![Y, H_\nabla]\!] \in \overline{\Lambda}^1(P, \alpha) \otimes V\mathrm{D}(P, \mathfrak{p}) \subset \overline{\Lambda}^1(P, \alpha) \otimes V\mathrm{D}(P, \alpha).$$

Now, $V\mathrm{D}(P, \mathfrak{p}) = \ker \omega$ by construction, and therefore

$$\begin{aligned} L_Y \eta = L_Y i_X i_\nabla \omega &= [L_Y, i_X] i_\nabla \omega + i_X L_Y i_\nabla \omega \\ &= i_{[Y,X]} i_\nabla \omega + i_X i_\nabla L_Y \omega + i_X [L_Y, i_\nabla] \omega \\ &= i_\nabla i_{[Y,X]} \omega + i_X i_{[\![Y, H_\nabla]\!]} \omega = 0. \end{aligned}$$

Since fibers of $\mathfrak{p}$ are connected we conclude that $\eta = \mathfrak{p}^*(\widetilde{\eta})$ for a unique $\widetilde{\eta} \in \Omega^0(\widetilde{P}, \widetilde{\alpha})$. Put

$$\widetilde{\omega}(\widetilde{\nabla})(\widetilde{X}) := \widetilde{\eta},$$

so that $\widetilde{\omega}$ is a well-defined element in $\Omega^2(\widetilde{P}, \widetilde{\alpha})$. Indeed, let $\nabla' \in C(P, \alpha)$ be also $\mathfrak{p}$-compatible with $\widetilde{\nabla}$ and let $X' \in V\mathrm{D}(P, \alpha)$ be another $\mathfrak{p}$-projectable vector field projecting onto $\widetilde{X}$. Then $\nabla' - \nabla \in \overline{\Lambda}^1(P, \alpha) \otimes V\mathrm{D}(P, \mathfrak{p})$ and $X' - X \in V\mathrm{D}(P, \mathfrak{p})$. Therefore,

$$\begin{aligned} \omega(\nabla')(X') = i_{X'} i_{\nabla'} \omega &= i_{X'} i_\nabla \omega + i_{X'} i_{\nabla' - \nabla} \underline{\omega} \\ &= i_X i_\nabla \omega + i_{X' - X} i_\nabla \omega = i_X i_\nabla \omega + i_\nabla i_{X' - X} \omega = \omega(\nabla)(X). \end{aligned}$$

Moreover, $\omega = \mathfrak{p}^*(\widetilde{\omega})$ by construction.

Let us compute $\ker \underline{\widetilde{\omega}}$. Thus, let $\widetilde{X} \in VD(\widetilde{P}, \widetilde{\alpha})$ be such that $i_{\widetilde{X}} \underline{\widetilde{\omega}} = 0$ and let $X \in VD(P, \alpha)$ be as above. Then $i_X \underline{\omega} = \mathfrak{p}^*(i_{\widetilde{X}} \underline{\widetilde{\omega}}) = 0$. This shows that $X \in VD(P, \mathfrak{p})$ and then $\widetilde{X} = 0$.

Finally, let σ be a local section of α, $\widetilde{\sigma} := \mathfrak{p} \circ \sigma$, $\widetilde{X} \in VD(\widetilde{P}, \widetilde{\alpha})$ and let X be as above. Compute

$$
\begin{aligned}
(i_{\dot{\widetilde{\sigma}}} \widetilde{\omega}|_{\widetilde{\sigma}})(\widetilde{X}|_{\widetilde{\sigma}}) &= i_{\dot{\widetilde{\sigma}}} (i_{\widetilde{X}} \widetilde{\omega})|_{\widetilde{\sigma}} = \widetilde{\sigma}^*(i_{\widetilde{X}} \widetilde{\omega}) = (\sigma^* \circ \mathfrak{p}^*)(i_{\widetilde{X}} \widetilde{\omega}) \\
&= \sigma^*(i_X \omega) = i_{\dot{\sigma}}(i_X \omega)|_\sigma = (i_{\dot{\sigma}} \omega|_\sigma)(X|_\sigma).
\end{aligned}
$$

This shows that $i_{\dot{\sigma}} \omega|_\sigma = 0$ if and only if $i_{\dot{\widetilde{\sigma}}} \widetilde{\omega}|_{\widetilde{\sigma}} = 0$. ■

Proposition 6.7 *There are natural isomorphisms of Lie algebras*

$$
\overline{\mathscr{S}}(\omega) \simeq \mathscr{S}(\widetilde{\omega}), \quad \mathscr{C}(\omega) \simeq \mathscr{C}(\widetilde{\omega}).
$$

Proof First of all let $f \in \mathscr{C}(\omega)$ and $X \in \mathscr{S}(\omega)$ be relative to each other. Then $f = \mathfrak{p}^*(\widetilde{f})$ for some $\widetilde{f} \in \Omega^0(\widetilde{P}, \widetilde{\alpha})$ and X is $\mathfrak{p}$-projectable. Indeed, for all $Y \in \ker \omega$,

$$
L_Y f = i_Y \delta f + \delta i_Y f = i_Y i_X \omega = i_{[Y,X]} \omega = 0.
$$

Moreover,

$$
\mathfrak{p}^*(\delta \widetilde{f}) = \delta \mathfrak{p}^*(\widetilde{f}) = \delta f = i_X \omega = \mathfrak{p}^*(i_{\widetilde{X}} \widetilde{\omega}),
$$

where $\widetilde{X}$ denotes the $\mathfrak{p}$-projection of X, and, therefore, $\delta \widetilde{f} = i_{\widetilde{X}} \widetilde{\omega}$, *i.e.*, $\widetilde{f} \in \mathscr{C}(\widetilde{\omega})$ and $\widetilde{X} \in \mathscr{S}(\widetilde{\omega})$ is a PD Noether symmetry relative to it. Thus, maps

$$
\overline{\mathscr{S}}(\omega) \ni X + \ker \omega \longmapsto \widetilde{X} \in \mathscr{S}(\widetilde{\omega}), \tag{6.1}
$$

