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Norms of Complex Harmonic Projection Operators

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Abstract. In this paper we estimate the $(L^p - L^2)$-norm of the complex harmonic projectors $\pi_{\ell\ell'}$, $1 \le p \le 2$, uniformly with respect to the indexes ℓ, ℓ'. We provide sharp estimates both for the projectors $\pi_{\ell\ell'}$, when ℓ, ℓ' belong to a proper angular sector in $\mathbb{N} \times \mathbb{N}$, and for the projectors $\pi_{\ell 0}$ and $\pi_{0\ell}$. The proof is based on an extension of a complex interpolation argument by C. Sogge. In the appendix, we prove in a direct way the uniform boundedness of a particular zonal kernel in the L^1 norm on the unit sphere of $\mathbb{R}^{2n}$.

1 Introduction

According to a classical result every square-integrable function on the unit sphere Σ^{n-1} of $\mathbb{R}^n$, $n \ge 2$, admits a unique decomposition as

$$f = \sum_{k=0}^{+\infty} Y^k,$$

where $Y^k \in \mathcal{H}^k$ for every $k \ge 0$ and $\mathcal{H}^k$ denotes the space of spherical harmonics of degree k ([8]). The subspaces $\mathcal{H}^k$ satisfy remarkable properties, such as orthogonality and invariance under the rotation group.

By identifying the unit sphere S^{2n-1} in $\mathbb{C}^n$ with the real unit sphere Σ^{2n-1} of $\mathbb{R}^{2n}$, we may obtain an analogous decomposition for the functions from $L^2(S^{2n-1})$ by means of real spherical harmonics. In the complex framework, however, it is possible to obtain a finer direct sum decomposition for $L^2(S^{2n-1})$ by imposing the weaker condition that the subspaces are preserved by the unitary group. It turns out that if $f \in L^2(S^{2n-1})$, then

$$f = \sum_{\ell,\ell'=0}^{+\infty} Y^{\ell\ell'},$$

where $Y^{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$ for every $\ell, \ell' \ge 0$ and $\mathcal{H}^{\ell\ell'}$ consists of the restrictions to S^{2n-1} of polynomials $p(z,\bar{z}) = p(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$ homogeneous of degree ℓ in $z_1, \dots, z_n$, homogeneous of degree ℓ' in $(\bar{z}_1, \dots, \bar{z}_n)$ and harmonic ([5]).

Given $f \in L^2(\Sigma^{n-1})$, the harmonic projection operator π_k is defined by

$$\pi_k \colon L^2(\Sigma^{n-1}) \ni f \mapsto Y^k \in \mathcal{H}^k.$$

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A sharp estimate for the $(L^p - L^2)$-norm of π_k, $1 \le p \le 2$, was given by C. Sogge in 1986 ([7]) (for an earlier, but less precise result of this kind, see [1, p. 249]).

Sogge proved, in particular, that $\|\pi_k\|_{(p,2)} = O(k^{\tilde\gamma(\frac{1}{p})})$, where $\tilde\gamma$ is the affine function given by

$$\tilde\gamma\Big(\frac{1}{p}\Big) := \begin{cases} (n-1)\frac{1}{p} - \frac{1}{2}n & \text{if } 1 \le p \le p_n \\ \frac{1}{2}(n-2)(\frac{1}{p} - \frac{1}{2}) & \text{if } p_n \le p \le 2, \end{cases}$$

with $p_n = \frac{2n}{n+2}$. Afterwards, sharp estimates for the norm of harmonic projection operators on compact Lie groups were found by E. Giacalone and F. Ricci ([4], [3]).

The aim of this paper is to prove analogous sharp estimates for the $(L^p - L^2)$-norm of the complex harmonic projector $\pi_{\ell\ell'}$, $1 \le p \le 2$, defined by

$$\pi_{\ell\ell'} \colon L^2(S^{2n-1}) \ni f \mapsto Y^{\ell\ell'} \in \mathcal{H}^{\ell\ell'}.$$

We are particularly interested in estimates that depend on the length of $\ell + \ell'$, uniformly with respect to the indexes $\ell, \ell' \in \mathbb{N}$. If $(\ell, \ell') \in \mathbb{N} \times \mathbb{N}$, such uniform estimates do not hold, with the exception of the case $n = 2$. Anyway, if (ℓ, ℓ') belongs to a proper angular sector in $\mathbb{N} \times \mathbb{N}$ (that is if $\varepsilon_0 \ell \le \ell' \le M\ell$ for some $0 < \varepsilon_0 \le M$), it is possible to estimate $\|\pi_{\ell\ell'}\|_{(p,2)}$, uniformly with respect to ℓ, ℓ'. Under this assumption, we prove in Theorem 3.5 that $\|\pi_{\ell\ell'}\|_{(p,2)} = O(\ell^{\gamma(\frac{1}{p})})$, where

$$\gamma\Big(\frac{1}{p}\Big) := \begin{cases} (2n-2)\frac{1}{p} + \frac{1}{2} - n & \text{if } 1 \le p \le \bar p \\ (n - \frac{3}{2})(\frac{1}{p} - \frac{1}{2}) & \text{if } \bar p \le p \le 2, \end{cases}$$

with $\bar p = 2\frac{2n-1}{2n+1}$. Observe that the values of $\gamma(\frac{1}{p})$ are equal to the corresponding values found by Sogge for Σ^{2n-2}, but this result cannot be expected *a priori*. Moreover, our proof in this case is analogous to that of Sogge and they are both inspired by the proof of the Stein-Tomas restriction theorem.

An essential tool in the proof of Theorem 3.5 is given by the uniform boundedness in $L^1(\Sigma^{2n-1})$ of a particular zonal kernel. This property may be deduced from some results on the functional calculus for the Laplace operator on compact manifolds ([10, Chapter XII]). Nonetheless we present in the appendix a more direct and elementary proof.

Secondly, we estimate the $(L^p - L^2)$-norm of $\pi_{\ell\ell'}$ when either ℓ or ℓ' are equal to zero. In this case, we first interpolate by means of the Riesz-Thorin Theorem with respect to trivial endpoint estimates, then we prove that this estimate cannot be improved and is no better than $\|\pi_{\ell 0}\|_{(p,2)} = \|\pi_{0\ell}\|_{(p,2)} = O(\ell^{(n-1)(\frac{1}{p} - \frac{1}{2})})$.

By using the classical formula relating real and complex projectors

$$\sigma_k = \sum_{\ell + \ell' = k} \pi_{\ell\ell'},$$

where σ_k denotes the real harmonic projector in $L^2(\Sigma^{2n-1})$ and $\pi_{\ell\ell'}$ denotes the complex harmonic projector in $L^2(S^{2n-1})$, we could recover an estimate for the $(L^p - L^2)$-norm of $\pi_{\ell\ell'}$ from the results by Sogge in $\mathbb{R}^{2n}$. However, the estimates presented in this paper are better than the estimates we could deduce from the real case (*cf.* Remarks 3.7 and 4.3).

2 Notation and Preliminaries

For $n \geq 2$, let $\mathbb{C}^n$ denote the n-dimensional complex space endowed with the scalar product $\langle z, w\rangle := z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$, $z, w \in \mathbb{C}^n$ and let S^{2n-1} denote the unit sphere in $\mathbb{C}^n$, that is

$$S^{2n-1} := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \langle z, z\rangle = 1\}.$$

The symbol $\mathbf{1}$ will denote the north pole of S^{2n-1}, that is $\mathbf{1} := (0, \ldots, 0, 1)$.

Following the notation in [5] we denote by $\mathcal{R}^{\ell\ell'}$ the linear space of all homogeneous polynomials $p(z, \bar{z}) = p(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ of homogeneity degree ℓ in $z_1, \ldots, z_n$ and of homogeneity degree ℓ' in $(\bar{z}_1, \ldots, \bar{z}_n)$, *i.e.*

$$p(az, b\bar{z}) = a^{\ell} b^{\ell'} p(z, \bar{z}), \quad a, b \in \mathbb{R}.$$

A polynomial p in $z, \bar{z}$ is said to be *harmonic* if

$$\Delta p := \frac{1}{4}\Big(\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \cdots + \frac{\partial^2}{\partial z_n \partial \bar{z}_n}\Big) p = 0.$$

We will denote the space of the restrictions to S^{2n-1} of harmonic polynomials from $\mathcal{R}^{\ell\ell'}$ by the symbol $\mathcal{H}^{\ell\ell'}$.

Let $L^2(S^{2n-1})$ the Hilbert space of functions on S^{2n-1} endowed with the inner product $(f, g) := \int_{S^{2n-1}} f(\xi)\overline{g(\xi)}\, d\sigma(\xi)$, where $d\sigma$ is the measure invariant under the action of $U(n)$.

We collect some standard facts on the spaces $\mathcal{H}^{\ell\ell'}$ which will be useful in the following.

