1

Paradoxes; *or*, "Here in the Presence of an Absurdity"

Wherein we officially meet some paradoxes: of sets, vagueness, and spatial boundaries. The chapter is expository, laying out intuitive arguments for thinking that some of these paradoxes are genuine; it will be the task of later chapters to see how much of this reasoning can be brought up to logical code.

1.1 Sets

A starting point for taking the paradoxes arch-seriously comes from naive set theory.[1] Set theory provides a very natural and intuitive language and basic toolkit for the rest of mathematics. It also provides an ontology for mathematics, insofar as (it is generally thought) any mathematical object can be reduced to, or at least modeled by, sets. Set theory is a *foundation*. Paradoxes there are paradoxes at the source.

1.1.1 An Analytic Definition

The concept of a set is simple to state. A *set* is any collection of objects that is itself an object, with its identity completely determined by its members. A set is the unique extension of a predicate or property.

Many textbooks open by claiming that "set" cannot be formally defined,[2] but this isn't so; we've just had a fine definition. This is the *naive set concept*, and it can be completely characterized by the following principles:

Abstraction: $x \in \{z : \varphi(z)\} \leftrightarrow \varphi(x)$ and
Extensionality: $x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)$.

[1] For standard presentations of set theory, some good "classic" sources are [Fraenkel, 1953; Levy, 1979], the more advanced [Kunen, 1980], and of course the compendious [Jech, 1974]. For more attention to philosophical issues, see [Potter, 2004]. Parts of this section go back to [Weber, 2009].

[2] E.g., [Quine, 1969] among others. "We cannot say with any kind of conviction what sort of things sets are, so we attempt a type of ostensive definition of them through axiomatization or 'listings'" [Hallett, 1984, p. 303].

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Abstraction says that something is in the set of φs if and only if it is a φ, without exception or qualification.[3] The axiom of extensionality vindicates the definite descriptor "the." These clauses fix the meanings of $\in$ and $=$, the only nonlogical parts of the vocabulary of set theory, giving *existence* and *uniqueness* conditions over the universe of sets. Existential generalization on abstraction gives the further quantified principle,

Comprehension: $\exists y \forall x (x \in y \leftrightarrow \varphi(x))$,

that for any property there is some set of all and only the things with that property.

Frege construed sets as the ontology of predication. In his *Grundgesetze* [Frege, 1903b], he stated the set concept in a single axiom, the infamous equivalence

Basic Law V: $\{x : \varphi(x)\} = \{x : \psi(x)\} \leftrightarrow \forall x(\varphi(x) \leftrightarrow \psi(x))$.

Frege's axiom looks obvious to the point of banality. The φs are the ψs exactly when all and only φs are ψs. Indeed. These clauses look very much like analytic definitions of predication. Peano's choice of the "$\in$" symbol, from the Greek verb $\epsilon\sigma\tau\iota\nu$, "to be," suggests that set membership is intended to capture the "is" of predication, or more metaphysically, property instantiation.[4] Sets are predicates in extension.[5]

These are *definitions*, in the old Socratic sense. For Socrates proposes to describe the world in terms of collections of things that share precise necessary and sufficient conditions, forms, or models that provide a standard by which we may be able to say that such and such an x is φ, such another not φ (e.g., *Euthyphro* 6e [Cohen et al., 2000, p. 95]). The naive set concept captures, in slogan form, sets as the metaphysics of definitions.

The reason textbooks claim that there can be no definition of set, then, is not that the concept is somehow opaque or ambiguous; and it is not because the concept is so familiar or primitive that it admits no definition. The reason is that the set concept is inconsistent. The concept is not indeterminate, or underdetermined; it is *over*determined, famously and paradoxically inconsistent.[6] After reviewing why, I will argue that naive set concept is

[3] Priest and Routley: "The naive notion of set is that of the extension of an arbitrary predicate .... This is as tight an account as can be expected from any fundamental notion. It was thought to be problematical only because it was assumed (under the ideology of consistency) that 'arbitrary' could not mean arbitrary. However, it does" [Priest et al., 1989, p. 499]. Or Priest again: "[A] set just is the extension of an arbitrary condition, and that's that" [Priest, 2006b, p. 29]. (Cf. Forster, in defense of a (consistent) universal set [Forster, 1995, ch. 1].) These statements motivate the *absolutely unrestricted* or *generalized comprehension scheme*, explicitly introduced by Routley and then Brady, which allows the set being defined to appear in its own defining property, "impredicative" instances of the form

$$x \in y \leftrightarrow \varphi(x, y);$$

[Routley, 1977, p. 915; Brady and Routley, 1989, p. 419; Brady, 2006, p. 177]. Others who endorse the naive set concept require a *restricted* form, where the set being defined *cannot* appear in the description defining it [Priest, 2006b, ch. 2]. But the distinction makes little difference, as circular sets can be produced by the sedate version as the unrestricted (Theorem 13 of Chapter 5); nothing much is gained or lost either way. The redundancy is pointed out at [Petersen, 2000, p. 383, footnote 14].

[4] In Peano's 1889 *Principles of Arithmetic*, at [van Heijenoort, 1967, p. 89].

[5] "By the law of excluded middle, ... for any predicate there is a set of all and only those things to which it applies (as well as a set of just those things to which it does not apply). ... Our thought might therefore be put: 'Any predicate has an extension'" [Boolos, 1971, p. 216].

[6] "Naive set theory is simple to state, elegant, initially quite credible, and natural in that it articulates a view about sets that might occur to one quite naturally. ... Alas, it is inconsistent" [Boolos, 1971, p. 217].

correct, and not in spite of but *because* of its paradoxicality. Its inconsistency cannot be removed without doing fatal damage to the concept.

1.1.2 The Antinomies

Naive set theory is full of paradoxes, or "antinomies." Consequently, the general consensus at the start of the twentieth century was of a *crisis in the foundations of mathematics*.[7] This is a story that has been told many, many times – even in a 2008 graphic novel, *Logicomix* – so I presume familiarity. Here then, in brief, are the famous paradoxes of naive set theory. It has seemed to many, from [Russell, 1905b] to [Priest, 2002a, ch. 8, 9], that they are all related (see Chapter 2).

1.1.2.1 The Liar [500 BCE]

Naive set theory includes naive truth theory. (One reasonable, but wrong, hypothesis about this is that naive set theory is inconsistent *because* it includes naive truth theory.) For any sentence (a closed formula, with no free variables) φ, just consider the set

$$\ulcorner\varphi\urcorner := \{x : \varphi\},$$

the set of all x such that φ, which exists by naive comprehension. If φ is true, then $\forall x(x \in \ulcorner\varphi\urcorner)$. For every φ there is such a set, and if $\ulcorner\varphi\urcorner = \ulcorner\psi\urcorner$, then by Basic Law V, $\varphi \leftrightarrow \psi$, so the naming is unique. So we can define a truth predicate, where for some arbitrary but fixed t,

$$\mathsf{T}(x) := t \in x,$$

and, in particular, $\mathsf{T}(\ulcorner\varphi\urcorner)$ is $t \in \ulcorner\varphi\urcorner$. Then $t \in \ulcorner\varphi\urcorner \leftrightarrow \varphi$ and naive set theory has vindicated the truth schema,

$$\mathsf{T}(\ulcorner\varphi\urcorner) \leftrightarrow \varphi.$$

By making the naive set assumption, we have already assumed naive truth theory [Priest, 2002b, p. 363; Beall, 2009, p. 114].

To get a liar, we prove a special case of a general fact about naive comprehension, the fixed point theorem (Theorem 13 of Chapter 5). Consider the open formula $\neg\mathsf{T}(x)$. By naive comprehension, there is a set L such that

$$x \in L \leftrightarrow \neg\mathsf{T}(\{z : t \in L\}).$$

So by instantiation, $t \in L \leftrightarrow \neg\mathsf{T}(\{z : t \in L\})$. Letting ℓ be the sentence $t \in L$, then $\ulcorner\ell\urcorner = \{z : t \in L\}$, and ergo

$$\ell \leftrightarrow \neg\mathsf{T}(\ulcorner\ell\urcorner)$$

is a liar sentence. The liar contradiction follows as in the Introduction.

