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AN EXTENSION OF SURY'S IDENTITY AND RELATED CONGRUENCES

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Abstract

In this paper we give an extension of a curious combinatorial identity due to B. Sury. Our proof is very simple and elementary. As an application, we obtain two congruences for Fermat quotients modulo p^3. Moreover, we prove an extension of a result by H. Pan that generalizes Carlitz's congruence.

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1. Introduction

For nonnegative integers n and m the binomial coefficients are defined by

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & n \geq k \\ 0 & n < k. \end{cases}$$

Sury [13] proved a most curious combinatorial identity, namely

$$\sum_{j=0}^{n} \frac{1}{\binom{n}{j}} = \frac{n+1}{2^n} \sum_{k=0}^{n} \frac{2^k}{k+1} = \frac{n+1}{2^n} \sum_{\substack{1 \leq j \leq n+1 \\ j \text{ odd}}} \frac{1}{j}\binom{n+1}{j}. \tag{1.1}$$

A polynomial analogue of the first identity in (1.1) is established in [14, Theorem 2.1] by Sury *et al.* In the proof of both above identities, as well as in the proof of [14, Theorem 2.1], the authors use the following integral expression for the inverses of binomial coefficients $\binom{n}{k}$ with $0 \leq k \leq n$:

$$\binom{n}{k}^{-1} = \frac{k!(n-k)!}{n!} = \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+1)} = (n+1)\int_0^1 t^k(1-t)^{n-k}\, dt.$$

We point out that the first equation of the identity (1.1) is identity no. 2.25 in Gould's collection [4] (due to Staver [8]). Combinatorial sums involving inverses of binomial coefficients have been studied by many authors; see, for instance, [16–18].

For a nonnegative integer n, let

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \quad n \geq 1,$$

be the nth *harmonic number* (we assume that $H_0 = 0$). In this note we establish a double extension of the identity (1.1) as follows.

THEOREM 1.1. *For any positive integer n,*

$$\frac{2^n}{n+1} \cdot \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \sum_{k=0}^{n} \frac{2^k}{k+1} = \sum_{\substack{1 \leq j \leq n+1 \\ j \text{ odd}}} \frac{1}{j}\binom{n+1}{j}$$
$$= \frac{1}{2}\sum_{i=1}^{n+1} \frac{1}{i}\binom{n+1}{i} + \frac{1}{2}H_{n+1} = 2^n H_{n+1} - \sum_{k=1}^{n} 2^{k-1} H_k. \tag{1.2}$$

In the next section we give a complete proof of Theorem 1.1. More precisely, if we denote the expressions in (1.2) by A, B, C, D and E in given order, then we prove the identities $C = D$, $D = B$, $B = E$ and $E = A$. The proof of each identity is derived easily by induction on n.

In Section 3 we present the proof of Theorem 1.5. Our proof is combinatorial in spirit and we additionally use some known congruences modulo prime powers related to harmonic numbers. As is shown at the end of this section, Theorem 1.7 easily follows by combining the first congruence of (1.3) and the first identity in (1.2).

The equalities $C = D = B$ in (1.2) clearly yield the following identity.

COROLLARY 1.2 [14, Corollary 2.3, identity (13)]. *For any positive integer n,*

$$\sum_{\substack{1 \leq j \leq n+1 \\ j \text{ even}}} \frac{1}{j}\binom{n+1}{j} = \sum_{k=1}^{n} \frac{2^k - 1}{k+1}.$$

REMARK 1.3. The proof of the first equation in the identity (1.2) immediately follows from the known binomial harmonic identity (2.2) given in Lemma 2.2 (see for example, identity no. 1.45 on page 6 in Gould's listing [4]). Observe also that this identity can itself be established directly from a few other results in [4] (for instance, setting $a = 1$ in identity no. 1.134, p. 17, or taking $m = n$ in identity no. 7.17, p. 60). Moreover, in Theorem 1.1 we give a simple induction proof of this identity which is based on another binomial coefficient identity.

REMARK 1.4. A natural question is as follows: is it possible to establish certain identities analogous to (1.2) involving the sum $\sum_{k=0}^{n} a^k/(k+1)$ with some integer $a \neq 2$?