(1) If $(\ell, \ell') \neq (m, m')$, then the spaces $\mathcal{H}^{\ell\ell'}$ and $\mathcal{H}^{mm'}$ are orthogonal in $L^2(S^{2n-1})$.
(2) The spaces $\mathcal{H}^{\ell\ell'}$ are $U(n)$-invariant for every ℓ, ℓ'.
(3) The dimension of $\mathcal{H}^{\ell\ell'}$ is given by

$$\dim \mathcal{H}^{\ell\ell'} = (n-1) \cdot \frac{\ell + \ell' + n - 1}{\ell\ell'} \binom{\ell + n - 2}{\ell - 1} \binom{\ell' + n - 2}{\ell' - 1}$$

for all $\ell, \ell' \geq 1$.
(4) $\mathcal{H}^{\ell 0}$ and $\mathcal{H}^{0\ell}$ coincide respectively with the spaces $\mathcal{R}^{\ell 0}$ and $\mathcal{R}^{0\ell}$, so that

$$\dim \mathcal{H}^{\ell 0} = \dim \mathcal{H}^{0\ell} = \binom{\ell + n - 1}{\ell}.$$

(5) The representation of $U(n)$ on the space $\mathcal{H}^{\ell\ell'}$ is irreducible.

It is a classical result ([5], volume II, Chapter 11) that if $f \in L^2(S^{2n-1})$ there exists a unique decomposition

$$f = \sum_{\ell,\ell'=0}^{+\infty} Y^{\ell\ell'}, \tag{2.1}$$

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where $Y^{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$ for every $\ell, \ell' \geq 0$ and the series at the right converges to f in the $L^2(S^{2n-1})$-norm.

The complex harmonic projector is defined by

$$\pi_{\ell\ell'} \colon L^2(S^{2n-1}) \ni f \mapsto Y^{\ell\ell'} \in \mathcal{H}^{\ell\ell'}.$$

A zonal function of degree (ℓ, ℓ') on S^{2n-1} is a function in $\mathcal{H}^{\ell\ell'}$, which is constant on the orbits of the stabilizer of $\mathbf{1}$ (which is isomorphic to $U(n-1)$). If f is zonal, we may associate to f a map $\hat{f}$ defined on the unit disk

$$f(\xi) = \hat{f}(\langle \xi, \mathbf{1} \rangle), \quad \xi \in S^{2n-1},$$

(by using the notation in [5, Section 11.1.5] we have $\langle \xi, \mathbf{1} \rangle = \xi_n = e^{i\varphi} \cos\theta$, where $\varphi \in [0, 2\pi]$ and $\theta \in [0, \frac{\pi}{2}]$).

Then the convolution between a zonal function f and an arbitrary function g on S^{2n-1} may be defined as

$$(f * g)(\xi) := \int_{S^{2n-1}} \hat{f}(\langle \xi, \eta \rangle) g(\eta) \, d\sigma(\eta).$$

In the following we shall write $f(\theta, \varphi)$ instead of $\hat{f}(e^{i\varphi} \cos\theta)$.

It is well-known ([5, Section 11.2]) that $\pi_{\ell\ell'} f = \mathbf{Z}_{\ell\ell'} * f$, where $\mathbf{Z}_{\ell\ell'}$ is the zonal function from $\mathcal{H}^{\ell\ell'}$, given by

$$\mathbf{Z}_{\ell\ell'}(\theta, \varphi) := \frac{n-1}{\omega_{2n-1}} \binom{\ell+n-2}{\ell-1} \binom{\ell'+n-2}{\ell'-1} \frac{\ell+\ell'+n-1}{\ell\ell'} \cdot \frac{q!(n-2)!}{(q+n-2)!} \times e^{i(\ell'-\ell)\varphi} (\cos\theta)^{|\ell-\ell'|} P_q^{(n-2,|\ell'-\ell|)}(\cos 2\theta),$$

$\varphi \in [0, 2\pi]$, $\theta \in [0, \frac{\pi}{2}]$, if $\ell, \ell' \geq 1$; here $q := \min(\ell, \ell')$, ω_{2n-1} denotes the surface area of S^{2n-1} and $P_q^{(n-2,|\ell'-\ell|)}$ is the Jacobi polynomial. If $\ell = 0$ or $\ell' = 0$, the zonal functions are given by

$$\mathbf{Z}_{\ell 0}(\theta, \varphi) := \frac{1}{\omega_{2n-1}} \binom{\ell+n-1}{\ell} \cdot e^{-i\ell\varphi} \cos^\ell \theta \quad \text{and}$$

$$\mathbf{Z}_{0\ell}(\theta, \varphi) := \frac{1}{\omega_{2n-1}} \binom{\ell+n-1}{\ell} \cdot e^{i\ell\varphi} \cos^\ell \theta.$$

Observe that according to our notation (and differently from that in [KV]) we have

$$\mathbf{Z}_{\ell\ell'}(\mathbf{1}) = \frac{\dim \mathcal{H}^{\ell\ell'}}{\omega_{2n-1}}. \tag{2.2}$$

We will use the following estimate, due to Darboux and Szegö ([9, pp. 169, 198]).

Lemma 2.1 *Let $\alpha, \beta > -1$. Fix $0 < c < \pi$. Then*

$$P_\ell^{(\alpha,\beta)}(\cos\theta) = \begin{cases} O(\ell^\alpha) & \text{if } 0 \leq \theta \leq \frac{c}{\ell} \text{ or } \pi - \frac{c}{\ell} \leq \theta \leq \pi, \\ \ell^{-\frac{1}{2}} k(\theta)[\cos(N_\ell \theta + \gamma) + (\ell \sin\theta)^{-1} O(1)] & \text{if } \frac{c}{\ell} \leq \theta \leq \pi - \frac{c}{\ell}, \end{cases}$$

where $k(\theta) := \pi^{-\frac{1}{2}} (\sin\frac{\theta}{2})^{-\alpha-\frac{1}{2}} (\cos\frac{\theta}{2})^{-\beta-\frac{1}{2}}$, $N_\ell := \ell + \frac{\alpha+\beta+1}{2}$, $\gamma := -(\alpha + \frac{1}{2})\frac{\pi}{2}$.

Finally, by the symbols $\mathcal{S}^k$ and σ_k we shall denote, respectively, the space of spherical harmonics of degree k in $\mathbb{R}^{2n}$ ([8, Chapter IV]) and the (real) harmonic projector from $L^2(\Sigma^{2n-1})$ onto $\mathcal{S}^k$. The relationship between $\mathcal{S}^k$ and the space $\mathcal{H}^{\ell\ell'}$ of complex spherical harmonics of degree ℓ and ℓ' in $\mathbb{C}^n$ is given by

$$\mathcal{S}^k = \sum_{\ell+\ell'=k} \oplus \mathcal{H}^{\ell\ell'},$$

yielding the following formula

$$\sigma_k = \sum_{\ell+\ell'=k} \pi_{\ell\ell'}. \tag{2.3}$$

3 The Harmonic Projectors $\pi_{\ell\ell'}$, With $\ell, \ell' \geq 1$

In this section we shall prove our main result, Theorem 3.5, concerning the $\big(L^p(S^{2n-1}), L^2(S^{2n-1})\big)$-operator norm of the complex harmonic projector $\pi_{\ell\ell'}$, if $1 \leq p \leq 2$ and ℓ and ℓ' belong to a proper angular sector, that is

$$\varepsilon_0 \ell \leq \ell' \leq M\ell \quad \text{for some } 0 < \varepsilon_0 \leq M. \tag{3.1}$$

To prove the theorem, we shall need some lemmata. The first one, which may be of independent interest, regards the uniform boundedness in $L^1(\Sigma^{2n-1})$ of a particular zonal kernel and may be deduced from some results in [10, Chapter XII] (see, in particular, Exercise 4.1, p. 322). Nonetheless, we can prove it in a direct and elementary way, by using only some representation formulae for the zonal harmonics in spaces of even dimension. In order not to burden the exposition, we defer the proof to an appendix.

In the statement, the symbol $\mathbf{Z}_j$ denotes the zonal harmonic of degree j in $L^2(\Sigma^{d-1})$, for $d \in \mathbb{N}$ [8, p. 143].

Lemma 3.1 *Fix k, $d \in \mathbb{N}$, with $d \geq 4$, d even. Let B a $\mathcal{C}^\infty$ function, compactly supported in $(0, 2]$. Then*

$$\Big\| \sum_j B\Big(\frac{j}{k}\Big) \mathbf{Z}_j \Big\|_{L^1(\Sigma^{d-1})} \leq C,$$

where the constant C does not depend on k.

Lemma 3.2 *Assume that condition (3.1) is satisfied for some $0 < \varepsilon_0 \leq M$. Let $\ell, \ell' \geq 1$, $\alpha \in (0, 1)$, $\beta > 1$. Set*

$$J^* := \{(j, j') \in \mathbb{N}^* \times \mathbb{N}^* : \alpha(\ell+\ell') < j + j' < \beta(\ell+\ell'), j' - j = \ell' - \ell\}.$$

If $\alpha > \max\{\frac{M-1}{M+1}, \frac{1-\varepsilon_0}{1+\varepsilon_0}\}$, then there exist $A_0, B_0, A_1, B_1 > 0$ such that

$$A_0\ell \leq j \leq B_0\ell \text{ and } A_1\ell \leq j' \leq B_1\ell \quad \text{for all } j, j' \in \mathbb{N}^*.$$

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Proof Take $(j, j') \in J^*$. Then $j' = j+\ell'-\ell$, so that $\alpha(\ell+\ell') < 2j+\ell'-\ell < \beta(\ell+\ell')$. By using $\varepsilon_0\ell \le \ell' \le M\ell$, we obtain

$$\frac{1}{2}[\alpha(1+M)+1-M]\ell < j < \frac{1}{2}[\beta(1+M)+1-M]\ell,$$

yielding the first part of the thesis with $A_0 = \frac{1}{2}[\alpha(1+M)+1-M]$ and $B_0 = \frac{1}{2}[\beta(1+M)+1-M]$.