[7] Of that crisis, Fraenkel et al. say that "a treatment of the logico-mathematical antinomies is a task that cannot be dodged" [Fraenkel et al., 1958, p. 5]. Though for an alternative view of the history, see Lavine [1994, p. 3].

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1.1.2.2 Cantor's Paradox [c. 1895]

Set theory became an independent discipline when Cantor proved in 1874 that the set of all subsets of X, the *powerset*

$$\mathscr{P}(X) = \{x : x \text{ is a subset of } X\},$$

must be bigger in size than X itself. For Cantor's great insight is that the sizes of infinite sets can be tracked, via the notion of a one-to-one correspondence. Even if we cannot *count* all the members of a set, we can say whether or not it is the same size as another set, by trying to pair off their members exactly. If any attempted pairing off between two sets fails, then they are not the same size, or *cardinality*.

Take then a function f from X to $\mathscr{P}(X)$. Consider the *diagonal subset*

$$r_X = \{x \in X : x \notin f(x)\}$$

of all the members of X that are not in the subset they map to. Noting that r_X is a subset of X, $r_X \in \mathscr{P}(X)$, Cantor proved that f cannot map anything from X to r_X. For if some $x \in X$ had the ill fortune to pair off with this diagonal subset, $f(x) = r_X$, then $x \in r_X$ if and only if (iff) $x \notin r_X$, which is (or classically entails) a contradiction. So, by reductio, nothing in X maps to this subset, and so the powerset $\mathscr{P}(X)$ has more members than X.

That is *Cantor's theorem*. But what about the universe of *all* sets, $\mathcal{V}$? Surely $\mathcal{V}$ is the biggest size there is: any set is *in* the universe (containment), and any set is a *subset* of the universe (inclusion). It would seem that

$$\mathscr{P}(\mathcal{V}) = \mathcal{V}.$$

If sets are *identical*, then of course they are the same size. So then by Cantor's theorem, the powerset of the universe is greater than, but not greater than, the universe itself. This is also known as Frege's paradox for Basic Law V.

1.1.2.3 Russell's Paradox [1902]

Russell found his eponymous paradox based on his own study of Cantor's proof. He transmitted it to Frege in 1902, and it is only a small exaggeration to say that this destroyed the latter's life's work.

Focus on a special case of Cantor's diagonal process, where X is instantiated by the universe of sets, and f is just the identity $f(x) = x$. Then Cantor's $r_{\mathcal{V}} = \{x \in \mathcal{V} : x \in f(x)\}$ is just

$$r = \{x : x \notin x\},$$

which is called the *Russell set*: the set of all sets that are not members of themselves (or "nonselfmembered," in pseudo-German). Then $r \in r$ iff $r \notin r$. Hence, by the law of excluded middle, $r \in r$ and $r \notin r$.

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Russell's antinomy is just the tip of the iceberg.[8]

1.1.2.4 Mirimanoff [1917]

A set M is well-founded (by $\in$) iff from M there is no infinitely descending membership chain

$$\cdots M_2 \in M_1 \in M_0 \in M.$$

A well-founded set M cannot be a member of itself, because if it were then $\cdots \in M \in M \in M$ would be an infinite chain. But the set of all well-founded sets

$$\mathfrak{M} = \{M : M \text{ is well-founded}\}$$

is itself well-founded, since for every $M \in \mathfrak{M}$ there is no infinite descent from M by definition. Therefore, $\mathfrak{M} \in \mathfrak{M}$. But then, by well-foundedness, $\mathfrak{M} \notin \mathfrak{M}$.[9]

1.1.2.5 Burali-Forti [1897]

Ordinals are a generalization on the natural numbers. The study of ordinals was one of the cornerstones of Cantorian set theory [Cantor, 1895], carried on by Hausdorff [Hausdorff, 2005]. Upon later reductions of mathematics to set theory, the ordinals came to form the central load-bearing column of the mathematical universe. The first ordinals are $0, 1, 2, \ldots$ followed by the first *transfinite* number greater than all of these, ω, followed by further transfinite successors and limits, with an order relation. The ordering has its members in a perfectly straight line, and any part of the line has a first (but not necessarily last) member; the ordinals are not only well-founded (as before) but *well-ordered*. And ordinals are *transitive*: anything that precedes an ordinal in the well-order is itself an ordinal.

Ordinals are the order types of well-ordered sets. Von Neumann found that it works very nicely to think of an ordinal number α recursively, as the *set* of all ordinals β that precede it, so that an ordinal is a well-ordered, transitive set of ordinals. But the set of all ordinals, On, is itself a well-ordered, transitive set of ordinals – so On is an ordinal, and, being the set of all ordinals, is also the *greatest* ordinal. But every ordinal has a *successor*, which is strictly greater. Therefore, the successor of On is strictly greater than, and also not strictly greater than, On.

Indeed, with the ordering relation on the ordinals represented by $\in$, and an ordinal taken to be the set of all the ordinals that come before it, $\alpha = \{\beta : \beta \in \alpha\}$, then

$$On = \{\alpha : \alpha \in On\}$$

is clearly an ordinal, and self-membered at that. But since the ordering relation $\in$ on ordinals is well-founded, too, there are no self-members; so the contradiction is just

$$On \in On \qquad \text{and} \qquad On \notin On,$$

which is what you should expect from the biggest number.

8 "[A]lthough logic forces us to accept that there isn't any such [Russell] set, it's highly paradoxical that there isn't. ... [I]sn't a collection or totality just the same thing as a set? How COULD there NOT be a set containing all and only the sets that don't contain themselves?" [Boolos, 1998, p. 148].

9 See Hallett [1984, § 4.4]; Barwise and Moss [1996]. A variant of this paradox is Smullyan's hypergame.

This is my favorite paradox. The ordinals are *designed* to be recursive, to be a foundation for arithmetic and transfinite induction. For the ordinals to do their job, they need to be self-reproducing in an orderly, automatic way. But once you start a flower forever blooming from within itself, it cannot be stopped.[10]

Several other more "semantically" flavored paradoxes appeared in a rush.[11]

1.1.2.6 König [1905]

A set is *countable* iff it is the same cardinality as the natural numbers. A number is *definable* iff there is a finite sequence of (English) words that refers to it. There are only finitely many words; so there are only countably many finite sequences of words; so there are only countably many definable numbers. So take the least indefinable number. The previous sentence just defined it.

1.1.2.7 Richard [1905]

Similarly to König's, consider the set of definable real numbers on the interval $[0, 1]$. By a diagonal construction, there is a definable real number not in this set.[12]

1.1.2.8 Berry [1906]

Consider the set of natural numbers definable by less than 19 syllables. There is a least such number; this number is defined by less than 19 syllables.

1.1.2.9 Grelling [1908]

A term is homological iff it exemplifies what it describes, e.g., the word "word" is a word. A word is heterological if not homological. Then "heterological" is heterological iff it is not.

There are many others, but you get the point. These are apparently sound arguments with apparently false conclusions. We must deny the reasoning, deny a premise, or accept the result. In all cases, the premise is naive set comprehension.

1.1.3 Naive Sets and Iterative Sets

1.1.3.1 Why Are There Set Theory Paradoxes?

The antinomies of naive set theory are overwhelming. Obviously true assumptions (in each case, that there is some set of things with some clearly defined property, such as "is an ordinal") lead by simple reasoning to contradictions. By 1925, Hilbert is apocalyptic, calling for a solution to these problems "not merely for the interests of the individual sciences, but rather for the *honour of the human understanding itself*"; for the paradoxes of

10 Burali-Forti himself thought this was a counterexample to trichotomy [van Heijenoort, 1967, p. 105]. Sometimes the contradiction is expressed as $On = On + 1$. See [Moore, 1982].

11 On dividing these paradoxes into different categories (set theoretic vs. semantic) as Ramsey suggested, and whether the division is good or not (it is not), see Priest [2002a, ch. 10].

12 See van Heijenoort [1967, p. 143]; and Priest [2002a, ch. 9].

set theory, appearing "ever more severely and ominously," had "a downright catastrophic effect in the world of mathematics":

> Let us admit that the situation in which we find ourselves with respect to the paradoxes is in the long run intolerable. Just think: this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdity. [Hilbert, 1925, p. 375; emphasis in the original text]

How could it be? I think the answer lies in thinking about what a set is.