For an integer a not divisible by p let $q_p(a) := (a^{p-1} - 1)/p$ be the Fermat quotient of p with base a. Then using the binomial formula,

$$\begin{aligned} aq_p(a) &= \frac{a^p - a}{p} = \frac{((a+1)-1)^p - a}{p} \\ &= \sum_{k=1}^{p-1} \binom{p}{k} \frac{(a+1)^k(-1)^{p-k}}{p} + \frac{(a+1)^p - (a+1)}{p} \\ &= \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}(a+1)^k(-1)^{p-k} + (a+1)q_p(a+1), \end{aligned}$$

whence, using the congruence

$$\frac{1}{p}\binom{p}{k} = \frac{1}{k}\binom{p-1}{k-1} \equiv \frac{(-1)^{k-1}}{k} \pmod{p}$$

(see (3.3) of Lemma 3.2),

$$aq_p(a) \equiv -\sum_{k=1}^{p-1} \frac{(a+1)^k}{k} + (a+1)q_p(a+1) \pmod{p},$$

which for $a = 1$ becomes Glaisher's congruence given in Remark 1.6, while for $a = -2$ it becomes the first part of the second congruence given in Remark 1.6.

Inductively on $a \geq 1$, the above congruence yields

$$q_p(a) \equiv -\frac{1}{a} \sum_{k=1}^{p-1} \frac{1^k + 2^k + \cdots + a^k}{k} \pmod{p}.$$

Analogously, for a negative integer $-a$ (instead of a), also by induction on $a \geq 1$,

$$q_p(a) \equiv -\frac{1}{a} \sum_{k=1}^{p-1} \frac{(-1)^k(1^k + 2^k + \cdots + (a-1)^k)}{k} \pmod{p}.$$

Notice that the term of the sum on the right-hand side of either of the above two congruences can also be expressed in terms of the Bernoulli polynomial of degree k. We believe that the above two congruences may be useful for establishing congruences involving the Fermat quotient $q_p(a)$ with $a \geq 3$ that are analogous to the congruences given in our Theorem 1.5.

As an application of Theorem 1.1, we establish the following two congruences.

THEOREM 1.5. *Let $p > 3$ be a prime. Then*

$$q_p(2) \equiv -\frac{1}{2} \sum_{k=1}^{p-1} \frac{2^k}{k} - \frac{7}{24} p^2 B_{p-3} \equiv \sum_{k=1}^{p-1} 2^{k-1} H_k + \frac{1}{24} p^2 B_{p-3} \pmod{p^3}. \tag{1.3}$$

In particular,

$$q_p(2) \equiv -\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k} \equiv \sum_{k=1}^{p-1} 2^{k-1}H_k \pmod{p^2}.$$

REMARK 1.6. Note that the first congruence of (1.3) was recently obtained by Sun in [10, Theorem 4.1(i)] using congruential properties of the Mirimanoff polynomial associated with p. We point out that this congruence is a generalization of a classical result due to Glaisher (see [3] or [5]) which asserts that for a prime $p \geq 3$,

$$q_p(2) \equiv -\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k} \pmod{p}.$$

Note also that the above congruence may be extended by the following well-known congruences (see, for example, [12, proof of Corollary 1.2]; see also our Remark 1.4)

$$q_p(2) \equiv \frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k} \equiv -\frac{1}{2}\sum_{j=1}^{(p-1)/2}\frac{1}{j} \pmod{p}.$$

Recently, Pan [7, Theorem 1.1] established a generalization of Carlitz's congruence [2] of the form

$$\sum_{k=0}^{p-1}(-1)^{(a-1)k}\binom{p-1}{k}^a \equiv 2^{a(p-1)} + \frac{a(a-1)(3a-4)}{48}p^3B_{p-3} \pmod{p^4}, \qquad (1.4)$$

where p is an odd prime and a a positive integer. Note that this congruence modulo p^3 is proved by Cai and Granville in [1, Theorem 6], and their result generalized Morley's congruence [6].

Surprisingly, here we show that the above congruence holds for $a = -1$; in other words, we have the following congruences for the sum of the reciprocals of binomial coefficients.