Analogously, the equality $j = j' - \ell' + \ell$ implies that

$$\frac{1}{2}[\varepsilon_0(\alpha+1)+\alpha-1]\ell < j' < \frac{1}{2}[(\beta-1)+M(\beta+1)]\ell.$$

By choosing $A_1 = \frac{1}{2}[\varepsilon_0(\alpha+1)+\alpha-1]$ and $B_1 = \frac{1}{2}[(\beta-1)+M(\beta+1)]$, we conclude the proof. ■

Lemma 3.3 *Fix $y \neq 0$. Let a_0, b_0, s be real constants, with $s \neq 0$, and $\delta > 0$. Then*

$$\Big| \int_\delta^{\frac{\pi}{2}-\delta} \cos(a_0\theta + b_0)\cdot(\sin 2\theta)^{-1-isy}\, d\theta \Big| \le C\Big(|y| + \frac{1}{|y|}\Big)$$

for some positive constant C, independent of δ, a_0 and b_0.

Proof In the following the symbol C will denote a constant, which may vary from one formula to the other. Set $\psi := a_0\theta + b_0$ and

$$J_1 := \int_\delta^{\frac{\pi}{2}-\delta} \cos\psi\cdot(\sin 2\theta)^{-1-isy}\, d\theta$$

and define $I_\delta^+ := [\delta, \frac{\pi}{2}-\delta]$ and $I_\delta^- := [-\frac{\pi}{2}+\delta, -\delta]$. Thus

$$\begin{aligned} J_1 &= \int_{I_\delta^+} \cos\psi\cdot[(\sin 2\theta)^{-1-isy} - (2\theta)^{-1-isy}]\, d\theta + \int_{I_\delta^+} \cos\psi\cdot(2\theta)^{-1-isy}\, d\theta \\ &=: J_1' + J_1''. \end{aligned}$$

We easily check that $|J_1'| \le C(1+|y|)$ for some positive constant C. If χ denotes the characteristic function of $I_\delta^+ \cup I_\delta^-$, then

$$\begin{aligned} J_1'' &= \frac{1}{2}\Big(\int_{I_\delta^+} + \int_{I_\delta^-}\Big) e^{i\psi}\cdot|2\theta|^{-1-isy}\, d\theta = \frac{1}{2}\int_{\mathbb{R}} e^{i\psi}\cdot|2\theta|^{-1-isy}\cdot\chi(\theta)\, d\theta \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{i\psi}\cdot|2\theta|^{-1-isy}\cdot\big(\chi(\theta)-\tilde\chi(\theta)\big)\, d\theta + \frac{1}{2}\int_{\mathbb{R}} e^{i\psi}\cdot|2\theta|^{-1-isy}\cdot\tilde\chi(\theta)\, d\theta \\ &=: R_1 + R_2, \end{aligned}$$

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$\tilde{\chi}$ being a $\mathcal{C}_0^\infty$ function, which coincides with χ on $I_\delta^+ \cup I_\delta^-$ and vanishes outside $[\frac{\delta}{4}, \frac{\pi}{2}] \cup [-\frac{\pi}{2}, -\frac{\delta}{4}]$. It is easy to check that $|R_1| \leq C$, for some $C > 0$; moreover we have

$$R_2 = \frac{1}{4} \cdot 2^{-isy} \cdot \mathfrak{F}(|\theta|^{-1-isy} \cdot \tilde{\chi})(\psi) = \frac{1}{4} \cdot 2^{-isy} \cdot \left(\mathfrak{F}(|\theta|^{-1-isy}) * \mathfrak{F}\tilde{\chi}\right)(\psi),$$

where $\mathfrak{F}$ denotes the Fourier transform. By using some well-known formulae for the Fourier transform of the distribution $\theta \mapsto |\theta|^{\zeta-1}$, $\zeta \in i\mathbb{R}$, ([2, p. 168]), we obtain

$$R_2 = \frac{1}{4} \cdot \sqrt{\pi} \cdot 2^{-2isy} \cdot \frac{\Gamma(-i\frac{s}{2}y)}{\Gamma(\frac{1}{2} + i\frac{s}{2}y)} (|\cdot|^{isy} * \mathfrak{F}\tilde{\chi})(\psi).$$

Since

$$\left| \frac{\Gamma(-i\frac{s}{2}y)}{\Gamma(\frac{1}{2} + i\frac{s}{2}y)} \right| \leq \begin{cases} \frac{C}{|y|} & \text{if } |y| < 1,\ y \neq 0 \\ C & \text{if } |y| \geq 1, \end{cases}$$

and $\left|(|\cdot|^{isy} * \mathfrak{F}\tilde{\chi})(\psi)\right| \leq C$, where the positive constants C do not depend on a_0, b_0 and y, we finally find that

$$|J_1| \leq C\left(|y| + \frac{1}{|y|}\right). \qquad \blacksquare$$

Lemma 3.4 *Assume that hypothesis (3.1) is satisfied. Let $n \geq 2$.*

(1) *If $q > 2\frac{n-1}{n-\frac{3}{2}}$, then*

$$\frac{\|\mathbf{Z}_{\ell\ell'}\|_q}{\|\mathbf{Z}_{\ell\ell'}\|_2} \geq C\ell^{n-\frac{3}{2}-\frac{2}{q}(n-1)}. \tag{3.2}$$

(2) *If $q \geq 2$ and $Q^{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$ is defined by $Q^{\ell\ell'}(z) := (z_1^\ell \cdot \bar{z}_2^{\ell'})_{|S^{2n-1}}$, then*

$$\frac{\|Q^{\ell\ell'}\|_q}{\|Q^{\ell\ell'}\|_2} \sim C\ell^{(n-\frac{3}{2})(\frac{1}{2}-\frac{1}{q})}. \tag{3.3}$$

Proof (1) We may suppose $\ell' \geq \ell$ (the case $\ell' \leq \ell$ is analogous). Then the zonal harmonic function is given by

$$\mathbf{Z}_{\ell\ell'}(\theta,\varphi) := \frac{1}{\omega_{2n-1}} \binom{\ell'+n-2}{\ell'-1} \frac{\ell+\ell'+n-1}{\ell'} \cdot e^{i(\ell'-\ell)\varphi}$$
$$\times (\cos\theta)^{\ell'-\ell} \cdot P_\ell^{(n-2,\ell'-\ell)}(\cos 2\theta), \quad \varphi \in [0,2\pi],\ \theta \in \left[0, \frac{\pi}{2}\right]$$

and for some $0 < c < \pi$

$$\begin{aligned}\|\mathbf{Z}_{\ell\ell'}\|_q^q &= \int_{S^{2n-1}} |\mathbf{Z}_{\ell\ell'}(\xi)|^q \, d\sigma(\xi) \\ &= C\ell^{(n-1)q} \int_0^{\frac{\pi}{2}} |(\cos\theta)^{\ell'-\ell} \cdot P_\ell^{(n-2,\ell'-\ell)}(\cos 2\theta)|^q (\sin\theta)^{2n-3} \cos\theta \, d\theta \\ &\geq C\ell^{(n-1)q}\ell^{(n-2)q} \int_0^{\frac{c}{\ell}} (\cos\theta)^{(\ell'-\ell)q+1} \cdot (\sin\theta)^{2n-3} \, d\theta = C\ell^{(2n-3)q-2n+2},\end{aligned}$$

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where we used the Darboux-Szegö estimate for $P_\ell^{(n-2,\ell'-\ell)}(\cos 2\theta)$. From the properties of zonal harmonics it follows that

$$\|\mathbf{Z}_{\ell\ell'}\|_2 = \sqrt{\frac{\dim \mathcal{H}^{\ell\ell'}}{\omega_{2n-1}}} \sim C\ell^{n-\frac{3}{2}}, \quad \text{so that}$$

$$\frac{\|\mathbf{Z}_{\ell\ell'}\|_q}{\|\mathbf{Z}_{\ell\ell'}\|_2} \geq C\frac{\ell^{2n-3-\frac{2}{q}(n-1)}}{\ell^{n-\frac{3}{2}}} = C\ell^{n-\frac{3}{2}-\frac{2}{q}(n-1)}.$$