An otherwise disparate collection is encircled by a predicate, a property, a definition. Birds become a flock, stairs a flight, people a crowd (a *Menge*), counting numbers the naturals $\mathbb{N}$. Objects become a set. A set is a multiplicity that forms a unity, a many that is also a one. Metaphysically speaking, a set is a *composition*. But this is not a definition; this is a mystery. Here *are* birds. Here *is* a flock. Even at the level of grammar, the former are plural, while the latter is singular.[13] How is a many also a one?

The beauty of naive set theory is that it provides just the right way to say that a flock is a *set* of some birds, not unlike Plato's forms in the *Republic* (596a6–7). The way this is possible is that sets play two roles at once, roles that are in tension with each other. A set is no more and no less than its members *together*. This is the ultimate reason why set theory is inconsistent. Let us return to the explication of the nature of sets; Bolzano takes arbitrary assemblages to be cohesive entities [Bolzano, 1973, p. 128]:

> I permit myself, then, to call any group you please, in which the nature of the connection among the parts is to be regarded as an indifferent matter, a *set* [*Inbegriff*].

And here, more famously, is Cantor echoing the same thought: in the 1895 *Beiträge* [Cantor, 1915, p. 85],

> A set is any collection into a whole of definite, well-distinguished objects of our intuition or thought.

Or earlier, in the 1883 *Grundlagen*,[14]

> By a "manifold" or "set" I understand generally any multiplicity which can be thought of as one [*jades Viele, welches sich als Eines denken lasst*], that is to say, any totality of definite elements which can be bound up into a whole by means of a law.... By this I believe I have defined something related to the Platonic $\epsilon\iota\delta o\varsigma$.

Sets are extensional collections, yes, but they are also grasped (the etymology of "inbegriff"), "by means of a law." This is our ingress:

On the one hand, sets are *extensional*, in that they are determined by their members. To pick out a set, there is nothing more one needs to know about it than its members. Set theory is often called the theory of extensions par excellence.[15] On the other hand, sets are *intensional*, too, in that they are each determined by a property. Within naive set theory is

[13] Cf. Lewis [Lewis, 1991, p. 81, emphasis in the original text] on compositions and the things that compose them: "It just *is* them. They just *are* it."
[14] From [Hallett, 1984, p. 33; Jané, 1995, p. 391].
[15] Forster calls "the most tough-minded expression" of extensionality, that "the only thing a set theorist can know about = is that it is a congruence relation with respect to ∈" [Forster, 1982, p. 2].

naive property theory, obtained just by dropping the extensionality axiom, and reading "$\in$" as property instantiation: $\varphi(x)$ just in case x instantiates the property (or form, or *eidos*) of φ. Sets *must* be more than extensional, if they are to be useful for any more than some (small) finite combinatorics. As Weyl aptly observes,

> No one can describe an infinite set other than by indicating properties which are characteristic of the elements of the set. *[Weyl, 1919, p. 23]*

A collection requires a predicate.[16] But sets *cannot* be more than extensional, since they are properties in extension. Governing naive set theory is a membership-based notion of identity, and a property-based notion of membership. It is *both* "top-down" and "bottom-up." And this means a collision of extensionality with intensionality.

Here then is the nexus of the paradox. Sets are themselves objects, intensions over and above their members, *more* than the sum of their parts.[17] Meanwhile, the extensionality principle governs all sets, forcing them to be *no more* than the sum of their parts. Since these together characterize sets, sets are both intensional and extensional. Frege's Basic Law V expresses, correctly, that there is a one-to-one correspondence between extensions and properties; Cantor's theorem expresses, correctly, that there are more properties than extensions.[18] This is unstable. This is Cantor's paradox: there are more objects than fit in the universe, because there are properties with the same extension; but because (by Basic Law V) there are exactly as many properties as extensions, the universe is bigger than itself.

My view is that this contradiction inherent in the notion of naive sets is a *good thing*. Naive sets provide a foundation that has both these qualities, intensional comprehension and extensional tractability. Theories that try to do without one or the other will inevitably be incomplete. The dual nature of naive sets is a feature, not a bug; they offer Platonic forms with precise identity conditions. And, most tendentiously, if the comprehension axiom turns out to be *true*, then this is an even more attractive reason to use it as a foundation.

1.1.3.2 From Twentieth-Century Set Theory to Twenty-First

The naive comprehension principle is an axiom in the old sense: simple, self-evident, inalienable. I suggest the mathematics of collections *cannot* abandon the naive view without ceasing to be a theory of all collections. To see this, let us review the past century's attempt to evade the paradoxes. This was done, first, by replacing naive set theory with *axiomatic* set theory (from 1908 to the 1920s); and then later (from the 1930s to the 1960s) by clarifying

16 Cf. Beall's diagnosis of the semantic paradoxes of truth theory: God would have no need of a truth predicate [Beall, 2009, p. 1], because God in his infinitude does not need to save time by generalizing. By these lights, God would have no need of a comprehension principle, either.

17 Unlike *pluralities*, e.g., some birds, which advocates of *plural logic* say do not always give a further entity, "a plurality," e.g., a flock of birds, over which to quantify. A "plurality" is not itself an entity; referring to "it" in the singular is only a *façon de parler* (cf. proper classes in set theory; Section 1.1.3.2). The most prominent argument that not all pluralities are sets is that, if they were, one could derive Russell's contradiction [Boolos, 1998, p. 67]. A naive set is also different from a *mereological sum/fusion* – the smallest portion of reality that has all the φs as *parts*. (*Mereology* is the theory of parthood.) A fusion exists, as a numerically distinct entity from its members, unlike plurals; but the parthood relation is more like subset than membership. See [Lewis, 1991]; Potter, 2004, ch. 2; [Cotnoir and Baxter, 2014].

18 Or just think of the difference between the properties "equilateral triangle" and "equiangular triangle" even though their extensions in Euclidean geometry are identical. This basis for the paradoxes of naive set theory is spelled out in [Zalta, 2007].

the "intended model" of the new axiomatic set theory, and urging that this was the real (consistent) set concept all along. How did this strategy fare?

Zermelo's 1908 axioms met with "intense criticism," not only over the axiom of choice but because of his decision to molest the comprehension principle; Schönflies, Bernstein, and Poincaré all rejected this possibility out of hand.[19] Zermelo's axioms name some key properties sets have, without explaining what a set *is*.[20] The aporia is apparent in the opening pages of most set theory books. Devlin opens as many do with exposition of basic Cantorian theory.

> In set theory, there is really only one fundamental notion: the ability to regard any collection of objects as a single entity (i.e. a set). *[Devlin, 1979, p. 1]*

Or, as a very rigorous textbook puts it,

> the idea of the collection of all objects having a specified property is so basic that we could hardly abandon it. *[Takeuti and Zaring, 1971, p. 9]*

Reformed set theory since the paradoxes retains "as many as possible of the naive set theoretic arguments which we remember with nostalgia from our days in Cantor's paradise" [Potter, 2004, p. 34]. And since the Gödel/Cohen independence results,[21] it has been known that the axioms of ZFC – Zermelo–Fraenkel set theory with the axiom of choice – are highly *incomplete*, leaving key questions about sets permanently unanswered.[22]

Now, the reader may wish to remind me, gently but firmly, that I appear to be trying to re-litigate a closed case. During the twentieth century, much work went in to a replacement idea, *iterative sets*, and their main formal theory, ZFC.[23] An iterative set is formed from *already* existing objects. Therefore, iterative sets cannot be, are not candidates to be, members of themselves. The set theoretic universe is a cumulative hierarchy, in which sets may only be formed from preexisting members, starting with the empty set – formed from "all" the sets that exist at the start.[24] Then the idea is that none of the antinomy-inducing collections are (iterative) sets. The solution is similar to Tarski's truth hierarchy, with the

19 [Moore, 1982, pp. 111, 117].