$$
\mathscr{C}(\omega) \ni f \longmapsto \widetilde{f} \in \mathscr{C}(\widetilde{\omega}) \tag{6.2}
$$

are well defined. Conversely, let $\widetilde{X}_1 \in \mathscr{S}(\widetilde{\omega})$, $\widetilde{f}_1 \in \mathscr{C}(\widetilde{\omega})$ be relative to each other, $X_1 \in VD(P, \alpha)$ be any $\mathfrak{p}$-projectable vector field, $\widetilde{X}_1 \in VD(\widetilde{P}, \widetilde{\alpha})$ be its projection, and let $f_1 := \mathfrak{p}^*(\widetilde{f}_1) \in \Omega^0(P, \alpha)$. Then $X_1 \in \mathscr{S}(\omega)$ and $f_1 \in \mathscr{C}(\omega)$ is a PD Noether current relative to it. Indeed,

$$
i_{X_1} \omega = \mathfrak{p}^*(i_{\widetilde{X}_1} \widetilde{\omega}) = \mathfrak{p}^*(\delta \widetilde{f}_1) = \delta \mathfrak{p}^*(\widetilde{f}_1) = \delta f_1.
$$

We conclude that (6.1) and (6.2) are inverted by

$$
\begin{aligned}
\mathscr{S}(\widetilde{\omega}) \ni \widetilde{X}_1 &\longmapsto X_1 + \ker \omega \in \overline{\mathscr{S}}(\omega), \\
\mathscr{C}(\widetilde{\omega}) \ni \widetilde{f}_1 &\longmapsto f_1 \in \mathscr{C}(\omega),
\end{aligned}
$$

respectively. ■

7 Examples

7.1 Non-Degenerate Examples

Let $\alpha\colon \mathbb{R}^{2n+1} \ni (x^1, \dots, x^n, u, u_1, \dots, u_n) \longmapsto (x^1, \dots, x^n) \in \mathbb{R}^n$, $n > 1$. Consider $T, V \in C^\infty(\mathbb{R}^{2n+1})$ of the form $T = T(u_1, \dots, u_n)$ and $V = V(u)$, respectively. The form

$$\omega := \frac{\partial^2 T}{\partial u_i \partial u_j} du_i (dud^{n-1}x_j - u_j d^n x) - V' dud^n x,$$

is a PD prehamiltonian system on α (here and in what follows a prime " $'$ " denotes differentiation with respect to u). The associated PD Hamilton equations read

$$\frac{\partial^2 T}{\partial u_i \partial u_j}\partial_j u_i + V' = 0, \qquad \partial_i u = u_i,$$

and are in turn equivalent to

$$\frac{\partial^2 T}{\partial u_i \partial u_j}\partial^2_{ij} u + V' = 0, \qquad \partial_i u = u_i, \tag{7.1}$$

$\partial^2_{ij} := \partial_i \partial_j$, $i, j = 1, \dots, n$. Moreover, if

$$\det\Big(\frac{\partial^2 T}{\partial u_i \partial u_j}\Big) \neq 0,$$

then ω is a PD Hamiltonian system. We will only consider this case in the following. Thus, put $T^{ij} := \frac{\partial^2 T}{\partial u_i \partial u_j}$, $i, j = 1, \dots, n$, and let (T_{ij}) be the inverse matrix of (T^{ij}). As examples, we note the following:

(1) For $T = \frac{1}{2} g^{ij} u_i u_j$,

$$(g^{ij}) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

(resp., $g^{ij} = \delta^{ij}$, $i, j = 1, \dots, n$), (7.1) reduces to the wave equation (resp., the Poisson equation) with a u-dependent potential V (including the f-Gordon equation as a particular example, if $n = 2$ and $f = -V'$).

(2) For $n = 2$, $T = \sqrt{1 + g^{ij} u_i u_j}$, $g^{ij} = \delta^{ij}$, $i, j = 1, 2$, and $V = 0$, (7.1) reduces to the equation for minimal surfaces in $\mathbb{R}^3$ transversal to the projection $\mathbb{R}^3 \ni (x_1, x_2, u) \longmapsto (x_1, x_2) \in \mathbb{R}^2$.

Let us search for PD Noether symmetries and currents of ω. Let $Y = U\frac{\partial}{\partial u} + U_i \frac{\partial}{\partial u^i} \in \mathrm{V}D$ and let $f = f^i d^{n-1} x_i \in \Omega^0$. Then

$$i_Y \omega = T^{ij}(U_i du - U du_i) d^{n-1} x_j - (T^{ij} u_i U_j + V' U) d^n x,$$

$$\delta f = \partial_i f^i d^n x + \frac{\partial}{\partial u} f^i dud^{n-1} x_i + \frac{\partial}{\partial u_k} f^i du_k d^{n-1} x_i.$$

Recall that Y and f are a PD Noether symmetry and a PD Noether current relative to each other, respectively, if and only if $i_Y\omega = \delta f$, *i.e.*,

$$\partial_i f^i + T^{ij}u_i U_j + V'U = 0, \tag{7.2}$$

$$\frac{\partial}{\partial u} f^i - T^{ij}U_j = 0, \tag{7.3}$$

$$\frac{\partial}{\partial u_j} f^i + T^{ij}U = 0. \tag{7.4}$$

It follows from (7.4) that $\frac{\partial}{\partial u_j} f^i = \frac{\partial}{\partial u_i} f^j$, $i, j = 1, \ldots, n$, and then

$$\frac{\partial^2}{\partial u_k \partial u_j} f^i = \frac{\partial^2}{\partial u_k \partial u_i} f^j, \quad i, j, k = 1, \ldots, n.$$

Now,

$$\frac{\partial^2}{\partial u_k \partial u_j} f^i = \frac{\partial}{\partial u_k}\frac{\partial}{\partial u_j} f^i = -\frac{\partial}{\partial u_k}(T^{ij}U)$$
$$= -\frac{\partial^3 T}{\partial u_k \partial u_i \partial u_j} U - T^{ij}\frac{\partial}{\partial u_k} U.$$

Similarly,

$$\frac{\partial^2}{\partial u_k \partial u_i} f^j = \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_k} f^j = -\frac{\partial^3 T}{\partial u_i \partial u_j \partial u_k} U - T^{jk}\frac{\partial}{\partial u_i} U.$$