(2) We have:

$$\begin{aligned}
\|Q^{\ell\ell'}\|_q^q &= \int_{S^{2n-1}} |Q^{\ell\ell'}(\xi)|^q \, d\sigma(\xi) \\
&= C\int_0^{\frac{\pi}{2}} (\sin\theta_{n-1})^{(\ell+\ell')q+2n-3} \cos\theta_{n-1}\, d\theta_{n-1} \\
&\quad \times \int_0^{\frac{\pi}{2}} (\sin\theta_{n-2})^{(\ell+\ell')q+2n-5} \cos\theta_{n-2}\, d\theta_{n-2} \\
&\quad \times \cdots \times \int_0^{\frac{\pi}{2}} (\sin\theta_2)^{(\ell+\ell')q+3} \cos\theta_2\, d\theta_2 \\
&\quad \times \int_0^{\frac{\pi}{2}} (\sin\theta_1)^{\ell q+1} (\cos\theta_1)^{\ell' q+1}\, d\theta_1 \\
&= C\frac{1}{(\ell+\ell')q+2n-2} \cdot \frac{1}{(\ell+\ell')q+2n-4} \cdots \frac{1}{(\ell+\ell')q+4} \\
&\quad \times \int_0^{\frac{\pi}{2}} \sin^{\ell q+1}\theta_1 \cos^{\ell' q+1}\theta_1 \, d\theta_1 \\
&= C\frac{1}{[(\ell+\ell')q]^{n-2}} \cdot \frac{\Gamma(\frac{\ell q}{2}+1)\Gamma(\frac{\ell' q}{2}+1)}{\Gamma(\frac{(\ell+\ell')q}{2}+2)}.
\end{aligned}$$

By using (3.1) and the Stirling formula, the thesis easily follows. ■

Theorem 3.5 *Let $n \geq 2$, $\ell, \ell' \geq 1$ and let hypothesis (3.1) be satisfied. Thus*

$$\|\pi_{\ell\ell'} f\|_2 \leq C(n,p)\ell^{\gamma(\frac{1}{p})}\|f\|_p, \tag{3.4}$$

where

$$\gamma\Big(\frac{1}{p}\Big) := \begin{cases} (2n-2)\frac{1}{p}+\frac{1}{2}-n & \text{if } 1 \leq p \leq 2\frac{2n-1}{2n+1} \\ (n-\frac{3}{2})(\frac{1}{p}-\frac{1}{2}) & \text{if } 2\frac{2n-1}{2n+1} \leq p \leq 2 \end{cases}$$

and the constant $C(p,n)$ depends only on p and n. Moreover, all of these estimates are sharp.

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Proof Observe that

$$\|\pi_{\ell\ell'} f\|_2 \le \|f\|_2,$$

since $\pi_{\ell\ell'}$ is an orthogonal projector, and

$$\|\pi_{\ell\ell'} f\|_2 \le \|\mathbf{Z}_{\ell\ell'}\|_2 \cdot \|f\|_1 = C\ell^{n-\frac{3}{2}}\|f\|_1,$$

as a consequence of the Young's inequality and the fact that $\|\mathbf{Z}_{\ell\ell'}\|_2 = O(\ell^{n-\frac{3}{2}})$.

Thus by the Riesz-Thorin convexity theorem it suffices to prove the theorem for $p = 2\frac{2n-1}{2n+1}$, that is

$$\|\pi_{\ell\ell'} f\|_2 \le C\ell^{\frac{2n-3}{4n-2}}\|f\|_{2\frac{2n-1}{2n+1}}.$$

Now, exactly as in [7, Theorem 4.1] we have that

$$\|\pi_{\ell\ell'} f\|_2^2 \le \|\pi_{\ell\ell'} f\|_{p'} \cdot \|f\|_p,$$

so that it is sufficient to prove the following inequality

$$\|\pi_{\ell\ell'} f\|_{2\frac{2n-1}{2n-3}} \le C\ell^{\frac{2n-3}{2n-1}}\|f\|_{2\frac{2n-1}{2n+1}}. \tag{3.5}$$

We do this by using Stein's theorem on analytic interpolation [8, pp. 205–209].

Fix a number $\beta \in (1,2]$. Choose a positive number $\alpha \in (0,1)$ such that $\alpha > \max\{\frac{M-1}{M+1}, \frac{1-\varepsilon_0}{1+\varepsilon_0}\}$, where M and ε_0 are the constants in (3.1).

Introduce a function $B \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $0 \le B \le 1$ and

$$B(t) = \begin{cases} 1 & \text{if } t \in (\frac{1+\alpha}{2}, \frac{1+\beta}{2}) \\ 0 & \text{outside } (\alpha, \beta). \end{cases}$$

Now define the zonal multiplier operator $M_{\ell\ell'}$ by

$$f = \sum_{j,j'=0}^{+\infty} \pi_{jj'} f \mapsto M_{\ell\ell'} f := \sum_{j,j'=0}^{+\infty} B\Big(\frac{j+j'}{\ell+\ell'}\Big)\pi_{jj'} f.$$

Observe that $B(\frac{j+j'}{\ell+\ell'}) \ne 0$ if and only if $\alpha(\ell+\ell') < j+j' < \beta(\ell+\ell')$, so that for fixed ℓ, ℓ' the sum at the right is finite. Let $m_1, \dots, m_s$ be the consecutive integer numbers such that

$$\alpha(\ell+\ell') < m_1 < m_2 < \cdots < m_s < \beta(\ell+\ell').$$

Thus as a consequence of formula (2.3) we obtain

$$\begin{aligned}
&\Big\| \sum_{j,j'=0}^{+\infty} B\Big(\frac{j+j'}{\ell+\ell'}\Big)\mathbf{Z}_{jj'}\Big\|_{L^1(S^{2n-1})} \\
&= \Big\| \sum_{j+j'=m_1} B\Big(\frac{j+j'}{\ell+\ell'}\Big)\mathbf{Z}_{jj'} + \sum_{j+j'=m_2} B\Big(\frac{j+j'}{\ell+\ell'}\Big)\mathbf{Z}_{jj'} \\
&\qquad + \cdots + \sum_{j+j'=m_s} B\Big(\frac{j+j'}{\ell+\ell'}\Big)\mathbf{Z}_{jj'}\Big\|_{L^1(S^{2n-1})} \\
&= \Big\| B\Big(\frac{m_1}{\ell+\ell'}\Big)\mathbf{Z}_{m_1} + B\Big(\frac{m_2}{\ell+\ell'}\Big)\mathbf{Z}_{m_2} + \cdots + B\Big(\frac{m_s}{\ell+\ell'}\Big)\mathbf{Z}_{m_s}\Big\|_{L^1(\Sigma^{2n-1})},
\end{aligned}$$

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where $\mathbf{Z}_{m_i}$, $i = 1, \dots, s$, denotes the zonal harmonic of degree m_i in $L^2(\Sigma^{2n-1})$. By using Lemma 3.1, we conclude that

$$\Big\| \sum_{j,j'=0}^{+\infty} B\Big(\frac{j+j'}{\ell+\ell'}\Big) \mathbf{Z}_{jj'} \Big\|_{L^1(S^{2n-1})} \le C, \tag{3.6}$$

where C does not depend on ℓ and ℓ'.

Let $F^z_{\ell\ell'}$ be the zonal function defined by

$$F^z_{\ell\ell'}(\theta,\varphi) := \frac{2n-1}{2}(1-z)\mathbf{Z}_{\ell\ell'}(\theta,\varphi)(\sin 2\theta)^{n-\frac{3}{2}-\frac{2n-1}{2}z},$$
$$\theta \in \Big[0, \frac{\pi}{2}\Big],\ \varphi \in [0, 2\pi],\ 0 \le \Re e z \le 1.$$

We define the analytic family of operators $\{T^z_{\ell\ell'}\}$ as

$$T^z_{\ell\ell'} f := F^z_{\ell\ell'} * M_{\ell\ell'} f, \quad f \in L^2(S^{2n-1}),\ 0 \le \Re e z \le 1.$$

Observe that if $t = \frac{2n-3}{2n-1}$ then $T^t_{\ell\ell'} f = \pi_{\ell\ell'} f$.

We shall estimate the $(L^1 - L^\infty)$-norm of $T^{0+iy}_{\ell\ell'}$ and the $(L^2 - L^2)$-norm of $T^{1+iy}_{\ell\ell'}$.

By Young's inequality we have if $\ell' \ge \ell$ (the other case is analogous)

$$\begin{aligned}
\|T^{0+iy}_{\ell\ell'} f\|_\infty &\le \Big\| F^{0+iy}_{\ell\ell'} * \sum_{j,j'=0}^{+\infty} B\Big(\frac{j+j'}{\ell+\ell'}\Big)\mathbf{Z}_{jj'} \Big\|_\infty \cdot \|f\|_1 \\
&\le \|F^{0+iy}_{\ell\ell'}\|_\infty \cdot \Big\| \sum_{j,j'=0}^{+\infty} B\Big(\frac{j+j'}{\ell+\ell'}\Big)\mathbf{Z}_{jj'} \Big\|_1 \cdot \|f\|_1 \\
&\le C\ell^{n-1}(1+|y|) \cdot \sup_{\theta\in[0,\frac{\pi}{2}]} |g_{\ell\ell'}(\theta)| \cdot \Big\| \sum_{j,j'=0}^{+\infty} B\Big(\frac{j+j'}{\ell+\ell'}\Big)\mathbf{Z}_{jj'} \Big\|_1 \cdot \|f\|_1,
\end{aligned}$$

where $g_{\ell\ell'}(\theta) := (\cos\theta)^{\ell'-\ell} \cdot P_\ell^{(n-2,\ell'-\ell)}(\cos 2\theta) \cdot (\sin 2\theta)^{n-\frac{3}{2}}$.