20 "Axiomatization went hand in hand with the divorce from any attempt to understand what sets are or what conceptual role they play" [Hallett, 1984, p. 303]. The point is made explicitly and overtly not least by Zermelo himself, at the outset of his 1908 axiomatization: "At present, the very existence of the discipline [of set theory] seems to be threatened by the existence of certain contradictions or 'antinomies' that can be derived from its principles – principles necessarily governing our thinking, it seems – and to which no entirely satisfactory solution has yet been found" [Zermelo, 1967, p. 200]. See Woods [2003, p. 334]. In 1914, Hausdorff expressed doubt about Zermelo's system: "At present, these extremely ingenious investigations cannot be regarded as completed, and introducing a beginner by this [axiomatic] approach would cause great difficulties. Thus we wish to permit the use of naive set theory here" [Hausdorff, 1957, p. 2]. Zermelo's axioms are supposed not even intelligible without naive acquaintance with sets [Devlin, 1979, p. 49]. "A survey of the the axioms does not suffice to reveal the source of their attraction" [van Aken, 1986, p. 992]. Von Neumann, at the close of his own sophisticated 1925 axiomatization, sighs that despite much work, still he must "entertain certain reservations" because "for the time being no way of rehabilitating this theory is known" [van Heijenoort, 1967, p. 413].

21 See [Cohen, 1966; Smullyan and Fitting, 1996].

22 As witness to the ongoing dissatisfaction with rejecting naive comprehension, there remains a steady stream of research attempting either to approximate it with ever-stronger new "large cardinal" axioms [Kanamori, 1994] or to restore some semblance of the principle, via modalities or other devices.

23 There are many other alternative set theories, before even considering nonclassical logic; [Holmes et al., 2012]. Most land on saying that some collection is not a set.

24 Note that this – *if there are no sets, then all sets are in* $\varnothing$ – is a form of explosion: $p \mathrel{.\cdot.} \neg p \supset q$. So the *ex nihilo* construction in this form is very classical.

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added weight of the legitimacy and importance of actual mathematics (say as it is needed for science) and therefore the pragmatic obligation to get some kind of working fix on the table so that we may get on with life.

Let us ask, though, whether iterative sets can be sufficient for *explaining* what sets are. Without impugning the impressive work of the last century, there is an obvious problem in explaining what sets are in terms of iteration: iterative sets are formed from *some collection* of preexisting members. It presupposes collections. And not only that, but it supposes those collections arise in an orderly, indexed process. For example, in [Potter, 2004], following an idea from Scott, it is shown how one can justify the axioms of ZF from some "stage axioms"; these axioms postulate (a) the existence of cumulative stages, (b) an ordering relation such as "earlier than," and (c) indices to keep track of the process. But collections, order relations, and a (transfinite) index set are *exactly the sorts of things developed from within set theory*. This is "assuming a considerable amount of 'set theory' in order to define our *set theory*" [Devlin, 1979, p. 49, emphasis in the original text]. The iterative notion is therefore not the primary intuition because it *presupposes set theory*, including the notion of ordinality. One of the central purposes of set theory is to deliver a theory of ordinals, not the other way around.[25] So the iterative notion is not the last word on sets [Weir, 1998, p. 780]. Like a Tarskian hierarchy, the iterative universe is simply unable to account for itself.

There are also problems with making the iterative conception more than an attractive metaphor. The iterative idea does appeal to our physical intuition about collecting up objects into a bag; you need to have objects *before* you can put them in a bag! Beyond small finite sets, though, the intuition is exhausted – "naturally we are not thinking of *actually building* sets in any sense" [Devlin, 1979, p. 43]. Talk of "collecting" must be taken metaphorically. And only metaphorically:

> The notion that an infinite set is a "gathering" brought together by infinitely many individual arbitrary acts of selection . . . is nonsensical. *[Weyl, 1919, p. 23]*

To replace the metaphor with mathematics, we can follow Hilbert's strategy in geometry [Hilbert, 1902a] of replacing Euclid's imperative phrasing ("now you construct a triangle like so") with a declarative ("triangles exist"). Instead of "forming" the ωth stage, we simply profess that it is there: *there is an ωth stage*. This is now mathematically cogent, but it leaves the constructive intuition behind and with it any explanatory power of the iterative view. It calls into question whether there is any nonmetaphorical content to the iterative view. Unless we are some version of strict constructive finitists, or believe some kind of "idealized" constructing angels,[26] then sets simply exist or they do not.

[25] Priest and Routley: "We do not deny that once one has a notion of set one can non-circularly produce . . . the cumulative hierarchy. But to suppose one can use the notion of an ordinal to produce a non-question-begging definition of 'set' is moonshine" [Priest et al., 1989, p. 500].

[26] Agents human enough to make the "gathering together" talk more than a pretty locution, but superhuman enough to perform transfinite tasks up to arbitrary places in the ordinals. See Potter [Potter, 2004, pp. 36–40] for discussion of construction metaphors, idealized constructors, etc.

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To see that the classical solution to the paradoxes is not entirely happy, I would gesture at two problematic ideas that underwrite ZFC. One is the tortured doctrine of *proper classes*, which introduces entities that are like sets (classes) in every way, except they have a special property – if they were sets, they would be inconsistent![27] The related doctrine of *limitation on size* says that some collections are "too big" to be sets – namely, those collections that are the same size as a proper class.[28] Suffice to say that both of these ideas are rather euphemistic – as Boolos says, "you can't get out of this paradox merely by substituting one word for another" – and could be criticized at some length if one were so inclined (e.g., the very enjoyable [Weir, 1998]). As Woods puts it, these problems are "central, deep, and disabling" [Woods, 2003, p. 162]. These doctrines suggest, on my reading, that practitioners who are committed to doing some mathematics mainly want to get past paradoxes for the sake of getting on to some results. And indeed *this* seems like one of the better arguments in favor of "classical" set theory. Fair enough. But the argument is irrelevant in the present context. The natural notion of set is inconsistent, and attempts to replace it either acknowledge the inconsistency or rest content with mathematical storytelling.

Rather than relitigating the past, or peevishly throwing stones at the towers of modern ZFC, I can find much to agree on in statements such as this one:

It should not be forgotten that the paradoxes never applied to any type structure, and in this sense they are *not* paradoxes of set theory .... If we call such things *properties* then it is clear that the paradoxes are a real problem to be dealt with before a thoroughgoing theory of properties can be developed. *[Drake, 1974, p. 14]*

Or, to a lesser extent, this one:

It has been pointed out that the paradoxes of Russell, Burali-Forti etc, never really caused a crisis in mathematics (where one deals only with unproblematic examples of sets) but rather in logic (and general set theory), where one tries to provide a general and universal frame for mathematics and in particular for arbitrary sets. *[Reinhardt, 1974, p. 190]*

These statements acknowledge unsolved problems in what is being called "property theory" and "logic" or "general set theory." So let "iterative set theory" carry on unperturbed, as long as it is admitted that ZFC sets are a subconcept of what collections are, as long as it is clear that iterative sets are not *all* the sets or have somehow *solved*, rather than deferred, the paradoxes.[29] After all, "simply saying that we ought never to have expected any property whatever to be collectivizing, even if true, leaves us well short of an account which will settle which properties are" [Potter, 2004, p. 27]. As Dummett puts it,

[27] As such, proper classes cannot, by definition, be members of anything. See [Maddy, 1983]. Here is Halmos: "[I]t seems a little harsh to be told that certain sets are not really sets and even their names must never be mentioned. Some approaches to set theory try to soften the blow by making systematic use of such illegal sets but just not calling them sets; the customary word is 'class'" [Halmos, 1974, p. 11]. Shapiro and Wright sum up neatly: "Invoking proper classes is an attempt to do the very thing we are intuitively barred from doing. ... *Set* is supposed to encompass the *maximally general category* of entities of the relevant kind" [Shapiro and Wright, 2006, p. 272].

[28] "A great diffculty of the theory is that it does not tell us how far up the series of ordinals it is legitimate to go" [Russell, 1905b, p. 44]. See [Hallett, 1984].

[29] Reinhardt goes on to say, "We now consider such a frame to have been provided for set theory by the clarification of the intuitive idea of the cumulative hierarchy" – a kind of disciplinary "monster barring" and "withdrawal to a safe domain" [Lakatos, 1976] by drawing the boundaries of "set theory" to exclude problems in "logic" and "general set theory."