THEOREM 1.7. *Let $p \geq 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}^{-1} \equiv 2^{1-p} - \frac{7}{24}p^3B_{p-3} \pmod{p^4}.$$

In particular,

$$\sum_{k=0}^{p-1}\binom{p-1}{k}^{-1} \equiv 2^{1-p} \pmod{p^3}.$$

REMARK 1.8. Since

$$-1+2^{1-p} = -1+\frac{1}{1+pq_p(2)}$$
$$= -\frac{pq_p(2)}{1+pq_p(2)} \equiv -pq_p(2)(1-pq_p(2)+p^2q_p(2)^2) \pmod{p^4},$$

the first congruence of Theorem 1.7 may be written as

$$\sum_{k=0}^{p-1}\binom{p-1}{k}^{-1} \equiv 1-pq_p(2)+p^2q_p(2)^2-p^3q_p(2)^3-\frac{7}{24}p^3B_{p-3} \pmod{p^4},$$

which setting $pq_p(2)=2^{p-1}-1$ reduces to

$$\sum_{k=0}^{p-1}\binom{p-1}{k}^{-1} \equiv 4-6\cdot 2^{p-1}+4^p-8^{p-1}-\frac{7}{24}p^3B_{p-3} \pmod{p^4},$$

while modulo p^3 and p^2 we have

$$\sum_{k=0}^{p-1}\binom{p-1}{k}^{-1} \equiv 3-3\cdot 2^{p-1}+4^{p-1} \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\binom{p-1}{k}^{-1} \equiv 2-2^{p-1} \pmod{p^2}.$$

REMARK 1.9. A computation in Mathematica suggests that the congruence (1.4) holds for all odd primes p when a is any negative integer. We point out that this result can be proved by using Pan's technique applied in [7, proof of Theorem 1.1].

It is also interesting to note that Cai and Granville [1, Theorem 6] also showed that for any prime $p>5$ and any positive integer a,

$$\sum_{k=0}^{p-1}(-1)^{ak}\binom{p-1}{k}^{a} \equiv \binom{ap-2}{p-1} \pmod{p^4}.$$

Recently, using some new identities for multiple harmonic sums, Tauraso [15, Theorem 1.1] generalized the above congruence modulo p^6, where a ranges over the set of all integers.

In the next two sections we give proofs of all auxiliary results and the proofs of Theorems 1.1, 1.5 and 1.7.

2. Proof of Theorem 1.1

LEMMA 2.1. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k} = \frac{1}{n+1}. \tag{2.1}$$

PROOF. By the identity

$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}$$

with $0 \le k \le n$, and the fact that by the binomial formula,

$$0 = (1-1)^{n+1} = \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j},$$

we find that

$$\begin{aligned}\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n+1}{k} &= \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1} \\ &= -\frac{1}{n+1}\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j} + \frac{1}{n+1} = \frac{1}{n+1},\end{aligned}$$

as desired. □

LEMMA 2.2 [4, identity no. 1.45, p. 6]. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k} = H_n. \tag{2.2}$$

PROOF. By induction on n, assuming that (2.2) holds, using the identity $\binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1}$ and applying (2.1),

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k}\binom{n+1}{k} - H_n &= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k}\binom{n+1}{k} - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k} \\ &= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k}\left(\binom{n+1}{k} - \binom{n}{k}\right) \\ &= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k}\binom{n}{k-1} \\ &= \sum_{k=0}^{n} \frac{(-1)^{k}}{k+1}\binom{n}{k} = \frac{1}{n+1},\end{aligned}$$

whence

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k}\binom{n+1}{k} = H_n + \frac{1}{n+1} = H_{n+1}.$$

This concludes the induction proof. □

Proof of Theorem 1.1. Denote the expressions from (1.2) by A, B, C, D and E in the corresponding order. We prove by induction the identities $C = D$, $D = B$, $B = E$ and $E = A$. First note that all these identities are satisfied for $n = 1$.

The identity $C = D$ from (1.2) may be reduced to

$$\frac{1}{2}\sum_{\substack{0\le j\le n+1\\ j \text{ odd}}}\frac{1}{j}\binom{n+1}{j}-\frac{1}{2}\sum_{\substack{0\le j\le n+1\\ j \text{ even}}}\frac{1}{j}\binom{n+1}{j}=\frac{1}{2}H_{n+1},$$

or equivalently,

$$\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n+1}{k}=H_{n+1}.$$

The last identity follows from (2.2) with $n+1$ instead of n, and so the identity $C = D$ is satisfied.