By using Lemma 2.1 we obtain

$$\begin{aligned}
\sup_{\theta\in[\frac{c}{\ell},\frac{\pi}{2}-\frac{c}{\ell}]} |g_{\ell\ell'}(\theta)| &\le \Big| \frac{(\cos\theta)^{\ell'-\ell}}{\sqrt{\pi\ell}(\sin\theta)^{n-\frac{3}{2}}(\cos\theta)^{\ell'-\ell+\frac{1}{2}}} \\
&\qquad \times \Big[\cos(2N_\ell\theta+\gamma) + \frac{O(1)}{\ell\sin(2\theta)}\Big](\sin 2\theta)^{n-\frac{3}{2}} \Big| \\
&= O(1)(\cos\theta)^{n-2}\ell^{-\frac{1}{2}} = O(1)\ell^{-\frac{1}{2}}, \quad \text{since } n \ge 2, \text{ and}
\end{aligned}$$

$$\begin{aligned}
\sup_{\theta\in[0,\frac{c}{\ell}]\cup[\frac{\pi}{2}-\frac{c}{\ell},\frac{\pi}{2}]} |g_{\ell\ell'}(\theta)| &\le \sup_{\theta\in[0,\frac{c}{\ell}]\cup[\frac{\pi}{2}-\frac{c}{\ell},\frac{\pi}{2}]} |(\cos\theta)^{\ell'-\ell} \cdot \ell^{n-2}(\sin 2\theta)^{n-\frac{3}{2}}| \\
&= O(1)\ell^{n-2} \cdot \ell^{-n+\frac{3}{2}} = O(1)\ell^{-\frac{1}{2}}.
\end{aligned}$$

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By collecting these results and by using (3.6) we deduce

$$\|T^{0+iy}_{\ell\ell'} f\|_\infty \le C(1+|y|)\ell^{n-\frac{3}{2}}\|f\|_1. \tag{3.7}$$

We shall now estimate the (L^2-L^2)-norm of $T^{1+iy}_{\ell\ell'}$. As a consequence of Plancherel's theorem for spherical harmonics, it suffices to estimate $\sup_{(j,j')\in J}|c_{j,j',y}|$, where

$$c_{j,j',y} := B\Big(\frac{j+j'}{\ell+\ell'}\Big)\frac{(F^{1+iy}_{\ell\ell'} * \mathbf{Z}_{jj'})(\mathbf{1})}{\mathbf{Z}_{jj'}(\mathbf{1})} \quad \text{and}$$

$$J := \{(j,j')\in\mathbb{N}^*\times\mathbb{N}^* : \alpha(\ell+\ell') < j+j' < \beta(\ell+\ell')\}.$$

By using the definition of zonal functions and formula (2.2) we have

$$\begin{aligned}
\|T^{1+iy}_{\ell\ell'}\|_{(2,2)} &\le \sup_{(j,j')\in J}|c_{j,j',y}| \le \sup_{(j,j')\in J}\Big|B\Big(\frac{j+j'}{\ell+\ell'}\Big)\int_{S^{2n-1}} F^{1+iy}_{\ell\ell'}(\xi)\frac{\overline{\mathbf{Z}_{jj'}(\xi)}}{\mathbf{Z}_{jj'}(\mathbf{1})}\,d\sigma(\xi)\Big| \\
&\le \sup_{(j,j')\in J} C\ell^{n-1}{q_j}^{2-n} \\
&\qquad \times \Big|B\Big(\frac{j+j'}{\ell+\ell'}\Big)\,y\int_0^{2\pi} e^{i\varphi(\ell'-\ell-j'+j)}\,d\varphi\cdot\int_0^{\frac{\pi}{2}} h_{j,j'}(\theta)\,d\theta\Big| \\
&\le \sup_{(j,j')\in J^*} C\ell^{n-1}{q_j}^{2-n}\cdot\Big|y\int_0^{\frac{\pi}{2}} h_{j,j'}(\theta)\,d\theta\Big|,
\end{aligned}$$

where J^* is defined as in Lemma 3.2 and

$$\begin{aligned}
h_{j,j'}(\theta) &:= (\cos\theta)^{|\ell-\ell'|+|j-j'|+1}\cdot P^{(n-2,|\ell-\ell'|)}_{q_\ell}(\cos 2\theta)\cdot P^{(n-2,|j-j'|)}_{q_j}(\cos 2\theta) \\
&\qquad \times (\sin\theta)^{2n-3}\cdot(\sin 2\theta)^{-1-\frac{2n-1}{2}iy},
\end{aligned}$$

with $q_j := \min\{j,j'\}$ and $q_\ell := \min\{\ell,\ell'\}$. By using Lemma 3.2 we see that

$$\ell^{n-1}{q_j}^{2-n} = C\ell. \tag{3.8}$$

If $y=0$ the right-hand side in the last term of the inequality above is zero; thus from now on we shall assume $y\neq 0$.

In order to estimate the integral $\int_0^{\frac{\pi}{2}} h_{j,j'}(\theta)\,d\theta$ we pose

$$\int_0^{\frac{\pi}{2}} h_{j,j'}(\theta)\,d\theta = \Big(\int_0^{\frac{c}{\ell}} + \int_{\frac{c}{\ell}}^{\frac{\pi}{2}-\frac{c}{\ell}} + \int_{\frac{\pi}{2}-\frac{c}{\ell}}^{\frac{\pi}{2}}\Big) h_{j,j'}(\theta)\,d\theta =: I_1+I_2+I_3.$$

Now we have

$$\begin{aligned}
|I_1| &= \Big|\int_0^{\frac{c}{\ell}} O(1){q_\ell}^{n-2}{q_j}^{n-2}(\sin\theta)^{2n-3}(\sin 2\theta)^{-1-\frac{2n-1}{2}iy}\,d\theta\Big| \\
&\le O(1)\ell^{2n-4}\int_0^{\frac{c}{\ell}}(\sin\theta)^{2n-4}\,d\theta = O(1)\ell^{-1},
\end{aligned} \tag{3.9}$$

where we used Lemma 2.1 and Lemma 3.2. The estimate for I_3 is analogous. Finally, we have

$$I_2 = \int_{\frac{c}{\ell}}^{\frac{\pi}{2}-\frac{c}{\ell}} \frac{1}{\pi\sqrt{q_\ell q_j}} \left[\cos(2N_\ell\theta+\gamma) + \frac{O(1)}{q_\ell \sin 2\theta}\right]\left[\cos(2N_j\theta+\gamma) + \frac{O(1)}{q_j \sin 2\theta}\right] \times (\sin 2\theta)^{-1-\frac{2n-1}{2}iy}\, d\theta$$

where N_ℓ, N_j and γ have been defined in Lemma 2.1.

We first estimate

$$I_2' := \int_{\frac{c}{\ell}}^{\frac{\pi}{2}-\frac{c}{\ell}} \cos(2N_\ell\theta+\gamma)\cdot\cos(2N_j\theta+\gamma)\cdot(\sin 2\theta)^{-1-\frac{2n-1}{2}iy}\, d\theta.$$

Since $I_2' = \frac{1}{2}\int_{\frac{c}{\ell}}^{\frac{\pi}{2}-\frac{c}{\ell}}[\cos(2N_\ell\theta+2N_j\theta+2\gamma)+\cos(2N_\ell\theta-2N_j\theta)]\cdot(\sin 2\theta)^{-1-\frac{2n-1}{2}iy}\, d\theta$, we may use Lemma 3.3 to prove that

$$|I_2'| \le C\left(|y| + \frac{1}{|y|}\right),$$

for some positive constant C independent of ℓ, j and y. Then $|y|\,|I_2'| = O(1)\times(1+|y|^2)$.

Since $\frac{1}{q_j\sin(2\theta)}$ and $\frac{1}{q_\ell\sin(2\theta)}$ are bounded from above uniformly with respect to j and ℓ when $\theta \in (\frac{c}{\ell}, \frac{\pi}{2}-\frac{c}{\ell})$, we may use Lemma 3.3 and conclude that

$$|y|\cdot|I_2| = O(1)\ell^{-1}\cdot(1+|y|^2),$$

which combined with (3.9) and (3.8) yields

$$\|T_{\ell\ell'}^{1+iy}\|_{(2,2)} \le C\cdot(1+|y|^2). \tag{3.10}$$

By interpolating (3.7) and (3.10) we obtain (3.5).

Finally, we shall prove that the estimate of the $(L^p - L^2)$-norm of $\pi_{\ell\ell'}$ is sharp, for $1 \le p \le 2$. By duality, this is equivalent to prove the sharpness for the estimate of the $(L^2 - L^q)$-norm of $\pi_{\ell\ell'}$, $2 \le q \le +\infty$, that is

$$\|\pi_{\ell\ell'} f\|_{(2,q)} = \begin{cases} O(\ell^{(n-\frac{3}{2})(\frac{1}{2}-\frac{1}{q})}) & \text{if } 2 \le q \le 2\frac{2n-1}{2n-3} \\ O(\ell^{(n-\frac{3}{2})-\frac{2}{q}(n-1)}) & \text{if } 2\frac{2n-1}{2n-3} \le q \le +\infty. \end{cases} \tag{3.11}$$

By Lemma 3.4(1) we have that

$$\frac{\|\mathbf{Z}_{\ell\ell'}\|_q}{\|\mathbf{Z}_{\ell\ell'}\|_2} \ge C\ell^{n-\frac{3}{2}-\frac{2}{q}(n-1)}$$

for $q > 2\frac{2n-2}{2n-3}$, yielding *a fortiori* the second part of (3.11). Analogously, Lemma 3.4 (2) gives the first part of (3.11). ■

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The following sharp result for the norms of complex spherical harmonics easily follows.