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A mere prohibition leaves the matter a mystery. . . . To say, 'If you persist in talking about the number of all cardinal numbers, you will run into a contradiction', is to wield the big stick, but not to offer an explanation. *[Dummett, 1991, p. 315]*

There remains a more general theory, naive set theory, with a lot of work still to do. If there is something true about collections that is not captured by a theory of collections, then that theory is eo ipso inadequate as a theory of collections – sets. If the naive set concept were not ineluctable, then its inconsistent consequences would be reason to reject it. But it "seem[s] forced upon us in such a way," writes Slaney, "that we should in all intellectual honesty take [it] seriously" [Slaney, 1989, p. 472].[30]

1.1.3.3 Prospects

Classical logic makes set comprehension absurd. It pushes us to say that there is no universal set, and thus that there is no such thing as universal quantification ("for all" claims) in absolute generality, because there is no universe to quantify over. This has been cause of much surprise and consternation. The consternation is due to the deep sense that comprehension is not absurd, because there is a collection – a set – of all sets, and we do indeed universally quantify. That, indeed, saying "all universal quantification is impossible" is, in the long run, intolerable.

At the end of an exhaustive survey of possible approaches to the paradoxes of set theory (the Burali-Forti paradox in particular), Shaprio and Wright say that "frankly, we do not see a satisfying position here," but they do mention one other option:

Allow the quantification and the predicates, allow the associated order-types, allow that they are ordinals as originally understood, . . . and just accept that there are ordinals that come later than all the ordinals. Cost: none – unless one demurs from the acceptance of contradiction. *[Shapiro and Wright, 2006, p. 293]*

As with the solutions to the liar, there is no option for resolving the paradoxes that gives us everything and costs us nothing. But the paraconsistent dialetheic approach dangles the possibility of an absolutely comprehensive theory of sets, where the paradoxes can finally be accepted for what they are, proofs.

Cantor knew that there exist what he called "inconsistent multiplicities"; he even used them in a proof in a letter to Dedekind, proving that every cardinality is an aleph.[31] Potter warns us to be wary of the Panglossian view that "the paradoxes are not really so paradoxical if we only think about them in the right way" [Potter, 2004, pp. 26, 37]. The paradoxes are paradoxes. Forster advises:

In the ZF world, . . . the paradoxes are viewed as large holes in the ground that one might fall into. . . . However, it is *always* a mistake to think of *anything* in mathematics as a *mere* pathology, for there are no such things in mathematics. . . . One should think of the paradoxes as supernatural creatures,

[30] For more pugilistic motivations for naive set theory, and attacks on ZFC and its associates, see [Priest and Routley, 1983]; cf. Priest [2006b, ch. 2].

[31] "As we can readily see, the 'totality of everything thinkable,' for example, is such a multiplicity" [Cantor, 1967, p. 114]. Cantor had been thinking about absolute paradoxes at least since 1895 [Lavine, 1994, p. 55].

oracles, minor demons – on whom one should keep a weather eye in case they make prophecies or by some other means divulge information from another world not normally obtainable otherwise. One should approach them as closely as is safe, and from as many different angles as possible. *[Forster, 1995, p. 11, emphasis in the original text]*

The question turns on just how close is still a safe distance. The idea of maintaining comprehension in a nonclassical logic goes back at least to Skolem [Skolem, 1963]. The idea of mathematics founded on self-evident axioms goes back even further. Naive set theory in paraconsistent logic is presented in Chapter 5.

1.2 Vagueness

Let's leave set theory for a while and go outside. It was raining very heavily earlier, but now we can go out. It is not raining, although it is still raining a bit. It is what in Ireland is sometimes called a "soft day."

Like most predicates, "is raining" is *vague*.[32] And vagueness gives rise to the sorites paradox. The sorites paradox, like the liar, is attributed to Eubulides in the fourth century BCE. The sorites paradox may seem different from the inconsistencies of naive set theory or truth theory. But the sorites exhibits some of the same and most important features, especially in the way that most solutions to it fall prey to revenge, and in the way it connects with the spatial/boundary paradoxes of Section 1.2.1.[33]

1.2.1 The Sorites Paradox

Olympus Mons is the tallest mountain in the known universe.[34] It is an extinct shield volcano on Mars, and its summit stands 26 kilometers over the surrounding plains – three times the height of Mt. Everest. Unlike Everest, though, Olympus Mons is much wider than it is tall, about 600 km wide, so its slope is generally extremely gentle, about 2.5 degrees around the caldera at the top and 5 degrees on the wider foothill; atmosphere notwithstanding, one could easily walk most of the path to the top, and down again, without need of climbing gear or any special athletic skill. A path down the gentle slope of the mountain (which, when viewed from the summit, extends beyond the horizon) can be sketched out by a discrete linear order,

$$a_0 < a_1 < \ldots < a_n,$$

with the indices natural numbers and a_n a point at surface level.[35] The top, point a_0, is very high up. The bottom of the mountain is not high up any more.

32 The discussion here is mostly neutral as to the locus of vagueness – whether is it linguistic (vague predicates), metaphysical (vague properties), or ontic (vague objects). (The idea of *ontic vagueness* is unpopular, but can be made at least an intelligible thesis [Barnes, 2010].) Throughout, as before, "predicate" can be substituted mutatis mutandis for "property," depending on your preferences.

33 Thanks to Mark Colyvan for collaboration on topics here [Weber and Colyvan, 2010].

34 Gary Hardegree (in conversation) objects that surely there are taller mountains out there in the big, big universe. Yes, probably. But they are not (yet) *known*.

35 Or "martian geodetic datum." Since Mars has no sea, it has no sea level, so measuring the elevation of Olympus Mons is more complicated than I've suggested. See Carr [2007, p. 51]; Frankel [2005, ch. 6].

Let's have our points spaced one centimeter apart. Then any two consecutive points are too similar in all relevant respects, too close to each other, for one to be high up but not the other. This is an instance of the *tolerance* principle;[36] any φ is tolerant when it obeys the following:

Tolerance: It is not the case that two things are very, very φ-similar, and yet one is φ, but the other is not.

Formalizing a bit, then, we have the (material) *conditional* version of the sorites paradox, following Hyde's very useful classification of the sorites paradoxes [Hyde and Raffman, 2018]: TOLERANCE is that, for all i, it is not that case that a_i is high up but a_{i+1} is not, and the soritical argument is spelled out:

$$
\begin{array}{ll}
 & a_0 \text{ is high up} \\
 & a_0 \text{ is high up } \supset a_1 \text{ is high up} \\
 & a_1 \text{ is high up } \supset a_2 \text{ is high up} \\
 & \vdots \\
 & a_{n-1} \text{ is high up } \supset a_n \text{ is high up} \\
\therefore & a_n \text{ is high up,}
\end{array}
$$

with $\supset$ the material conditional (Section 0.2.2.3), "a_i is not high up or else a_{i+1} is." Then the conclusion follows by a sufficient number of applications of disjunctive syllogism, or "material modus ponens":

$$
\begin{array}{lllll}
\text{high}(a_0) & & & & \\
\text{not high}(a_0), & \text{or high}(a_1) & & & \\
 & \text{not high}(a_1), & \text{or high}(a_2) & & \\
 & & \text{not high}(a_2), & \text{or high}(a_3) & \\
 & & & \ddots & \\
 & & & & \text{high}(a_n).
\end{array}
$$

But a_n is at the foot of the mountain. It is *not* high up. Generalizing, we get the *inductive* version of the paradox, which uses mathematical induction rather than material modus ponens:

$$
\begin{array}{ll}
 & a_0 \text{ is high up} \\
 & \text{For each } i \text{ in the sequence, } (a_i \text{ is high up } \supset a_{i+1} \text{ is high up}) \\
\therefore & \text{For every } i \text{ in the sequence, } (a_i \text{ is high up}).
\end{array}
$$

This argument has led from truth to falsity.

This is a *paradox* in the sense defined in the Introduction. It is an apparently sound argument with an apparently false conclusion. We must deny the reasoning, deny a premise,

36 The name "tolerance" comes from [Wright, 1976, p. 334]: a predicate φ is tolerant "if there is also some positive degree of change in respect of φ insufficient ever to affect the justice with which [the predicate] applies to a particular case."

or accept the result. In this case, we cannot accept the result: not all points in the path are high up. Accepting "all points are high up" would make a nonsense out of the predicate. And all the more so since this is just an example. If sound, the conditional inductive sorites would render any vague predicate completely vacuous. Everything is red, everyone is bald, all jokes are funny. In fact, if "vague" is itself a vague predicate (as some have urged [Sorensen, 1985; Hyde, 1994]), and we accept soritical reasoning such as the preceding, then it would follow that *all* predicates are vague, and in particular *truth*, and then, like all grains of sand are heaps, so too would everything be true. Absurd.