Therefore,

$$T^{ij}\frac{\partial}{\partial u_k} U - T^{jk}\frac{\partial}{\partial u_i} U = 0$$

Contracting with T_{ij} we find $(n-1)\frac{\partial}{\partial u_i} U = 0$ and, therefore,

$$U = U(x^1, \ldots, x^n, u),$$

so that (7.4) can be rewritten as

$$\frac{\partial}{\partial u_j}\left(f^i + \frac{\partial T}{\partial u_i} U \right) = 0.$$

We conclude that

$$f^i = -\frac{\partial T}{\partial u_i} U + A^i \tag{7.5}$$

for some $A^i = A^i(x^1, \ldots, x^n, u)$, $i = 1, \ldots, n$. Notice that (7.3) can be used to determine the U_j's from the f^i's via

$$U_j = T_{ji}\frac{\partial}{\partial u} f^i.$$

It remains to solve (7.2) which, in view of (7.5), reduces to

$$\left(\partial_i + u_i \frac{\partial}{\partial u}\right) A^i - \frac{\partial T}{\partial u_i}\left(\partial_i + u_i \frac{\partial}{\partial u}\right) U + V' U = 0. \tag{7.6}$$

We cannot continue solving (7.6) without further specifying T. In the following we will only consider two special cases.

Case 1 $T = \frac{1}{2} g^{ij} u_i u_j$, (g^{ij}) is a constant, nondegenerate, symmetric matrix with inverse (g_{ij}): In this case (7.6) reads

$$\partial_i A^i + V' U + \left(\frac{\partial}{\partial u} A^i - g^{ij}\partial_j U\right) u_i - \left(g^{ij} \frac{\partial}{\partial u} U\right) u_i u_j = 0. \tag{7.7}$$

The left-hand side of (7.7) is polynomial in $u_1, \ldots, u_n$. Thus, all the corresponding coefficients must vanish, *i.e.*,

$$\frac{\partial}{\partial u} U = 0, \tag{7.8}$$

$$\frac{\partial}{\partial u} A^i - g^{ij}\partial_j U = 0, \tag{7.9}$$

$$\partial_i A^i + V' U = 0. \tag{7.10}$$

From (7.8), $U = U(x^1, \ldots, x^n)$ and then from (7.9), $\frac{\partial^2}{\partial u^2} A^i = 0$, $i = 1, \ldots, n$, which in turn implies, using (7.9) again,

$$A^i = (g^{ij}\partial_j U) u + B^i$$

for some $B^i = B^i(x^1, \ldots, x^n)$. Finally, (7.10) implies

$$(g^{ij}\partial^2_{ij} U) u + \partial_i B^i + V' U = 0,$$

and differentiating once more with respect to u,

$$g^{ij}\partial^2_{ij} U + V'' U = 0.$$

Since U doesn't depend on u, we have the following cases:

(a) If $V''' \neq 0$, then $U = 0$ so that

$$f^i = \frac{1}{2}\partial_j B^{ji}, \quad U_j = 0$$

for some $B^{ij} = -B^{ji} = B^{ij}(x^1, \ldots, x^n)$, *i.e.*,

$$Y = 0 \quad \text{and} \quad f = d\beta,$$

$\beta = B^{ji} d^{n-2} x_{ji}$, where $d^{n-2} x_{ji} := i_{\partial_j} d^{n-1} x_i$, $i, j = 1, \ldots, n$. Therefore, ω does not possess PD Noether symmetries nor nontrivial PD Noether currents.

(b) If $V''' = 0$, then $V = \frac{1}{2}\mu u^2$ for some constant μ and

$$g^{ij}\partial^2_{ij}U + \mu U = 0, \quad f^i = g^{ij}(u\partial_j U - u_j U) + \frac{1}{2}\partial_j B^{ji}, \quad U_j = \partial_j U,$$

for some $B^{ij} = -B^{ji} = B^{ij}(x^1, \dots, x^n)$. Thus,

$$Y = U\frac{\partial}{\partial u} + \partial_j U \frac{\partial}{\partial u_j} \quad \text{and} \quad f = g^{ij}(u\partial_j U - u_j U)d^{n-1}x_i + d\beta,$$

$\beta = B^{ji}d^{n-2}x_{ji}$, where U is any solution of the PD Hamilton equation

$$g^{ij}\partial^2_{ij}u + \mu u = 0. \tag{7.11}$$

Let us compute the PD Poisson bracket. Consider two solutions of (7.11), say U_1, U_2, the corresponding PD Noether symmetries Y_1, Y_2 and associated PD Noether currents f_1, f_2. Then

$$\{f_1, f_2\} = L_{Y_1} f_2 = g^{ij}(U_1\partial_j U_2 - U_2 \partial_j U_1)d^{n-1}x_i,$$

which, as can be easily checked, is a trivial conservation law.

Case 2 $n = 2$, $T = \sqrt{1 + \delta^{ij}u_i u_j}$ and $V = 0$: In this case, (7.6) reads

$$\tau^{1/2}\left(\partial_i + u_i\frac{\partial}{\partial u}\right)A^i = \delta^{ij}u_j\left(\partial_i + u_i\frac{\partial}{\partial u}\right)U, \tag{7.12}$$

where $\tau = 1 + \delta^{ij}u_i u_j$. Squaring both sides of (7.12) we get

$$\tau\left[\left(\partial_i + u_i\frac{\partial}{\partial u}\right)A^i\right]^2 - \left[\delta^{ij}u_j\left(\partial_i + u_i\frac{\partial}{\partial u}\right)U\right]^2 = 0,$$

whose left-hand side is polynomial in u_1, u_2. Collecting homogeneous terms we get

$$\begin{aligned}
&\left[\left(\frac{\partial}{\partial u}U\right)^2\delta^{ij} - \left(\frac{\partial}{\partial u}A^i\right)\left(\frac{\partial}{\partial u}A^j\right)\right]\delta^{kl}u_i u_j u_k u_l \\
&\quad + 2\delta^{ij}\left[\delta^{kl}\left(\frac{\partial}{\partial u}U\right)(\partial_l U) - \left(\frac{\partial}{\partial u}A^k\right)(\partial_l A^l)\right]u_i u_j u_k \\
&\quad - \left[\delta^{ij}(\partial_k A^k)^2 + \left(\frac{\partial}{\partial u}A^i\right)\left(\frac{\partial}{\partial u}A^j\right) - \delta^{ik}\delta^{jl}(\partial_k U)(\partial_l U)\right]u_i u_j \\
&\quad + 2(\partial_j A^j)^2\left(\frac{\partial}{\partial u}A^i\right)u_i + (\partial_i A^i)^2 = 0.
\end{aligned} \tag{7.13}$$