Proposition 3.6 *Let $n \geq 2$. Assume condition (3.1) is satisfied. Then there exists a positive constant $C = C(n)$ such that for any polynomial $P_{\ell\ell'} = P_{\ell\ell'}(z, \bar{z})$ harmonic, homogeneous of degree ℓ in z and of degree ℓ' in $\bar{z}$*

$$\|P_{\ell\ell'}\|_{L^q(S^{2n-1})} \leq \begin{cases} C(n)\ell^{(n-\frac{3}{2})(\frac{1}{2}-\frac{1}{q})}\|P_{\ell\ell'}\|_{L^2(S^{2n-1})} & \text{if } 2 \leq q \leq 2\frac{2n-1}{2n-3} \\ C(n)\ell^{n-\frac{3}{2}-\frac{2}{q}(n-1)}\|P_{\ell\ell'}\|_{L^2(S^{2n-1})} & \text{if } 2\frac{2n-1}{2n-3} \leq q \leq +\infty, \end{cases}.$$

Proof By duality. ∎

Remark 3.7 We compare now the results presented above with those obtained by Sogge in $\mathbb{R}^{2n}$.

According to [7], the critical point in $\mathbb{R}^{2n}$ is given by $p_{2n} = \frac{2n}{n+1}$, with critical exponent $\frac{n-1}{2n}$. By interpolation Sogge obtained the following sharp estimate for the norm of the harmonic projector σ_k in $\mathbb{R}^{2n}$

$$\|\sigma_k\|_{(p,2)} = O(k^{r(\frac{1}{p})}), \quad \text{where}$$
$$r\Big(\frac{1}{p}\Big) := \begin{cases} (2n-1)[\frac{1}{p} - \frac{n}{2n-1}] & \text{if } 1 \leq p \leq p_{2n} \\ (n-1)(\frac{1}{p} - \frac{1}{2}) & \text{if } p_{2n} \leq p \leq 2. \end{cases}$$

Then formula (2.3) yields

$$\|\pi_{\ell\ell'}\|_{(p,2)} \leq \|\sigma_{\ell+\ell'}\|_{(p,2)} \leq C \cdot (\ell^{r(\frac{1}{p})}) \tag{3.12}$$

under the assumption (3.1), and it is easy to check that the estimate (3.4) is better that (3.12).

Remark 3.8 According to our result, there is a correspondence between the complex harmonic projectors in $\mathbb{C}^n$ and the real ones in $\mathbb{R}^{2n-1}$; indeed the critical point $p = 2\frac{2n-1}{2n+1}$ given by Theorem 3.5 coincides with the critical point p_{2n-1} found by Sogge in $\mathbb{R}^{2n-1}$.

4 The Harmonic Projectors $\pi_{\ell 0}$ and $\pi_{0\ell}$

In this section we consider the special case of the harmonic projectors $\pi_{\ell 0}$ and $\pi_{0\ell}$.

$\mathcal{H}^{\ell 0}$ and $\mathcal{H}^{0\ell}$ denote respectively the space of polynomials $p(z) = p(z_1, \dots, z_n)$ homogeneous of degree ℓ in $z_1, \dots, z_n$ and the space of the polynomials $p(\bar{z}) = p(\bar{z}_1, \dots, \bar{z}_n)$ homogeneous of degree ℓ in $(\bar{z}_1, \dots, \bar{z}_n)$.

The symbols $\pi_{0\ell}$ and $\pi_{\ell 0}$ will denote the harmonic projections on the spaces $\mathcal{H}^{0\ell}$ and $\mathcal{H}^{\ell 0}$. It may be proved that $\pi_{\ell 0} f = \mathbf{Z}_{\ell 0} * f$ and $\pi_{0\ell} = \mathbf{Z}_{0\ell} * f$, where the zonal harmonics have been defined in Section 2. Since $\mathbf{Z}_{\ell 0}$ is the complex conjugate of $\mathbf{Z}_{0\ell}$, the norms of $\pi_{\ell 0}$ and $\pi_{0\ell}$ coincide.

Observe moreover that $\|\mathbf{Z}_{\ell 0}\|_2 = \sqrt{\frac{\dim \mathcal{H}^{\ell 0}}{\omega_{2n-1}}} = O(\ell^{\frac{n-1}{2}})$.

Theorem 4.1 *Let $n \geq 2$. Then*

$$\|\pi_{\ell 0} f\|_2 \leq C \ell^{(n-1)(\frac{1}{p}-\frac{1}{2})} \|f\|_p \quad \text{for all } 1 \leq p \leq 2, \tag{4.1}$$

where C depends only on p and n and the estimate is sharp.

Proof Exactly as in Theorem 3.4 we have $\|\pi_{\ell 0} f\|_2 \leq \|f\|_2$, and by Young's inequality $\|\pi_{\ell 0} f\|_2 \leq \|\mathbf{Z}_{\ell 0}\|_2 \cdot \|f\|_1 = C\ell^{\frac{n-1}{2}} \|f\|_1$.

Thus (4.1) follows by the Riesz-Thorin convexity theorem.

We prove now by duality arguments that this estimate is sharp. Consider the function $Q^{\ell 0}(z) := (z_1^\ell)_{|S^{2n-1}}$. $Q^{\ell 0}$ belongs to $\mathcal{H}^{\ell 0}$ and

$$\begin{aligned}
\|Q^{\ell 0}\|_q^q &= \int_{S^{2n-1}} |Q^{\ell 0}(\xi)|^q \, d\sigma(\xi) \\
&= C \int_0^{\frac{\pi}{2}} (\sin\theta_{n-1})^{\ell q + 2n-3} \cos\theta_{n-1} \, d\theta_{n-1} \\
&\qquad \times \int_0^{\frac{\pi}{2}} (\sin\theta_{n-2})^{\ell q + 2n-5} \cos\theta_{n-2} \, d\theta_{n-2} \\
&\qquad \times \cdots \times \int_0^{\frac{\pi}{2}} (\sin\theta_2)^{\ell q+3} \cos\theta_2 \, d\theta_2 \cdot \int_0^{\frac{\pi}{2}} (\sin\theta_1)^{\ell q+1} \cos\theta_1 \, d\theta_1 \\
&= C \cdot (\ell q)^{-(n-1)}.
\end{aligned}$$

Thus

$$\frac{\|Q^{\ell 0}\|_q}{\|Q^{\ell 0}\|_2} \sim C \ell^{(\frac{1}{2}-\frac{1}{q})(n-1)}$$

for all $q \geq 2$, showing that (4.1) is sharp. ∎

By duality, the following result is immediate.

Proposition 4.2 *Let $n \geq 2$, $\ell \geq 1$. Then there exists a positive constant $C = C(n)$ such that if P_ℓ is a polynomial, either in z or in $\bar{z}$, homogeneous of degree ℓ, then*

$$\|P_\ell\|_{L^q(S^{2n-1})} \leq C(n) \ell^{(n-1)(\frac{1}{2}-\frac{1}{q})} \|P_\ell\|_{L^2(S^{2n-1})} \quad \text{for all } q \geq 2.$$

Remark 4.3 As observed at the end of Section 3, formula (2.3) yields

$$\|\pi_{\ell 0}\|_{(p,2)} \leq \|\sigma_\ell\|_{(p,2)},$$

so that we could apply the results by Sogge in $\mathbb{R}^{2n}$ and obtain

$$\|\pi_{\ell 0}\|_{(p,2)} \leq C \ell^{r(\frac{1}{p})} \quad \text{for all } 1 \leq p \leq 2, \tag{4.2}$$

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where r has been defined in Remark 3.7. In this case, the exponent $r(\frac{1}{p})$ found by Sogge coincides with the exponent of ℓ given by Theorem 4.1 for $p_{2n} \leq p \leq 2$. However, for $1 \leq p \leq p_{2n}$ it is easy to check that estimate (4.1) is better than (4.2).

Finally, let us consider the special case $n = 2$. In this case, we may find an estimate for $\|\pi_{\ell\ell'}\|_{(p,2)}$ depending on the length of $\ell + \ell'$, uniformly with respect to $\ell, \ell' \geq 0$.

Since $\|\mathbf{Z}_{\ell\ell'}\|_2 = \sqrt{\frac{\ell+\ell'+1}{\omega_3}}$, if we are content with the estimate obtained by interpolating with respect to the trivial endpoint estimates

$$\|\pi_{\ell\ell'}\|_{(1,2)} \leq C(n)(\ell + \ell')^{\frac{1}{2}} \quad \text{and} \quad \|\pi_{\ell\ell'}\|_{(2,2)} \leq 1,$$

we obtain

$$\|\pi_{\ell\ell'}\|_{(p,2)} = O\big((\ell + \ell')^{\frac{1}{p}-\frac{1}{2}}\big) \quad \text{for all } 1 \leq p \leq 2,$$

which coincides with (4.1). Thus the following can be stated.