Further, using the well-known identities

$$\binom{k}{i}=\frac{k}{i}\binom{k-1}{i-1},\qquad \sum_{k=i}^{n}\binom{k-1}{i-1}=\binom{n}{i},$$

with $i \le k \le n$, and the fact that $\binom{k}{i} = 0$ when $i > k$,

$$\begin{aligned}
\sum_{k=1}^{n}\frac{2^k}{k}&=\sum_{k=1}^{n}\frac{(1+1)^k}{k}=\sum_{k=1}^{n}\frac{\sum_{i=0}^{k}\binom{k}{i}}{k}=\sum_{k=1}^{n}\frac{\sum_{i=1}^{k}\binom{k}{i}+1}{k}\\
&=\sum_{k=1}^{n}\sum_{i=1}^{n}\frac{1}{k}\binom{k}{i}+\sum_{k=1}^{n}\frac{1}{k}=\sum_{i=1}^{n}\sum_{k=i}^{n}\frac{1}{k}\binom{k}{i}+H_n\\
&=\sum_{i=1}^{n}\sum_{k=i}^{n}\frac{1}{i}\binom{k-1}{i-1}+H_n=\sum_{i=1}^{n}\frac{1}{i}\sum_{k=i}^{n}\binom{k-1}{i-1}+H_n\\
&=\sum_{i=1}^{n}\frac{1}{i}\binom{n}{i}+H_n.
\end{aligned}$$

The above identity with $n+1$ instead of n immediately gives

$$\sum_{k=0}^{n}\frac{2^k}{k+1}=\frac{1}{2}\sum_{k=1}^{n+1}\frac{2^k}{k}=\frac{1}{2}\sum_{i=1}^{n+1}\frac{1}{i}\binom{n+1}{i}+\frac{1}{2}H_{n+1},$$

which implies $B = D$.

In order to prove $B = E$, note that by setting $H_{n+1} = H_n + 1/(n+1)$, this identity can be written as

$$\sum_{k=0}^{n-1}\frac{2^k}{k+1}=2^nH_n-\sum_{k=1}^{n}2^{k-1}H_k. \tag{2.3}$$

If (2.3) is satisfied, then by induction we have

$$\begin{aligned}\sum_{k=0}^{n}\frac{2^k}{k+1} &= \sum_{k=0}^{n-1}\frac{2^k}{k+1}+\frac{2^n}{n+1} = 2^nH_n-\sum_{k=1}^{n}2^{k-1}H_k+\frac{2^n}{n+1}\\ &= 2^nH_n+2^nH_{n+1}-\sum_{k=1}^{n+1}2^{k-1}H_k+\frac{2^n}{n+1}\\ &= 2^nH_n+2^n\Big(H_n+\frac{1}{n+1}\Big)+\frac{2^n}{n+1}-\sum_{k=1}^{n+1}2^{k-1}H_k\\ &= 2^{n+1}\Big(H_n+\frac{1}{2(n+1)}+\frac{1}{2(n+1)}\Big)-\sum_{k=1}^{n+1}2^{k-1}H_k\\ &= 2^{n+1}\Big(H_n+\frac{1}{n+1}\Big)-\sum_{k=1}^{n+1}2^{k-1}H_k\\ &= 2^{n+1}H_{n+1}-\sum_{k=1}^{n+1}2^{k-1}H_k,\end{aligned}$$

as desired.

It remains to prove $E = A$. Suppose that this identity holds for $n-1$, that is,

$$\frac{2^{n-1}}{n}\cdot\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}} = 2^{n-1}H_n-\sum_{k=1}^{n-1}2^{k-1}H_k. \tag{2.4}$$

Note that by the identities

$$\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1} \quad\text{and}\quad \binom{n}{k+1}=\frac{n}{k+1}\binom{n-1}{k}$$

with $0 \le k \le n-1$,

$$\begin{aligned}\frac{1}{n+1}\left(\frac{1}{\binom{n}{k+1}}+\frac{1}{\binom{n}{k}}\right) &= \frac{1}{n+1}\cdot\frac{\binom{n}{k}+\binom{n}{k+1}}{\binom{n}{k+1}\cdot\binom{n}{k}} = \frac{1}{n+1}\cdot\frac{\binom{n+1}{k+1}}{\binom{n}{k+1}\cdot\binom{n}{k}}\\ &= \frac{1}{n+1}\cdot\frac{\frac{n+1}{k+1}\cdot\binom{n}{k}}{\frac{n}{k+1}\cdot\binom{n-1}{k}\cdot\binom{n}{k}} = \frac{1}{n}\cdot\frac{1}{\binom{n-1}{k}}.\end{aligned} \tag{2.5}$$