All coefficients on the left-hand side of (7.13) must vanish. It follows that

$$\frac{\partial}{\partial u}U = \partial_1 U = \partial_2 U = 0, \quad \frac{\partial}{\partial u}A^1 = \frac{\partial}{\partial u}A^2 = 0, \quad \partial_1 A^1 + \partial_2 A^2 = 0,$$

i.e., U is a constant while $A^1 = \partial_2 B$, $A^2 = -\partial_1 B$ for some $B = B(x^1, x^2)$. Thus,

$$Y = U\frac{\partial}{\partial u}, \quad f = U\tau^{-1/2}(u_2 dx^1 - u_1 dx^2) + dB.$$

It is obvious that the PD Poisson bracket is also trivial in this case.

7.2 A Degenerate, Constrained Example

The example in this subsection is taken from [33]. Let

$$\alpha\colon \mathbb{R}^{3m+2}\times\mathbb{R}_+ \ni (q^1,\dots,q^m,s_1,\dots,s_m,t_1\dots,t_m,s,t;e)\longmapsto (s,t)\in\mathbb{R}^2.$$

The form

$$\omega := -dt_\alpha dq^\alpha ds + ds_\alpha dq^\alpha dt - \delta^{\alpha\beta}(et_\alpha dt_\beta - s_\alpha ds_\beta)dsdt - \varepsilon dedsdt,$$

where $\varepsilon := \frac{1}{2}(\delta^{\alpha\beta}t_\alpha t_\beta - 1)$, is a PD prehamiltonian system on α. The associated PD Hamilton equations read

$$\begin{gathered}\frac{\partial}{\partial t}t_\alpha + \frac{\partial}{\partial s}s_\alpha = 0,\\ \frac{\partial}{\partial t}q^\alpha = e\delta^{\alpha\beta}t_\beta,\quad \frac{\partial}{\partial s}q^\alpha = -\delta^{\alpha\beta}s_\beta,\\ \varepsilon = 0,\end{gathered}$$

$\alpha = 1,\dots,m$, which are in turn equivalent to

$$\begin{gathered}e^{-1}\frac{\partial^2}{\partial t^2}q^\alpha - \frac{\partial^2}{\partial s^2}q^\alpha = e^{-2}\Big(\frac{\partial}{\partial t}q^\alpha\Big)\Big(\frac{\partial}{\partial t}e\Big),\\ e^2 = \delta_{\alpha\beta}\Big(\frac{\partial}{\partial t}q^\alpha\Big)\Big(\frac{\partial}{\partial t}q^\beta\Big),\\ t_\alpha = e^{-1}\delta_{\alpha\beta}\frac{\partial}{\partial t}q^\beta,\quad s_\alpha = \delta_{\alpha\beta}\frac{\partial}{\partial s}q^\beta.\end{gathered}$$

Notice that $\underline{D}^\omega$ is generated by $\frac{\partial}{\partial e}$, while

$$D^\omega_y = \begin{cases}\mathbf{0} & \text{for } \varepsilon(y)\neq 0,\\ \big\langle \frac{\partial}{\partial e}\big|_y\big\rangle & \text{for } \varepsilon(y)=0,\end{cases}\qquad y\in P$$

we conclude that $P_{(1)}$ is the hypersurface defined by $\delta^{\alpha\beta}t_\alpha t_\beta = 1$. It is easy to see that, actually, $\check{P} = P_{(1)}$.

Let us search for PD Noether symmetries and currents of ω. Let $Y = Q^\alpha\frac{\partial}{\partial q^\alpha} + S_\alpha\frac{\partial}{\partial s_\alpha} + T_\alpha\frac{\partial}{\partial t_\alpha} + E\frac{\partial}{\partial e}\in VD$ and let $f = \alpha ds + \beta dt \in \Omega^0$. Then $i_Y\omega = \delta f$ if and only if

$$\begin{gathered}\frac{\partial}{\partial s}\beta - \frac{\partial}{\partial t}\alpha = \delta^{\beta\gamma}(s_\beta S_\gamma - et_\beta T_\gamma) - \varepsilon E,\\ \frac{\partial}{\partial q^\alpha}\alpha = -T_\alpha,\quad \frac{\partial}{\partial q^\alpha}\beta = S_\alpha,\quad \frac{\partial}{\partial t_\alpha}\alpha = \frac{\partial}{\partial s_\alpha}\beta = Q^\alpha\\ \frac{\partial}{\partial s_\alpha}\alpha = \frac{\partial}{\partial t_\alpha}\beta = 0,\quad \frac{\partial}{\partial e}\alpha = \frac{\partial}{\partial e}\beta = 0,\end{gathered}\tag{7.14}$$

$\alpha = 1, \ldots, m$. Equations (7.14) can be easily solved and give quite large $\mathscr{S}(\omega)$ and $\mathscr{C}(\omega)$. Namely,

$$\alpha = C^\alpha t_\alpha + A \quad \beta = C^\alpha s_\alpha + B,$$

and

$$\begin{aligned}
Q^\alpha &= C^\alpha, \quad T_\alpha = -\frac{\partial C^\beta}{\partial q^\alpha} t_\beta - \frac{\partial A}{\partial q^\alpha}, \quad S_\alpha = \frac{\partial C^\beta}{\partial q^\alpha} s_\beta + \frac{\partial B}{\partial q^\alpha}, \\
\varepsilon E &= \frac{\partial C^\alpha}{\partial s} s_\alpha - \frac{\partial C^\alpha}{\partial t} t_\alpha + \frac{\partial B}{\partial s} - \frac{\partial A}{\partial t} \\
&\quad - \delta^{\alpha\beta}\left[s_\alpha \left(\frac{\partial C^\gamma}{\partial q^\beta} s_\gamma + \frac{\partial B}{\partial q^\beta} \right) + e t_\alpha \left(\frac{\partial C^\gamma}{\partial q^\beta} t_\gamma + \frac{\partial A}{\partial q^\beta} \right) \right]
\end{aligned}$$

where $A, B, \ldots, C^\alpha, \ldots, D^{\alpha\beta}, \ldots, E^\alpha, \ldots$ are arbitrary functions of the only s, t, q^β.