So, as Sorensen is careful to explain [Sorensen, 2001, ch. 1], the obvious solution to the paradox now is to deny the conclusion, and instead derive the falsity of TOLERANCE, via a *line-drawing* form of the argument. Informally, on Olympus Mons, the soritical situation is thus described with three true sentences:

Some point is high up.
Not all points are high up.
∴ Some point is high up while the very next one is not.

Here the conclusion is the negation of TOLERANCE. This is a sound argument, stemming from a truism about finite sets of natural numbers: if some number is such and so, then there is a *least* such number. Not only is the argument valid, but the premises are true. Since the vague predicate eventually ceases to apply, there *must* be a line – a sharp cutoff – between what is high up and what is not.

Except TOLERANCE is not so easily dismissed. Our new conclusion, that "high up" comes abruptly to an end at some exact centimeter-length patch of martian hillside, smacks just as false, just as unacceptable as the initial conclusion that "high up" never ends. This is especially pressing when we observe that the centimeter-length patches could be made as small as we like. However it is that a baby becomes a non-baby, it is not in the duration of a millionth of a picosecond, right? It is unavoidable – there *must* be such a line, a counterexample to an otherwise disastrous proof. But all the same, TOLERANCE seems basic: vague predicates do not, almost by definition, have sharp cutoff points. That's a truth of the world we live in. Here then is the appearance of *revenge*, and the point at which the paradox becomes genuine.

The original problem from the conditional/inductive sorites was that it would be absurd if all points were equally "high up" (bracketing any mystical epiphanies). The only (apparent) option was to affirm that not all points are high up, and rejig the argument accordingly, from the conditional version with a false conclusion (of the form $p, q \supset r \therefore s$) to its contrapositive, the line drawing version ($p, \neg s \therefore q \ \& \ \neg r$). But now there is a new problem, which is that the line-drawing version has a false conclusion, too: it seems absurd to think there is some exact spot – right there, and nowhere else – at which a predicate like "high up" cuts off. The proposed solution is every bit as counterintuitive as the original, since the original was driven by the conviction that TOLERANCE is true. This is how we know we are

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dealing with a genuinely paradoxical problem. Like the derivation of a Kantian antinomy, there are parallel reasons that

TOLERANCE is true and TOLERANCE is false

– a contradiction.

As with all serious paradoxes, our intuitions combine with the facts to box us into a corner, between impossibility and inevitability. Some mountains are tall; some are not; and the notion is tolerant: there aren't two mountains that are extremely close in height, but one of them is tall and the other is not. The notion is, nevertheless, as prone to having a "cutoff" as any other: since tallness gives out, it must give out *somewhere*. The existence of a cutoff point is inevitable. It is also incredible. It is a paradox.[37]

Going forward, it would be good to have a (provisional) definition of what it means to be vague. Susceptible to the sorites? Satisfying tolerance? Many accounts of vagueness begin by describing the phenomenon as a kind of *indeterminacy* – a predicate is vague when there are cases where it is too hard to say whether, e.g., a mountain is tall or not. But this is already a theory-laden diagnosis,[38] casting the whole situation in an epistemological light, and one pushing toward a gappy solution. I suggest defining a predicate φ as *vague* in a more theory neutral way, if and only if the following are true:[39]

- φ is *nontrivial*: something is φ, and something is not φ.
- φ satisfies TOLERANCE.

A nice consequence of the definition is duality: if φ is vague, then $\neg\varphi$ is vague. A more basic consequence is that, given some simple mathematics, a contradiction follows, as sketched previously; the specific mathematical principles involved will be laid out in Section 3.3.3.2.

1.2.1.1 A Continuous Sorites

Before moving to the options, let's look at a generalization of the paradox, a *continuous sorites*, which concerns a smooth (rather than incremental) transition. Increasing the level of abstraction in this way forces attention on cutoff points of vague predicates. This generalization shows that nothing about the paradox relies on the discrete nature of the presentation. Indeed, in presenting the previous paradox for "high up on Olympus Mons," I needed to "digitize" it in order to express the paradox – taking the continuous slope of the mountain and breaking it into centimeter-sized pieces. Vague properties as they are found

37 It is worth noticing just how many apparently independent philosophical problems and positions have the sorites paradox as a key part: *the problem of the many* [Unger, 1980]; *universalism* and *nihilism* as answers to the special composition question [van Inwagen, 1990, ch. 12], based on the *argument from vagueness* [Lewis, 1986, p. 212]; and four-dimensionalism about time as a solution to the Ship of Theseus [Sider, 2001], to name a few.

38 See [Bueno and Colyvan, 2012]. Thanks to Mark Colyvan here.

39 Thanks to Lloyd Humberstone for useful suggestions here. And also, once upon a time, for teaching me the correct way to use the abbreviation "cf.," among other things.

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in nature are often not broken up into units; but there still seems to be the same "slippery slope" problem about continuous properties – maybe even more so. We ought to be able to formulate the sorites paradox in terms of continuous transitions and not merely discretise continuous cases; otherwise, it would look like the problem is to do with $\mathbb{N}$, not vagueness. But the smaller the increments, the more compelling the sorites argument.

The sorites may be generalized to the continuous case.[40] The argument, which is entirely classical, draws out consequences of two properties of the real numbers $\mathbb{R}$, which is that (i) any set of reals bounded from above has a least upper bound, its supremum or sup; and (ii) the reals are dense, in the sense that if $x < y$, then there is a real z such that $x < z < y$. From (i), every set of reals bounded from below has a greatest lower bound, its infimum, or inf. Now consider a vague predicate φ mapped onto a real-number interval $[0, 1]$, exhaustively partitioned into two nonempty sets,

$$A = \{x \in [0, 1] : \varphi(x)\}$$
$$B = \{x \in [0, 1] : \neg\varphi(x)\},$$

with $a < b$ for all $a \in A, b \in B$. We assume that $\varphi(0)$ and $\neg\varphi(1)$. The nonempty set A has a least upper bound, call it $\sup A$. Now, φ is vague, hence TOLERANT; therefore, since points vanishingly close to $\sup A$ are φ, then also $\varphi(\sup A)$. By a symmetrical argument, $\neg\varphi(\inf B)$. By the linear order on $\mathbb{R}$, one of the following must be true:

$$\begin{array}{ll} & \sup A < \inf B \\ \textit{or} & \inf B < \sup A \\ \textit{or} & \sup A = \inf B. \end{array}$$

Since the reals are dense (between any two reals is another one), we have the following contradiction. If $\sup A$ and $\inf B$ are different numbers, then there is some z between them, $\sup A < z < \inf B$ or $\inf B < z < \sup A$. But then φz and $\neg\varphi z$, by examining both cases: if $\sup A < z < \inf B$, then anything less than $\inf B$ is φ but anything greater than $\sup A$ is not; if $\inf B < z < \sup A$, then anything less than $\sup A$ is φ but anything greater than $\inf B$ is not. On the other hand, if $\sup A = \inf B$, then again $\varphi \sup A$ and $\neg\varphi \sup A$. This exhausts all the cases. Therefore, there is a point both φ and $\neg\varphi$, a contradiction.[41]

What can we learn from this version of the paradox? We see how the sorites can be constructed so that it relies upon a property of the real line – the property of being *connected* (see Section 1.3.1.2). Because the reals are connected, a continuous path must cross over from A to B *at some distinct point*. If A and B are partitioned by a vague property, then that point of crossing is inconsistent. A very common response to the discrete forms of the sorites paradox is to see a problem with exclusively and exhaustively separating objects into two closed categories, φ and not. This problem is well expressed in terms of connectedness,

40 Due to James Chase [typescript], via Mark Colyvan.
41 The argument can be represented in analogy to the discrete inductive form of the sorites, via Cauchy sequences, appealing to what Priest calls the Leibniz continuity condition [Priest, 2006b, ch. 11]: whatever is going on arbitrarily close to some limiting point is also going on at the limiting point; *natura non facit saltus*. See Weber and Colyvan [2010, p. 316].

and is the key in generalizing from the discrete to the continuous. We can use this property to generalize again, for a metric-free topological version, but this will be better to return to at the end of the chapter.