Next, summing (2.5) over k,

$$\frac{1}{n+1}\left(\sum_{k=0}^{n-1}\frac{1}{\binom{n}{k+1}}+\sum_{k=0}^{n-1}\frac{1}{\binom{n}{k}}\right)=\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}},$$

or equivalently,

$$\frac{1}{n+1}\left(2\sum_{k=0}^{n}\frac{1}{\binom{n}{k}}-2\right)=\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}}. \tag{2.6}$$

Then multiplying (2.6) by 2^{n-1}, and combining this with the induction hypothesis given by (2.4),

$$\begin{aligned}\frac{2^n}{n+1}\cdot\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} &= \frac{2^n}{n+1}+\frac{2^{n-1}}{n}\cdot\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}}\\ &= \frac{2^n}{n+1}+2^{n-1}H_n-\sum_{k=1}^{n-1}2^{k-1}H_k\\ &= \frac{2^n}{n+1}+2^{n-1}H_n+2^{n-1}H_n-\sum_{k=1}^{n}2^{k-1}H_k\\ &= 2^n\left(H_n+\frac{1}{n+1}\right)-\sum_{k=1}^{n}2^{k-1}H_k\\ &= 2^nH_{n+1}-\sum_{k=1}^{n}2^{k-1}H_k.\end{aligned}$$

This concludes the induction proof. □

3. Proof of Theorems 1.5 and 1.7

For the proof of Theorem 1.5, we need some auxiliary results.

LEMMA 3.1. *Let p be an odd prime. Then*

$$q_p(2)=\frac{1}{2}\sum_{k=1}^{p-1}\frac{1}{k}\binom{p}{k}+\frac{1}{2}H_{p-1}-\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k}. \tag{3.1}$$

PROOF. Taking $n=p-1$ into the identity $D=B$ from (1.2),

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{p}\frac{1}{k}\binom{p}{k}+\frac{1}{2}H_p &= \frac{1}{2}\sum_{j=0}^{p-1}\frac{2^{j+1}}{j+1}\\ &= \frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k}+\frac{1}{2}\cdot\frac{2^p}{p}.\end{aligned} \tag{3.2}$$

Substituting $H_p=H_{p-1}+1/p$ into (3.2), we immediately obtain (3.1). □

LEMMA 3.2. *If p is an odd prime, then*

$$\binom{p-1}{k} \equiv (-1)^k - (-1)^k pH_k \pmod{p^2} \tag{3.3}$$

for each $k = 1, 2, \dots, p-1$.

PROOF. For a fixed $1 \le k \le p-1$ we have

$$(-1)^k\binom{p-1}{k} = \prod_{i=1}^{k}\Big(1-\frac{p}{i}\Big) \equiv 1 - \sum_{i=1}^{k}\frac{p}{i} \pmod{p^2},$$

which is actually the congruence (3.3). □

Recall that the *Bernoulli numbers* B_k are defined by the generating function

$$\sum_{k=0}^{\infty} B_k\frac{x^k}{k!} = \frac{x}{e^x-1}.$$

It is easy to find the values $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_n = 0$ for odd $n \ge 3$. Furthermore, $(-1)^{n-1}B_{2n} > 0$ for all $n \ge 1$.

LEMMA 3.3. *Let $p > 3$ be a prime and let $m \le p-3$ be a positive integer. Then*

$$\sum_{k=1}^{p-1}\frac{1}{k^m} \equiv \begin{cases} -\dfrac{m(m+1)}{2(m+2)}p^2B_{p-2-m} \pmod{p^3} & \text{if } m \text{ is odd} \\ \dfrac{m}{m+1}pB_{p-1-m} \pmod{p^2} & \text{if } m \text{ is even,} \end{cases} \tag{3.4}$$

$$\sum_{j=1}^{(p-1)/2}\frac{1}{j^2} \equiv \frac{7}{3}pB_{p-3} \pmod{p^2}, \tag{3.5}$$