Compute the PD Poisson bracket. Let f_1, f_2 be PD Noether currents determined by functions $A_1, B_1, \ldots, C_1^\alpha, \ldots$ and $A_2, B_2, \ldots, C_2^\alpha, \ldots$ respectively. A straightforward computation shows that

$$\{f_1, f_2\} = (C^\alpha t_\alpha + A)ds + (C^\alpha s_\alpha + B)dt$$

with

$$\begin{aligned}
A &= C_1^\beta \frac{\partial}{\partial q^\beta} A_2 - C_2^\beta \frac{\partial}{\partial q^\beta} A_1, \\
B &= C_1^\beta \frac{\partial}{\partial q^\beta} B_2 - C_2^\beta \frac{\partial}{\partial q^\beta} B_1, \\
C^\alpha &= C_1^\beta \frac{\partial}{\partial q^\beta} C_2^\alpha - C_2^\beta \frac{\partial}{\partial q^\beta} C_1^\alpha,
\end{aligned}$$

$\alpha = 1, \ldots, m.$

7.3 A Degenerate, Unconstrained Example

Finally, we present an example of reduction. Consider the cotangent bundle

$$\pi \colon T^*\mathbb{M} \ni A_i dx^i|_{(x^1,\ldots,x^n)} \longmapsto (x^1, \ldots, x^n) \in \mathbb{M}$$

and let

$$\alpha := \pi_1 : (x^1, \ldots, x^n, \ldots, A_i, \ldots, A_{i,j}, \ldots) \ni J^1\pi \longmapsto (x^1, \ldots, x^n) \in \mathbb{M},$$

with $\mathbb{M}$ being the n-dimensional Minkowski space. As such, $\mathbb{M}$ is endowed with the metric $g := g_{ij} dx^i \cdot dx^j$ where

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In the following we will raise and lower indexes using g. Let

$$\omega := 2dA^{[j,i]}\left(\frac{1}{2}A_{i,j}d^n x - dA_i d^{n-1}x_j\right).$$

Then ω is a PD prehamiltonian system on π whose PD Hamilton equation reads

$$\begin{aligned}\partial_k A^{[i,k]} &= 0,\\ \partial_{[j}A_{i]} &= A_{[i,j]},\end{aligned}$$

$i, j = 1, \ldots, n$, which are equivalent to Maxwell equations for the vector potential

$$\begin{aligned}(\partial_k\partial^k)A_i - \partial_i\partial_k A^k &= 0,\\ A_{[i,j]} &= \partial_{[j}A_{i]}.\end{aligned}$$

Notice that

$$\ker\omega = \ker\underline{\omega} = \left\langle \ldots, \frac{\partial}{\partial A_{i,j}} + \frac{\partial}{\partial A_{j,i}}, \ldots \right\rangle.$$

Therefore ω is "degenerate and unconstrained". Moreover, leaves of $D^\omega = \underline{D}^\omega$ are given by $A_{[i,j]} = \text{const}$. We conclude that $J^1\pi$ "reduces" via

$$\begin{aligned}\mathfrak{p}\colon J^1\pi &\to T^*\mathbb{M}\times_M \wedge^2 T^*\mathbb{M} \simeq \mathbb{R}^{n(n+3)/2}\\ (x^1, \ldots, x^n, \ldots, A_i, \ldots, A_{i,j}, \ldots) &\longmapsto (x^1, \ldots, x^n, \ldots, A_i, \ldots, F_{ij}, \ldots)\end{aligned}$$

where $F_{ij} = F_{[ij]}$, $\mathfrak{p}^*(F_{ij}) := 2A_{[j,i]}$ and $\omega = \mathfrak{p}^*(\widetilde{\omega})$, with

$$\widetilde{\omega} := dF^{ij}\left(\frac{1}{4}F_{ji}d^n x - dA_i d^{n-1}x_j\right)$$

is a PD Hamiltonian system on

$$\widetilde{\alpha}\colon \mathbb{R}^{n(n+3)/2} \ni (x^1, \ldots, x^n, \ldots, A_i, \ldots, F_{ij}, \ldots) \longmapsto (x^1, \ldots, x^n) \in \mathbb{R}^n,$$

whose PD Hamilton equations read

$$\begin{aligned}\partial_k F^{ik} &= 0,\\ \partial_{[j}A_{i]} &= 2F_{ji},\end{aligned}$$

which are Maxwell equations for the field strength.

8 PD Hamiltonian Systems in Mathematical Physics and Geometry

A system of PDEs is multisymplectic if it is in the form

$$K^i_{ab}\partial_i y^a = \partial_b H, \tag{8.1}$$

with $K^i_{ab} = -K^i_{ba}(y^1, \dots, y^m)$ and $H = H(y^1, \dots, y^m)$ given functions and $\kappa^i := K^i_{ab} dy^a dy^b$ a symplectic form for all $i = 1, \dots, n$. Multisymplectic systems of PDEs where first introduced in [5] to study the interaction and stability of non-linear waves. More generally, a multi-symplectic formulation of a PDE proved to be a useful tool in the stability analysis. Multisymplectic integrators have been also introduced [6], generalizing the symplectic methods so popular in numerical Hamiltonian dynamics.