Proposition 4.4 *If $n = 2$ and $\ell, \ell' \geq 0$, then*

$$\|\pi_{\ell\ell'} f\|_{(p,2)} \leq C(p)(\ell + \ell')^{\frac{1}{p}-\frac{1}{2}} \quad \textit{for all } 1 \leq p \leq 2,$$

and the estimate is sharp if either $\ell = 0$ or $\ell' = 0$.

Appendix

Proof of Lemma 3.1 The proof is inspired by that of [4, Lemma 2] in the framework of compact Lie groups.

We may assume, without loss of generality, that $\operatorname{supp} B \subseteq [\alpha, 2]$, for some $0 < \alpha < 2$.

We recall (see [1, p. 249]) that

$$\mathbf{Z}_j(\cos\theta) = \frac{j + \lambda}{\lambda} \cdot P_j^{(\lambda)}(\cos\theta),$$

where $\lambda = \frac{d-2}{2}$ and $P_j^{(\lambda)}$ denotes the Gegenbauer polynomial, expressed, since the dimension d is even, by

$$P_j^{(\lambda)}(\cos\theta) = 2\gamma_j \sum_{\nu=0}^{\lambda-1} \delta_\nu \frac{\cos[(j - \nu + \lambda)\theta - (\nu + \lambda)\frac{\pi}{2}]}{(2\sin\theta)^{\nu+\lambda}},$$

with $j \in \mathbb{N}$, $\lambda \in \mathbb{N}^*$, $\gamma_j := \binom{j + \lambda - 1}{j}$ and

$$\delta_\nu := \begin{cases} \gamma_\nu \cdot \frac{(1-\lambda)(2-\lambda)\cdots(\nu-\lambda)}{(j+\lambda-1)(j+\lambda-2)\cdots(j+\lambda-\nu)} & \text{if } 1 \leq \nu \leq \lambda - 1 \\ 1 & \text{if } \nu = 0 \end{cases}$$

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(see [9, p. 208]). Thus

$$\Big\| \sum_{\alpha k<j<2k} B\Big(\frac{j}{k}\Big)\mathbf{Z}_j \Big\|_{L^1(\Sigma^{d-1})}$$

$$= \int_0^{\frac{\pi}{2}} \Big| \sum_j B\Big(\frac{j}{k}\Big) \frac{j+\lambda}{\lambda} P_j^{(\lambda)}(\cos\theta) \Big| \sin^{d-2}\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \Big| \sum_j B\Big(\frac{j}{k}\Big) \frac{j+\lambda}{\lambda} 2\gamma_j \sum_{\nu=0}^{\lambda-1} \delta_\nu \times \frac{\cos[(j-\nu+\lambda)\theta-(\nu+\lambda)\frac{\pi}{2}]}{2^{\nu+\lambda}} \sin^{\lambda-\nu}\theta \Big| \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \Big| \sum_j B\Big(\frac{j}{k}\Big) \sum_{\nu=0}^{\lambda-1} c_{j,\nu} \cdot \frac{\cos[(j-\nu+\lambda)\theta-(\nu+\lambda)\frac{\pi}{2}]}{2^{\nu+\lambda}} \sin^{\lambda-\nu}\theta \Big| \, d\theta,$$

where

$$c_{j,\nu} := 2\frac{j+\lambda}{\lambda}\cdot\gamma_j\cdot\delta_\nu.$$

Observe that $c_{j,\nu} < 0$ if and only if ν is odd.

The proof requires now a distinction between the cases $\lambda-\nu$ even and $\lambda-\nu$ odd.

(a) If $\lambda-\nu = 2m$ for some $m \in \mathbb{N}^*$, then $(\lambda+\nu)\frac{\pi}{2} = (\lambda-m)\pi$ and

$$\cos\Big[(j-\nu+\lambda)\theta-(\nu+\lambda)\frac{\pi}{2}\Big] = \begin{cases} \cos\big((j+2m)\theta\big) & \text{if } \lambda-m \text{ even} \\ -\cos\big((j+2m)\theta\big) & \text{if } \lambda-m \text{ odd.} \end{cases}$$

Now, by induction we may easily prove the following formula

$$2^{2m}\cos\big((j+2m)\theta\big)(\sin\theta)^{2m} = (-1)^m\Big[\cos\big((j+4m)\theta\big) - \binom{2m}{1}\cos\big((j+4m-2)\theta\big) + \binom{2m}{2}\cos\big((j+4m-4)\theta\big) + \cdots + \cos(j\theta)\Big].$$

Thus

$$c_{j,\nu}\cdot\frac{\cos[(j-\nu+\lambda)\theta-(\nu+\lambda)\frac{\pi}{2}]}{2^{\nu+\lambda}}\cdot\sin^{\lambda-\nu}\theta$$

$$= \frac{1}{4^\lambda}\cdot c_{j,\nu}\cdot\varepsilon_m\cdot 2^{2m}\cos[(j+2m)\theta]\cdot\sin^{2m}\theta$$

$$= \frac{1}{4^\lambda}\cdot c_{j,\nu}\cdot\varepsilon_m\cdot(-1)^m\Big[\cos\big((j+4m)\theta\big) - \binom{2m}{1}\cos\big((j+4m-2)\theta\big) + \binom{2m}{2}\cos\big((j+4m-4)\theta\big) + \cdots + \cos(j\theta)\Big],$$

where $\varepsilon_m = \pm 1$, according that $\lambda - m$ is even or odd.

It will now be proved that the coefficients $c_{j,\nu} \cdot (-1)^m \cdot \varepsilon_m$ are always positive.

Indeed, if λ is odd, then either $\lambda - m$ is even and m is odd, or $\lambda - m$ is odd and m is even, so that $(-1)^m \cdot \varepsilon_m = -1$. Moreover, since $\lambda - \nu = 2m$ for some $m \in \mathbb{N}$ and λ is odd, ν is odd as well, and therefore the coefficients $c_{j,\nu}$ are negative, as observed above.

On the other hand, if λ is even, then either $\lambda - m$ and m are even or $\lambda - m$ and m are odd, so that $(-1)^m \cdot \varepsilon_m = 1$. In this case the coefficients $c_{j,\nu}$ are positive, since ν is even.

In both cases we conclude that $c_{j,\nu} \cdot (-1)^m \cdot \varepsilon_m > 0$.

Since $|c_{j,\nu}| = O(j^{\lambda-\nu})$, from now on we shall denote $\frac{1}{4^\lambda} \cdot c_{j,\nu} \cdot (-1)^m \cdot \varepsilon_m$ by $C_\nu \cdot j^{2m}$, for some positive constant $C_\nu > 0$.

(b) If $\lambda - \nu = 2m + 1$ for some $m \in \mathbb{N}$, then $\lambda + \nu$ is odd as well, so that

$$\cos\Big[(j - \nu + \lambda)\theta - (\nu + \lambda)\frac{\pi}{2}\Big] = \begin{cases} -\sin\big((j + 2m + 1)\theta\big) & \text{if } \lambda - m \text{ even} \\ \sin\big((j + 2m + 1)\theta\big) & \text{if } \lambda - m \text{ odd.} \end{cases}$$

Observe now that

$$\sin\big((j+2m+1)\theta\big) \cdot \sin^{2m+1}\theta = \frac{1}{2}\Big[\cos\big((j+2m)\theta\big) - \cos\big((j+2m+2)\theta\big)\Big] \cdot \sin^{2m}\theta,$$

so that we may proceed exactly as in (a).

Thus we have

$$\begin{aligned}
&\Big\| \sum_{\alpha k < j < 2k} B\Big(\frac{j}{k}\Big) \mathbf{Z}_j \Big\|_{L^1(\Sigma^{d-1})} \\
&\quad \le \int_0^{\frac{\pi}{2}} \Big| \sum_{\substack{\nu \in [0, \lambda-1] \\ \lambda - \nu = 2m}} \sum_{\alpha k < j < 2k} C_\nu \cdot B\Big(\frac{j}{k}\Big) j^{2m} \Big[\cos\big((j + 4m)\theta\big) \\
&\qquad\qquad - \binom{2m}{1} \cos\big((j + 4m - 2)\theta\big) + \cdots + \cos(j\theta)\Big] \Big| \, d\theta \\
&\qquad + \int_0^{\frac{\pi}{2}} \Big| \sum_{\substack{\nu \in [0, \lambda-1] \\ \lambda - \nu = 2m+1}} \sum_j C_\nu \cdot B\Big(\frac{j}{k}\Big) j^{2m+1} \\
&\qquad\qquad \times \cos\Big[(j - \nu + \lambda)\theta - (\nu + \lambda)\frac{\pi}{2}\Big] \cdot \sin^{\lambda-\nu}\theta \Big| \, d\theta \\
&\quad =: I_e + I_o. \qquad \text{(A1)}
\end{aligned}$$

In the light of what has been observed above for the case $\lambda - \nu$ odd, it suffices to estimate the first term I_e. Moreover, there is no restriction by assuming that $k > \frac{2\lambda}{2-\alpha}$ (so that $k > \frac{4m}{2-\alpha}$ for every m).