$$\sum_{j=1}^{(p-1)/2}\frac{1}{j^3} \equiv -2B_{p-3} \pmod{p}. \tag{3.6}$$

PROOF. The first congruence in [9, Theorem 5.1(a)] asserts that

$$\sum_{k=1}^{p-1}\frac{1}{k^m} \equiv \frac{m(m+1)}{2}\cdot\frac{B_{p-2-m}}{p-2-m}p^2 \pmod{p^3} \quad \text{if } m < p-1,$$

which immediately implies the first congruence from (3.4). Similarly, the second congruence in (3.4) is in fact the first congruence in [9, Corollary 5.1] with even $k \le p-3$.

The congruences (3.5) and (3.6) are the congruences (a) with $k = 2$ and (b) with $k = 3$ in [9, Corollary 5.2], respectively. □

Lemma 3.4 [11, (2.1) in Lemma 2.1]. *Let p be an odd prime. Then*

$$H_{p-k} \equiv H_{k-1} \pmod{p} \tag{3.7}$$

for every $k = 1, \dots, p-1$.

Proof. For a fixed k we have

$$H_{p-k} = \sum_{j=1}^{p-k} \frac{1}{j} = \sum_{i=k}^{p-1} \frac{1}{p-i} \equiv -\sum\nolimits_{i=k}^{p-1} \frac{1}{i} \pmod{p},$$

whence

$$H_{k-1} - H_{p-k} \equiv \sum_{i=1}^{k-1} \frac{1}{i} + \sum_{i=k}^{p-1} \frac{1}{i} = H_{p-1} \equiv 0 \pmod{p}.$$

This yields (3.7). □

Lemma 3.5. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^2} H_{k-1} \equiv -\frac{1}{4} B_{p-3} \pmod{p}. \tag{3.8}$$

Proof. Substituting $H_{k-1} = H_k - 1/k$ into (3.7) of Lemma 3.4 and multiplying this by $(-1)^k/k^2$, for $k = 1, \dots, p-1$,

$$\frac{(-1)^k}{k^2} H_k - \frac{(-1)^k}{k^3} \equiv \frac{(-1)^k}{k^2} H_{p-k} = -\frac{(-1)^{p-k}}{k^2} H_{p-k} \pmod{p},$$

whence, because $k^2 \equiv (p-k)^2 \pmod{p}$,

$$\frac{(-1)^k}{k^2} H_k + \frac{(-1)^{p-k}}{(p-k)^2} H_{p-k} \equiv \frac{(-1)^k}{k^3} \pmod{p}. \tag{3.9}$$

Summing (3.9) over k,

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} H_k + \sum_{k=1}^{p-1} \frac{(-1)^{p-k}}{(p-k)^2} H_{p-k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \pmod{p},$$

or

$$2 \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} H_k \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \pmod{p}. \tag{3.10}$$

Further, using (3.4) with $m = 3$ and (3.6) of Lemma 3.3,

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} &= 2 \sum_{\substack{1 \le j \le p-1 \\ 2|k}} \frac{1}{k^3} - \sum_{k=1}^{p-1} \frac{1}{k^3} \\ &= \frac{1}{4} \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} - \sum_{k=1}^{p-1} \frac{1}{k^3} \equiv -\frac{1}{2} B_{p-3} \pmod{p}\end{aligned}$$

which inserting in (3.10) immediately yields

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} H_k \equiv -\frac{1}{4} B_{p-3} \pmod{p}.$$

Finally, taking $H_{k-1} = H_k - 1/k$, the previous two congruences give

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^2} H_{k-1} &= \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^2} H_k - \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^3} \\ &= -\sum\nolimits_{k=1}^{p-1} \frac{(-1)^k}{k^2} H_k + \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \\ &\equiv \frac{1}{4} B_{p-3} - \frac{1}{2} B_{p-3} = -\frac{1}{4} B_{p-3} \pmod{p}.\end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.5. Using the identity $\binom{p}{k} = (p/k)\binom{p-1}{k-1}$, the congruence (3.3) from Lemma 3.2 yields

$$\frac{1}{k}\binom{p}{k} = \frac{p}{k^2}\binom{p-1}{k-1} \equiv \frac{(-1)^{k-1}}{k^2} p - \frac{(-1)^{k-1}}{k^2} p^2 H_{k-1} \pmod{p^3} \tag{3.11}$$

for each $k = 1, 2, \ldots, p-1$. Summing (3.11) over k and using the congruences (3.4) with $m = 2$, (3.5) and (3.8),