Notice that multisymplectic PDEs (8.1) are actually PD Hamilton equations of the PD Hamiltonian system

$$\omega = \frac{1}{2}\kappa^i d^{n-1}x_i + dH d^n x. \tag{8.2}$$

PD Hamiltonian systems of the form (8.2) are of a special kind. Indeed, they are autonomous in two respects. Neither their kinematics (encoded by the symplectic forms κ^i) nor their dynamics (encoded by the "Hamiltonian function" H) depend on space-time coordinates. While the latter condition can be always achieved by adding auxiliary coordinates, without loss of generality, the former is a special feature of multisymplectic PDEs among general PD Hamilton equations. We conclude that PD Hamiltonian systems provide an intrinsic (coordinate independent) geometric formalization of the theory of multisymplectic PDEs.

Many equations of fluid dynamics (and, more generally, of continuum mechanics), including the Euler equation as an instance, are multisymplectic PDEs, and, therefore, PD Hamilton equations (possibly after a suitable change of coordinates) [8, 42]. Notice that fluid dynamics on a general Riemannian manifold may not possess a multisymplectic formulation, while still possessing a PD Hamiltonian one, since, in this case, the "kinematics" depends on the metric and, therefore, on the space-time.

Systems of hydrodynamic type, with their Dubrovin–Novikov Poisson structures [16, 17], are also multisymplectic (see, for instance, [45]), and, therefore, PD Hamiltonian. For the latter systems, the relation between the multisymplectic structure and the Poisson structure has been discussed at least in the integrable, 1-dimensional case of the KdV equation (see [27] for details).

As already mentioned, a Lagrangian field theory in the bundle $\pi\colon E \to M$, with Lagrangian density $\mathscr{L}$ locally given by $\mathscr{L} = L d^n x$, $L = L(x^1, \dots, x^n, u^\alpha_i)$, where the u^α_i's denotes "space-time derivatives" of the field variables u^α's, determines canonically a PD Hamiltonian system $\omega_{\mathscr{L}}$ in the bundle $J^1\pi \to M$, locally given by

$$\omega_{\mathscr{L}} = d\frac{\partial L}{\partial u^\alpha_i} du^\alpha d^{n-1}x_i - dE d^n x, \quad E := u^\alpha_i \frac{\partial L}{\partial u^\alpha_i} - L.$$

In many cases the PD Hamilton equations of $\omega_{\mathscr{L}}$ are equivalent to the Euler–Lagrange equations (even in presence of gauge symmetries). The (functional) space of solutions of the Euler–Lagrange equations carries a canonical (pre)symplectic structure ω

[9, 58] whose degeneracy distribution is made of gauge symmetries [41]. Therefore, the (classical) gauge reduction of a field theory basically amounts to the symplectic reduction of ω. This remark is at the very basis of the BV formalism for the quantization of gauge theories [35]. Interesting examples may be found at the frontier between theoretical physics and geometry. For instance, the reduction of the space of flat connections in a principal bundle over a Riemannian 3-manifold to the *moduli space* of (gauge equivalent) flat connections amounts to the symplectic reduction of the presymplectic structure of the Chern–Simons theory. Similarly, the Atiyah–Hitchin manifold is a moduli space of (gauge equivalent) magnetic monopoles. As such it inherits the symplectic structure from the presymplectic structure of the Yang–Mills–Higgs theory. This is precisely the symplectic sector of the Atiyah–Hitchin hyper-Kähler structure. Now ω can be understood as a cohomology class of a suitable complex and $\omega_{\mathscr{L}}$ as a cocycle representing it [56]. For completeness, we present the (reduced) PD Hamiltonian system corresponding to the Yang–Mills–Higgs theory.

Let $\mathbb{M}$ be the n-dimensional Minkowski space with coordinates x^i and let $\mathfrak{g}$ be a Lie algebra with a basis σ_a. Consider the following (vector) bundles:

(1) $\mathbb{M} \times \mathfrak{g} \to \mathbb{M}$, with linear bundle coordinates ϕ^a;
(2) a copy of $T^*\mathbb{M} \otimes \mathfrak{g} \to \mathbb{M}$, with bundle coordinates ψ^a_i;
(3) a second copy of $T^*\mathbb{M} \otimes \mathfrak{g} \to \mathbb{M}$, with bundle coordinates A^a_i;
(4) $\wedge^2 T^*\mathbb{M} \otimes \mathfrak{g} \to \mathbb{M}$, with bundle coordinates F^a_{ij}, $i < j$.
(5) The fibered product $\alpha \colon P \to M$ of the above.

In α there is the following natural PD Hamiltonian system

$$\omega_{YMH} = k_{ab}\Big[dF^{aij}\Big(dA^b_j + \frac{1}{2}[A_l, A_j]^b dx^l \Big) + d\psi^{ai}(\phi^b + [A_l, \phi]^b dx^l)\Big] d^{n-1}x_i - Ld^n x,$$

where

$$L = k_{ab}\left(\frac{1}{4}F^{aij}F^b_{ij} + \frac{1}{2}\psi^{ai}\psi^b_i \right),$$

and (k_{ab}) is the Killing form of $\mathfrak{g}$. The PD Hamilton equations of ω_{YMH} are

$$\partial_i\psi^{ai} + [A_i, \psi]^a = 0, \qquad \partial_i\phi^a + [A_i, \phi]^a = \psi^a_i,$$
$$\partial_i F^{aij} + [A_i, F^{ij}]^a = 0, \qquad \partial_i A^a_j - \partial_j A^a_i + [A_i, A_j]^a = F^a_{ij},$$

which are basically the Yang–Mills–Higgs equations.

References

[1] V. Aldaya, and J. de Azcárraga, *Higher Order Hamiltonian Formalism in Field Theory.* J. Phys. A: Math. Gen. **13**(1982), 2545. http://dx.doi.org/10.1088/0305-4470/13/8/004

[2] R. J. Alonso-Blanco and A. M. Vinogradov, *Green Formula and Legendre Transformation.* Acta Appl. Math. **83**(2004), 149. http://dx.doi.org/10.1023/B:ACAP.0000035594.33327.71

[3] A. Awane, *k-Symplectic Structures.* J. Math. Phys. **32**(1992), 4046. http://dx.doi.org/10.1063/1.529855

[4] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor'kova, I. S. Krasil'shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov, *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics.* Transl. Math. Mon. **182**, Amer. Math. Soc., Providence, 1999.