1.2.2 The Options

In the Introduction we looked at dialectics of dealing with paradox – the options breaking into classical, paracomplete, and paraconsistent directions. The patten repeats here. There are already many impressive surveys on vagueness and analyses of the problem, so as with truth, I don't attempt that scholarship here.[42] The aim of the discussion is to suggest that, like the liar, all standard solutions to sorites (except, perhaps, nihilism) face revenge, either at the first level or higher orders, because they are committed to sharp cutoffs in some way. That means they deny TOLERANCE, which I think is getting it half-right.[43]

1.2.2.1 Classical Solutions

According to a venerable tradition – Frege, Russell, and Quine – vague predicates are not amenable to logic. The sorites paradox is a reductio against the existence, as far as logic is concerned, of most ordinary properties. Another tradition extends this nihilism to ontology, as mentioned in the Introduction: mereological nihilists take the sorites paradox as reductio against the existence of most ordinary objects. Any "object" that gives rise to sorites arguments does not even exist. Nihilists give some sophisticated explanations as to why our ordinary beliefs are in such massive error. But it would take us off-topic to dwell on this option. Nihilism is its own sort of revenge. The world I live in includes tables and chairs and tallish mountains and rainy days, and these things can be reasoned about using logic. Our job is to explain, not explain away. Stories about why we are wrong about almost everything are about some other world.

Another classical solution to the sorites paradox is to accept the conclusion of the line-drawing argument – there is indeed a sharp cutoff point for vague predicates – but to explain our incredulity about this by positing that the cutoff is *unknowable*. This is called epistemicism. Vagueness is hence a knowledge problem [Williamson, 1994; Sorensen, 2001]. Since I agree on much of the epistemicist's setup of the problem – that our job is not to get rid of cutoffs but rather to explain why they are "embarrassing" – all that matters for the purposes of this chapter is that epistemicism by design includes the existence of sharp cutoffs for vague predicates.

1.2.2.2 Gappy Solutions

Vagueness is, perhaps, the most widely accepted reason for dipping into nonclassical logic. And the most widely accepted way to do that is to deny bivalence, or the law of the

[42] See, e.g., [Keefe, 2000], [Hyde, 2008], and [Smith, 2008].

[43] Priest urges that a solution to the sorites paradox can *only* be in form of explaining why the existence of a cutoff is counterintuitive [Priest, 2003] in [Beall, 2003]. A different sort of option I don't discuss is *contextualism* (e.g., [Shapiro, 2006]).

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excluded middle. The basic idea is that some people are bald, some are not, and some are *indeterminate*, neither bald nor not bald. The aim here is to accept TOLERANCE in some sense, but the vague property does not spread everywhere, because somewhere along the way the vague property falls into a gap. For some points, the answer to whether they are high up or not is "neither."

There are many ways to go with this idea, but as with truth, the main problem for gappy solutions is revenge. For these approaches all still end up with a sharp cutoff point for the vague predicate somewhere.

- According to the popular *supervaluationism* of van Frassen and others, being, e.g., bald admits of different evaluations. Someone is "super"-bald iff they are bald on every admissible valuation. Then there is a cut between bald and "super"-bald – the first number of head-hairs that come out as bald on every valuation.
- In *subvalutationism*, someone is supra-bald iff they are bald on *some* admissible valuation.[44] This position is symmetrical to supervaluationism, as all involved attest, and so comes with sharp cutoffs, too.
- Using the *fuzzy* logic of Hajek, or the fuzzy set theory of Zadeh, baldness comes in degrees. Then there is a cut between those who are bald to degree 1, and those who are not, because "is bald to degree 1" is itself determinate, not fuzzy.[45]

This is called the problem of *higher-order* vagueness. It is a clear case of revenge: eliminating one cutoff gives rise to two new ones. The very notion invoked to solve a problem (that some things are neither φ nor not φ) then gives rise to a problem of exactly the same sort. See [Colyvan, 2008b].

Gap solutions to the sorites attempt to preserve TOLERANCE, at least in spirit, but at some level must (like everyone) deny it, by allowing sharp cuts.[46] I think if that's what is going to happen anyway, then as with the liar paradox we should be upfront about it. The gap theory takes only one horn of the dilemma, that there is something wrong with saying 'this is *exactly* what it takes to be a tall mountain'. But this is a dilemma with two horns. Let's look at the glutty solution.

1.2.2.3 Glutty Solutions

Because there is a difference between being high up and not, the difference must begin somewhere, and so there must be cutoffs. TOLERANCE is true, but it is also false. The idea of the glutty solution is just to accept the existence of the cutoff and explain it.[47] Because TOLERANCE is both true and false, borderline cases of φ are both φ and not φ. The guiding

44 See [Hyde, 1997; Beall and Colyvan, 2001a].
45 For a formidable elaboration of a fuzzy approach, see [Smith, 2008].
46 Although maybe not so for gap theorists who are so thorough as to adopt a different view of mathematics. In a Brouwerian setting, the intermediate value theorem fails [Bishop and Bridges, 1985] and with it the argument that a continuous shading off from red to not red must cross a sharp line somewhere; similarly for Heyting arithmetic, where the least number principle is weakened [Heyting, 1956].
47 See [Priest, 2019a]. From their start, paraconsistent logics were intended for application to vagueness [Jaśkowski, 1969]; cf. Priest and Routley [1989a, p. 389]. See also [Hyde and Sylvan, 1993] and [Beall and Colyvan, 2001b] on ontic (inconsistent) vagueness and seeing contradictions.

picture in the background is of the extension of a vague predicate φ overlapping with its complement, which looks like this:

$$\overbrace{a_0 < \ldots < a_{j-1} < a_j < \ldots < a_k}^{\varphi} < a_{k+1} < \ldots < a_n$$

$$a_0 < \ldots < a_{j-1} < \underbrace{a_j < \ldots < a_k < a_{k+1} < \ldots < a_n}_{\neg\varphi}$$

Vagueness is not indeterminacy, but *overdeterminacy*. The point of the approach is to admit the joint truth of two claims: TOLERANCE and *nontriviality* – the empirical fact that some points on Olympus Mons are high up, but others are not. Put these together with a mathematical principle asserting that if a change occurs, it must occur somewhere, and a contradiction follows.

How can this be coherently maintained? TOLERANCE is extensional: it just says that *either* n cm is not high up, or else $n + 1$ is. And since in a paraconsistent logic, disjunctive syllogism is invalid (the material conditional does not obey modus ponens; see Section 0.2.2.3), the apocalyptic conclusion that all points are high up does not follow.[48] Reconsider the classical conditional sorites premises: they present pairs, a_i, a_{i+1}, such that either a_i is not high up or a_{i+1} is high up. At each pair, to conclude that a_{i+1} is high up would be to take as valid the argument

$$\varphi(a_i), \neg\varphi(a_i) \vee \varphi(a_{i+1}) \therefore \varphi(a_{i+1}),$$

implicitly assuming that it couldn't be that a_i both is high up and is not. If there can be, or is, an inconsistency, then the conclusion that everything is high up does not follow, and the sorites comes to an abrupt but natural end. Rather than the soritical reasoning running unchecked to an absurdity, on this account, the sorites argument halts because it reaches an inconsistency. The inconsistency is *revealed* by the soritical reasoning: TOLERANCE on vagueness, which we must respect if we are trying to understand the phenomenon, insists that being high up does not abruptly come to a halt; since it does, one of the sorites premises is false as well as true, and nontriviality is preserved.

I will put more detail on this in a moment. For now, the crucial element of "saving the appearances" while escaping revenge is again a major virtue of the dialetheic approach. Revenge for the other approaches is an attempt to deny cutoffs, only to have them recur at higher levels. The dialetheic approach simply accepts the cutoff, and makes this possible via inconsistency. A baby becomes a non-baby by being, for a while, both a baby and not a baby. If you've ever lived with a baby, this should not seem so implausible.

1.2.3 Paraconsistency, Definite Descriptions, and Uniqueness

In the previous section, I suggested that the phenomenon of vagueness is that tolerance is both true and false; as such, vague predicates like "is high up" have *inconsistent cutoff*

48 What about phrasing tolerance with a conditional that does obey modus ponens? See [Beall and Colyvan, 2001a]. Depending on the conditional, that would deliver a principle that is simply false. See Section 2.2.3.5 and [Weber et al., 2014].

points. Some altitude is both high up and not high up. I will now argue that, correlatively, there can be *more than one* "first" high up point, and that the abundance of legitimate cutoffs is *why* it seems vague: there are so many "the first" to choose from, we can't believe any of them are really the first. But they are.