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Define

$$\begin{aligned} J_0 &:= (\alpha k + 4m, 2k),\\ J_1^\ell &:= (\alpha k + 4m - 2, \alpha k + 4m],\\ J_1^r &:= [2k, 2k+2),\\ &\vdots\\ J_{2m}^\ell &:= (\alpha k, \alpha k + 2],\\ J_{2m}^r &:= [2k + 4m - 2, 2k + 4m). \end{aligned}$$

For k and m fixed in $\mathbb{N}$, define

$$\begin{aligned} \eta_{k,m}(s) &:= B\Big(\frac{s}{k}\Big)\cdot s^{2m} \quad s \in \mathbb{R},\\ a_{k,m}(s) &:= \eta_{k,m}(s-4m) - \binom{2m}{1}\eta_{k,m}(s-4m+2) + \cdots + \eta_{k,m}(s), \quad s \in \mathbb{R}_+ \text{ and}\\ b_{k,m}(s) &:= a_{k,m}(|s|), \quad s \in \mathbb{R}. \end{aligned}$$

For notational simplicity, we replace the symbols $\eta_{k,m}$, $a_{k,m}$ and $b_{k,m}$ by η, a and b.

Then

$$\begin{aligned} &\sum_{\alpha k < j < 2k} B\Big(\frac{j}{k}\Big) j^{2m}\Big[\cos\big((j+4m)\theta\big) - \binom{2m}{1}\cos\big((j+4m-2)\theta\big) + \cdots + \cos(j\theta)\Big]\\ &= \sum_{j\in J_0} a(j)\cos(j\theta) + \sum_{j \in J_1^\ell}\Big[-\binom{2m}{1}B\Big(\frac{j-4m+2}{k}\Big)\cdot(j-4m+2)^{2m}\\ &\qquad + \binom{2m}{2}B\Big(\frac{j-4m+4}{k}\Big)\\ &\qquad \times (j-4m+4)^{2m} + \cdots + B\Big(\frac{j}{k}\Big)\cdot j^{2m}\Big]\cos(j\theta)\\ &\quad + \sum_{j\in J_1^r}\Big[B\Big(\frac{j-4m}{k}\Big)\cdot(j-4m)^{2m} - \binom{2m}{1}B\Big(\frac{j-4m+2}{k}\Big)\\ &\qquad \times (j-4m+2)^{2m} + \cdots - \binom{2m}{2m-1}B\Big(\frac{j-2}{k}\Big)\cdot(j-2)^{2m}\Big]\cos(j\theta)\\ &\quad + \cdots + \sum_{j\in J_{2m}^\ell} B\Big(\frac{j}{k}\Big) j^{2m}\cos(j\theta) + \sum_{j \in J_{2m}^r} B\Big(\frac{j-4m}{k}\Big)\cdot(j-4m)^{2m}\cos(j\theta)\\ &= \sum_{j\in J_0} a(j)\cos(j\theta) + \sum_{j\in J_1^\ell} a(j)\cos(j\theta) + \sum_{j \in J_1^r} a(j)\cos(j\theta) \end{aligned}$$

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$$+\cdots+\sum_{j\in J^{\ell}_{2m}} a(j)\cos(j\theta)+\sum_{j\in J^{r}_{2m}} a(j)\cos(j\theta)$$
$$=\sum_{\alpha k<j<2k+4m} a(j)\cos(j\theta)=\sum_{j\in\mathbb{N}} a(j)\cos(j\theta)=\frac{1}{2}\sum_{j\in\mathbf{Z}} b(j)e^{ij\theta},$$

where we repeatedly used the fact that the supports of B and a are contained, respectively, in $(\alpha,2)$ and in $(\alpha k, 2k+4m)$.

Now

$$I_e := \int_0^{\frac{\pi}{2}} \Big| \sum_{\substack{\nu\in[0,\lambda-1]\\ \lambda-\nu=2m}} C_\nu \sum_{j\in\mathbf{Z}} b(j)e^{ij\theta}\Big|\, d\theta$$
$$\leq C \sum_{\substack{\nu\in[0,\lambda-1]\\ \lambda-\nu=2m}} \Big\|\sum_{j\in\mathbf{Z}} b(j)e^{ij\theta}\Big\|_{L^1(\mathbb{T})}.$$

Observe that the function b belongs to the space $\mathcal{C}^\infty_c(\mathbb{R})$. Let $h_{k,m}$ (in the following, h) be the function in $\mathcal{C}^\infty(\mathbb{R})$ whose Fourier transform is

$$\mathfrak{F}h(s) := b(s).$$

By the Poisson summation formula ([8, p. 250]) we have

$$\Big\|\sum_j b(j)e^{ij\theta}\Big\|_{L^1(\mathbb{T})} \leq \|h\|_{L^1(\mathbb{R})}. \tag{A2}$$

Since the support of $\mathfrak{F}h$ is contained in an interval of radius r_k comparable with k, from the Plancherel's theorem we have

$$\|h\|_{L^1(\mathbb{R})} \leq C\big(\|\mathfrak{F}h\|_\infty + k\cdot\|(\mathfrak{F}h)'\|_\infty\big), \tag{A3}$$

where C does not depend on k or m (for a more general version of this result, due to Hörmander, see [4, Lemma 1]), so that in order to estimate I_e we have only to estimate $\|b\|_\infty$ and $\|b'\|_\infty$.

By the mean value theorem ([6, p. 52, pb. 98]) we have

$$|b(s)| \leq C_m \cdot |\eta^{(2m)}(\tau)|,$$

for all $s\in\mathbb{R}$, for some $\tau\in(-r_k, r_k)$ and some positive constant C_m, and therefore by a direct computation $\|b\|_\infty \leq C_m$.

Assume for simplicity $s > 0$. Then

$$\begin{aligned} b'(s) = \frac{1}{k}\Big[B'\Big(\frac{s-4m}{k}\Big)\cdot(s-4m)^{2m} - \binom{2m}{1} B'\Big(\frac{s-4m+2}{k}\Big)\cdot(s-4m+2)^{2m} \\ + \binom{2m}{2} B'\Big(\frac{s-4m+4}{k}\Big)\cdot(s-4m+4)^{2m} + \cdots + B'\Big(\frac{s}{k}\Big)\cdot s^{2m}\Big] \\ + 2m\Big[B\Big(\frac{s-4m}{k}\Big)\cdot(s-4m)^{2m-1} - \binom{2m}{1} B\Big(\frac{s-4m+2}{k}\Big) \\ \times (s-4m+2)^{2m-1} + \cdots + B\Big(\frac{s}{k}\Big)\cdot s^{2m-1}\Big] \\ =: S_1 + S_2. \end{aligned}$$

Observe that $|S_1| \le \frac{C_m}{k}$, since the expression in the square brackets is similar to b, with the function B replaced by B', and therefore may be treated as b.

In order to estimate S_2, we may define the function $\tilde{\eta}_{k,m}(s) := B(\frac{s}{k})s^{2m-1}$; by using again [6, p. 52, pb. 98] and by arguing as above, we easily check that $|S_2| \le \frac{C_m}{k}$.

It follows from (A3) that

$$\|h\|_{L^1(\mathbb{R})} \le C_m,$$

where C_m does not depend on k. Then by (A2) we have that

$$I_e \le C \sum_{\substack{\nu\in[0,\lambda-1]\\ \lambda-\nu=2m}} \Big\| \sum_{j\in\mathbf{Z}} b(j)e^{ij\theta} \Big\|_{L^1(\mathbb{T})} \le C',$$

for some constant C' independent of k.

Since, as observed before, an analogous estimate may be given for the term I_o, (A1) yields the thesis. ■

References

[1] A. Bonami and J. L. Clerc, *Sommes de Cesàro et moltiplicateurs des developpements en harmonic sphériques*. Trans. Amer. Math. Soc. **183**(1973), 223–263.

[2] I. M. Gelfand and G. E. Shilov, *Generalized Functions*. Academic Press, N. Y., 1964.

[3] E. Giacalone, *Stime uniformi per proiettori armonici su gruppi di Lie compatti di rango* 2. Boll. Un. Mat. Ital. B (7) **6**(1992), 205–216.

[4] E. Giacalone and F. Ricci, *Norms of harmonic projection operators on compact Lie groups*. Math. Ann. **280**(1988), 21–31.

[5] A. U. Klimyk and N. Ja. Vilenkin, *Representation of Lie Groups and Special Functions*. Kluwer Academic Publishers, 1993.

[6] G. Pólya and G. Szegö, *Problems and Theorems in Analysis, Vol. II*. Springer Verlag, Fourth Ed., 1971.

[7] C. Sogge, *Oscillatory integrals and spherical harmonics*. Duke Math. J. **53**(1986), 43–65.

[8] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, N.J., 1971.

[9] G. Szegö, *Orthogonal Polynomials*. Amer. Math. Soc. Colloq. Publ. **23**, Amer. Math. Soc., 4th ed., Providence, R.I., 1974.
[10] M. Taylor, *Pseudodifferential Operators*. Princeton University Press, Princeton, N.J., 1981.

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