[5] T. J. Bridges, *Multi-symplectic Structures and Wave Propagation.* Math. Proc. Camb. Philos. Soc. **121**(1997), 147. http://dx.doi.org/10.1017/S0305004196001429

[6] T. J. Bridges and S. Reich, *Multi-symplectic Integrators: Numerical Schemes for Hamiltonian PDEs that Preserve Symplecticity.* Phys. Lett. **A284**(2001), 184. http://dx.doi.org/10.1016/S0375-9601(01)00294-8

[7] F. Cantrijn and A. Ibort. M. de León, *On the Geometry of Multisymplectic Manifolds.* J. Austral. Math. Soc. Ser. A **66**(1999), 303. http://dx.doi.org/10.1017/S1446788700036636

[8] C. J. Cotter, D. D. Holm, and P. E. Hydon, *Multisymplectic Formulation of Fluid Dynamics Using the Inverse Map.* Proc. Roy. Soc. **A463**(2007), 2671. http://dx.doi.org/10.1098/rspa.2007.1892

[9] C. Crnković and E. Witten, *Covariant Description of Canonical Formalism in Geometrical Theories.* In: Three Hundred Years of Gravitation (eds. S. W. Hawking and W. Israel), Cambridge University Press, Cambridge, 1987, 676.

[10] P. Dedecker, *On the Generalization of Symplectic Geometry to Multiple Integrals in the Calculus of Variations.* Lecture Notes in Math. **570**, Springer, Berlin, 1977, 395.

[11] M. de León, J. Marín-Solano, and J. C. Marrero, *The Constraint Algorithm in the Jet Formalism.* Diff. Geom. Appl. **6**(1996), 275. http://dx.doi.org/10.1016/0926-2245(96)82423-5

[12] ———, *A Geometrical Approach to Classical Field Theories: a Constraint Algorithm for Singular Theories.* Math. Appl. **350**, Kluwer, Dordrecht, 1996, 291.

[13] M. de León, J. Marín-Solano, J. C. Marrero, M. C. Muñoz-Lecanda, and N. Román-Roy, *Singular Lagrangian on Jet Bundles.* Fort. Phys. **50**(2002), 103. arxiv:math-ph/0105012

[14] M. de León, D. Martin de Diego, and A. Santamaria-Merino, *Symmetries in Classical Field Theory.* Int. J. Geom. Methods Mod. Phys. **1**(2004), 651. http://dx.doi.org/10.1142/S0219887804000290

[15] M. de León, J. Marín-Solano, J. C. Marrero, M. C. Muñoz-Lecanda, and N. Román-Roy, *Pre-Multisymplectic Constraint Algorithm for Field Theories.* Int. J. Geom. Methods Mod. Phys. **2**(2005), 839. http://dx.doi.org/10.1142/S0219887805000880

[16] B. A. Dubrovin and S. P. Novikov, *Hamiltonian Formalism of One-Dimensional Systems of Hydrodynamic Type and the Bogolyubov–Whitham Averaging Method.* Dokl. Akad. Nauk SSSR **270**(1983), 781–785; Soviet Math. Dokl. **27**(1983), 665.

[17] ———, *On Poisson Brackets of Hydrodynamic Type.* Dokl. Akad. Nauk SSSR **279**(1984), 294–297; Soviet Math. Dokl. **30**(1984), 651.

[18] A. Echeverría-Enríquez, M. C. Muñoz-Lecanda, and N. Román-Roy, *Geometry of Multisymplectic Hamiltonian First-Order Field Theories.* J. Math. Phys. **41**(2000), 7402. http://dx.doi.org/10.1063/1.1308075

[19] ———, *Geometry of Lagrangian First-Order Classical Field Theories.* Forts. Phys. **44**(1996), 235. http://dx.doi.org/10.1002/prop.2190440304

[20] M. Forger and L. Gomes, *Multisymplectic and Polysymplectic Structures on Fiber Bundles.* arxiv:0708.1596

[21] M. Forger, C. Paufler, and H. Römer, *The Poisson Bracket for Poisson Forms in Multisymplectic Field Theory.* Rev. Math. Phys. **15**(2003), 705. http://dx.doi.org/10.1142/S0129055X03001734

[22] ———, *A General Construction of Poisson Brackets on Exact Multisymplectic Manifolds.* Rep. Math. Phys. **51**(2003), 187. http://dx.doi.org/10.1016/S0034-4877(03)80012-5

[23] ———, *Hamiltonian Multivector Fields and Poisson Forms in Multisymplectic Field Theory.* J. Math. Phys. **46**(2005), 112903. http://dx.doi.org/10.1063/1.2116320

[24] M. Forger and H. Römer, *A Poisson Bracket on Multisymplectic Phase Space.* Rep. Math. Phys. **48**(2001), 211. http://dx.doi.org/10.1016/S0034-4877(01)80081-1

[25] M. Forger and S. Romero, *Covariant Poisson Brackets in Geometric Field Theory.* Commun. Math. Phys. **256**(2005), 375. http://dx.doi.org/10.1007/s00220-005-1287-8

[26] H. Goldshmidt and S. Sternberg, *The Hamilton–Cartan Formalism in the Calculus of Variations.* Ann. Inst. Fourier **23**(1973), 203. http://dx.doi.org/10.5802/aif.451

[27] M. J. Gotay, *A Multisymplectic Approach to the KdV Equation.* In: Differential Geometric Methods in Mathematical Physics (eds. K. Bleuler and M. Werner), Kluwer, Amsterdam, 1988, 295.