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{1}{k}\binom{p}{k} &\equiv p \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^2} - p^2 \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^2} H_{k-1} \pmod{p^3} \\ &= p\Big(\sum_{k=1}^{p-1} \frac{1}{k^2} - 2 \sum_{\substack{1\le j\le p-1 \\ 2\mid k}} \frac{1}{k^2}\Big) - p^2 \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^2} H_{k-1} \\ &= p\Big(\sum_{k=1}^{p-1} \frac{1}{k^2} - \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k^2}\Big) - p^2 \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^2} H_{k-1} \\ &\equiv p\Big(\frac{2}{3} p B_{p-3} - \frac{1}{2} \cdot \frac{7}{3} p B_{p-3}\Big) + \frac{1}{4} p^2 B_{p-3} = -\frac{1}{4} p^2 B_{p-3} \pmod{p^3}.\end{aligned} \tag{3.12}$$

Finally, since by (3.4) with $m = 1$,

$$H_{p-1} = \sum_{k=1}^{p-1} \frac{1}{k} \equiv -\frac{1}{3} p^2 B_{p-3} \pmod{p^3},$$

substituting this and (3.12) into the identity (3.1),

$$
\begin{aligned}
q_p(2) &\equiv -\frac{1}{2}\cdot\frac{1}{4}p^2B_{p-3} - \frac{1}{2}\cdot\frac{1}{3}p^2B_{p-3} - \frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k}\\
&= -\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k} - \frac{7}{24}p^2B_{p-3} \pmod{p^3}.
\end{aligned}
\tag{3.13}
$$

This proves the first congruence of (1.3).

To prove the second congruence of (1.3), note that the last identity in (1.2) for $n = p-1$ easily reduces to

$$
\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k} = 2^{p-1}H_{p-1} - \sum_{k=1}^{p-1}2^{k-1}H_k,
$$

whence, setting $2^{p-1}\equiv 1 \pmod{p}$ and $H_{p-1}\equiv -\frac{1}{3}p^2B_{p-3} \pmod{p)^3}$,

$$
\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k} \equiv -\frac{1}{3}p^2B_{p-3} - \sum_{k=1}^{p-1}2^{k-1}H_k \pmod{p^3},
$$

as desired. This concludes the proof. □

Proof of Theorem 1.7. A simple calculation shows that both congruences in Theorem 1.7 hold for $p = 3$. Assume that $p \geq 5$. Note that the first identity in (1.2) for $n = p-1$ easily reduces to

$$
\frac{2^{p-1}}{p}\cdot\sum_{k=1}^{p-1}\frac{1}{\binom{p-1}{k}} = \frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k}. \tag{3.14}
$$

Multiplying (3.14) by $p\cdot 2^{1-p}$,

$$
\sum_{k=1}^{p-1}\frac{1}{\binom{p-1}{k}} = \frac{p\cdot 2^{1-p}}{2}\cdot\sum_{k=1}^{p-1}\frac{2^k}{k}. \tag{3.15}
$$

Further, from the first congruence of (1.3) in Theorem 1.5,

$$
\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k} \equiv -q_p(2) - \frac{7}{24}p^2B_{p-3} \pmod{p^3}.
$$

Substituting this into (3.15),

$$
\sum_{k=1}^{p-1}\frac{1}{\binom{p-1}{k}} \equiv -p2^{1-p}\Big(q_p(2) + \frac{7}{24}p^2B_{p-3}\Big) \pmod{p^4},
$$

whence using the identity $p \cdot 2^{1-p} \cdot q_p(2) = 1 - 2^{1-p}$ and the fact that by Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod{p}$,

$$\sum_{k=1}^{p-1} \frac{1}{\binom{p-1}{k}} \equiv -p \cdot 2^{1-p} \cdot q_p(2) - \frac{7}{24 \cdot 2^{p-1}} p^3 B_{p-3} \equiv -1 + 2^{1-p} - \frac{7}{24} p^3 B_{p-3} \pmod{p^4}.$$

Obviously, the above congruence coincides with the first congruence of Theorem 1.7, and the proof is completed. □

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