[28] M. J. Gotay, J. Isenberg, and J. E. Marsden, *Momentum Maps and Classical Relativistic Fields. I: Covariant Field Theory*. arxiv:physics/9801019

[29] M. J. Gotay, J. M. Nester, and G. Hinds, *Presymplectic Manifolds and the Dirac–Bergmann Theory of Constraints.* J. Math. Phys. **19**(1978), 2388. http://dx.doi.org/10.1063/1.523597

[30] K. Grabowska, *A Tulczyjew Triple for Classical Fields.* J. Phys. A: Math. Theor. **45**(2012), 145207. http://dx.doi.org/10.1088/1751-8113/45/14/145207

[31] K. Grabowska, J. Grabowski, and P. Urbański, *AV-Differential Geometry: Poisson and Jacobi Structures.* J. Geom. Phys. **52**(2004), 398. http://dx.doi.org/10.1016/j.geomphys.2004.04.004

[32] ———, *AV-Differential Geometry: Euler–Lagrange Equations*. J. Geom. Phys. **57**(2007), 1984. http://dx.doi.org/10.1016/j.geomphys.2007.04.003

[33] X. Gracia, R. Martin, and N. Román-Roy, *Constraint Algorithm for k-Presymplectic Hamiltonian Systems. Application to Singular Field Theories.* Int. J. Geom. Methods Mod. Phys. **6**(2009), 851. http://dx.doi.org/10.1142/S0219887809003795
[34] F. Hélein and J. Kouneiher, *Covariant Hamiltonian Formalism for the Calculus of Variations with Several Variables: Lepage–Dedecker versus De Donder–Weyl.* Adv. Theor. Math. Phys. **8**(2004), 565.
[35] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems.* Princeton University Press, Princeton, 1992.
[36] I. V. Kanatchikov, *On Field Theoretic Generalization of a Poisson Algebra.* Rep. Math. Phys. **40**(1997), 225. http://dx.doi.org/10.1016/S0034-4877(97)85919-8
[37] J. Kijowski, *A Finite-Dimensional Canonical Formalism in the Classical Field Theory.* Commun. Math. Phys. **30**(1973), 99. http://dx.doi.org/10.1007/BF01645975
[38] J. Kijowski and W. Szczyrba, *Multisymplectic Manifolds and the Geometrical Construction of the Poisson Bracket in Field Theory.* In: Géométrie Symplectique et Physique Mathématique (ed. J.-M. Souriau), Colloq. Internat. C. N. R. S. **237**(1975), 347.
[39] I. Kolář, *A Geometric Version of the Higher Order Hamilton Formalism in Fibered Manifolds.* J. Geom. Phys. **1**(1984), 127. http://dx.doi.org/10.1016/0393-0440(84)90007-X
[40] O. Krupkova, *Hamiltonian Field Theory.* J. Geom. Phys. **43**(2002), 93. http://dx.doi.org/10.1016/S0393-0440(01)00087-0
[41] J. Lee and R. Wald, *Local Symmetries and Constraints.* J. Math. Phys. **31**(1990), 725. http://dx.doi.org/10.1063/1.528801
[42] J. Marsden, S. Pekarsky, S. Shkoller, and M. West, *Variational Methods, Multisymplectic Geometry and Continuum Mechanics.* J. Geom. Phys. **38**(2001), 253. http://dx.doi.org/10.1016/S0393-0440(00)00066-8
[43] G. Martin, *A Darboux Theorem for Multisymplectic Manifolds.* Lett. Math. Phys. **16**(1988), 133. http://dx.doi.org/10.1007/BF00402020
[44] P. W. Michor, *Topics in Differential Geometry.* Graduate Stud. in Math. **93**, Amer. Math. Soc., Providence, 2008.
[45] O. I. Mokhov, *Symplectic and Poisson Geometry on Loop Spaces of Manifolds and Nonlinear Equations.* Uspekhi Mat. Nauk **53**(1998), 85–192; (English) Russian Math. Surveys **53**(1998), 515. http://dx.doi.org/10.4213/rm19
[46] G. Moreno, A. M. Vinogradov, and L. Vitagliano, *Integrals and Cohomology.* In preparation.
[47] C. Paufler and H. Römer, *Geometry of Hamiltonian n-Vectors in Multisymplectic Field Theory.* J. Geom. Phys. **44**(2002), 52. http://dx.doi.org/10.1016/S0393-0440(02)00031-1
[48] ———, *de Donder–Weyl Equations and Multisymplectic Geometry.* Rep. Math. Phys. **49**(2002), 325. http://dx.doi.org/10.1016/S0034-4877(02)80030-1
[49] N. Román-Roy, *Multisymplectic Lagrangian and Hamiltonian Formalism of First-Order Classical Field Theories.* SIGMA **5**(2009), 100. arxiv:math-ph/0506022
[50] D. J. Saunders, *Jet Fields, Connections and Second-Order Differential Equations.* J. Phys. A: Math. Gen. **20**(1987), 3261. http://dx.doi.org/10.1088/0305-4470/20/11/029
[51] ———, *The Geometry of Jet Bundles.* Cambridge University Press, Cambridge, 1989.
[52] ———, *A Note on Legendre Transformations.* Diff. Geom. Appl. **1**(1991), 109. http://dx.doi.org/10.1016/0926-2245(91)90025-5
[53] D. J. Saunders and M. Crampin, *On the Legendre Map in Higher-Order Field Theories.* J. Phys. A: Math. Gen. **23**(1990), 3169. http://dx.doi.org/10.1088/0305-4470/23/14/016
[54] W. F. Shadwick, *The Hamiltonian Formulation of Regular r-th Order Lagrangian Field Theories.* Lett. Math. Phys. **6**(1982), 409. http://dx.doi.org/10.1007/BF00405859
[55] A. M. Vinogradov, *The $\mathscr{C}$–Spectral Sequence, Lagrangian Formalism and Conservation Laws I, II.* J. Math. Anal. Appl. **100**(1984), 1. http://dx.doi.org/10.1016/0022-247X(84)90071-4
[56] L. Vitagliano, *Secondary Calculus and the Covariant Phase Space.* J. Geom. Phys. **59**(2009), 426. http://dx.doi.org/10.1016/j.geomphys.2008.12.001
[57] ———, *The Lagrangian–Hamiltonian Formalism for Higher Order Field Theories.* J. Geom. Phys. **60**(2010), 857. http://dx.doi.org/10.1016/j.geomphys.2010.02.003
[58] G. J. Zuckerman, *Action Principles and Global Geometry.* In: Mathematical Aspects of String Theory (ed. S. T. Yau), World Scientific, Singapore, 1987, 259.

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