How can there be more than one "the first" cutoff point? *Definite descriptions* such as "the first high-up point" can, on a paraconsistent analysis, be satisfied by multiple objects. Russell's 1905 analysis [Russell, 1905a] suggests that "a_k is the unique φ" should be glossed as

a_k is φ, and for all j, if a_j is φ then $a_j = a_k$.

Russell uses a material conditional. Unpacking the description, then:

a_k is φ, and for all j, either a_j is not φ, or $a_j = a_k$.

Now establishing that $a_j = a_k$, given that a_j is φ, would require an application of disjunctive syllogism. If, however, it is possible that a_j is both φ and not, then such an inference would be incorrect. What does this portend?

Using the idea of definite descriptions, we will say that some z is a *cutoff* for φ on the condition that it is the first non-φ, e.g., z is not φ but nothing before z is not φ, or

$$\neg\varphi z \ \& \ \forall y(y < z \supset \varphi y).$$

Now, consider what counts as a cutoff in the following stripped down scenario:[49]

$$\overbrace{a \quad < \quad b}^{\varphi}\underbrace{\quad < \quad c}_{\neg\varphi}$$

It may appear that b is the obvious cutoff point, and that c is not a cutoff. But this would not be the whole story. Because b is overdetermined, it will turn out that both b and c are cutoffs for φ, even while they are numerically distinct. For everything before b, namely a, is high up, but b is not high up. So b is (at least) a cutoff. On the other hand, everything before c is high up (because b also *is* high up), but c is not, so c is a cutoff. So both b and c are cutoffs. On assumption, b was distinct from c. Therefore, being a cutoff does not imply being unique. Uniqueness of the cutoff is not necessary.

If we read the Russellian analysis as suggested – that a_k is *the* φ – then there can be *more than one least* such and so. A definite description, "the first φ," may be satisfied by more than one object. According to the paraconsistent picture of sorites, a Russellian definite description holds, as it should, without the further consequences that are only drawn if inconsistency is discounted. If vagueness behaves anything like the picture presented here, drawing these further consequences about uniqueness would be as disastrous as inferring that everyone is bald. The notion of "least" is not univocal in inconsistent contexts.[50]

49 A four-element model was first suggested by Sam Butchardt. Is there a one-element model that would do the same job? Stay tuned.

50 What about rephrasing Russell's definite description scheme with a conditional $\rightarrow$ that *does* obey modus ponens, e.g., "$\varphi(x) \ \& \ \forall y(\varphi(y) \rightarrow x = y)$"? As with the issue of rephrasing the sorites argument with a detachable conditional, this

One more feature of the model should be pointed out. Both b and c are cutoffs, and not. That is, b is high up, so b is also not a cutoff:

$$\neg\varphi b \mathbin{\&} \forall y(y < b \supset \varphi y) \quad \text{and} \quad \neg(\neg\varphi b \mathbin{\&} \forall y(y < b \supset \varphi y))).$$

Meanwhile, b is not high up, so some things before c are not high up and therefore c is (at least) not a cutoff:

$$\neg\varphi c \mathbin{\&} \forall y(y < c \supset \varphi y) \quad \text{and} \quad \neg(\neg\varphi c \mathbin{\&} \forall y(y < c \supset \varphi y)).$$

Thus it is false, as well as true, that they are cutoffs; but they are the only cutoffs. While it is therefore true that in the model there is a cutoff for φ,

$$\exists z(z \text{ is a cutoff for } \varphi),$$

it is, consequently, also false, and hence has a true negation:

$$\neg\exists z(z \text{ is a cutoff for } \varphi).$$

Read literally, this says that *there is no cutoff at all*. This accounts for our incredulity about the existence of a cutoff point: no n is the cutoff point, because $\exists k(\varphi k \mathbin{\&} \neg\varphi(k-1))$ comes out as both true and false. There is more than one cutoff point, and none.

How does this *explain* the sorites? In part, I think the pull of TOLERANCE is due to an implicit, utterly reasonable question. If TOLERANCE is false and there really is is a cutoff point, *where* could it be? Intuitions cry out that it is *impossible* to believe that vague predicates have sharp boundaries. It is incredible – there *cannot* be such a sharp boundary; otherwise, why can't we identify it? One has the feeling that if there were a sharp boundary, we could say where it is. And while I suspect this sort of worry places too much emphasis on epistemic accessibility,[51] I do think part of any approach to the sorites paradox needs to offer some sort of answer to this reasonable question. The model here goes some way in doing so. Why can't we *identify* cutoffs? Because we expect any such identification to be *unique*, fixing the one-and-only first φ, and such cutoffs are not unique. In the three-element model, just by dint of cutoff b being overdetermined, c turns out to be a cutoff. Despite appearances, one cutoff point cannot be said to be a more natural or obvious cutoff point than the other, and this is how to vindicate our competing intuitions: *the first* φ is not the *only* φ. The progression is like descending from Olympus Mons itself. A single inconsistent cutoff appears; but then, by inconsistency, there cannot be just one cutoff; and so finally, since any one cutoff is a defeater for all the others, *there is no cutoff*, which is what we expected all along.

depends on the conditional (e.g., for using relevant implication in related applications, see [Dunn, 1987]) but the upshot is that anything stronger than the material hook is implausible as an expression of "the." We will return to this in the context of mathematical functions versus relations in Chapter 5.

51 Williamson writes: "When we conceive that something is so, we tend to imagine finding out that it is so. We are uneasy with a fact on which we cannot attain such a first-personal perspective. We have no idea how we ever could have found out that the vague statement is true, or that it is false, in an unclear case; we are unable to imagine finding out that it is true, or that it is false; we fallaciously conclude that it is inconceivable that is it true, and inconceivable that it is false" [Williamson, 1994, p. 3].

"How did you go bankrupt?" asks a character in Hemingway. "Two ways," is the answer: "Gradually, then suddenly." The sorites is a *paradox*, because the situation it describes is unbelievable, even while the principles generating that description make it unavoidable. The sorites paradox is that vague predicates cannot have sharp boundaries, even as they must. The dialetheic paraconsistent solution to the paradox is that it is true. Every TOLERANCE pair $\varphi_n \supset \varphi_{n+1}$ is true, but some things are φ and some things are not. There is some cutoff point, but there is also no such cutoff, since both tolerance and its negation are true. The intuitive data are (putatively) preserved, without absurdity, and the sorites is explained as a feature of more fundamental inconsistency. The paraconsistent mathematics to back this up is in Part III. Connections between sorites and the other paradoxes are discussed at the end of Section 1.4.

1.3 Boundaries

It is getting dark. Let's go back inside. It is twilight now, both day and night, but soon it will be only night.

Night is when we are in the the shadow of the Earth. But we do not think of the darkness of the night as one big shadow. Why not? Casati suggests it is because the shadow of night has no *edges*, or at least not ones we usually see,[52] and a shadow is ordinarily thought of as something with edges, or boundaries. Shadows are strange in themselves;[53] and yet, even for entities that minimally qualify as entities, there is a clear intuition that shadows must have an edge in order to exist. Boundaries are integral in some way to there being objects. Euclid [Book 1, Def 13–14, emphasis added] [Euclid, 1956]; emphasis added:

> A boundary [terma] is the extremity of any thing. A boundary is the limit within which anything is contained. *A figure is given by its boundaries*.

Without the limit of a figure, there is no figure at all.

Even more so than with sets, truth, or vagueness, objects with boundaries are so much a part of our everyday experience that it is at first highly implausible that they be paradoxical. We navigate boundaries *all the time*, from not bumping into people when we walk down the street, to turning off the water tap before the sink overflows. The boundaries of human bodies are basic to our phenomenal experience.[54] We are inclined to assume, incorrectly, that if something is a commonplace, then it could not be problematic; but material objects in space are every bit as puzzling – and paradoxical in *the same way* – as sets, semantics, and sorites.[55]

52 In [Casati, 2003]. A planet's *terminator* is the edge of the night.
53 Cf. the *Yale shadow paradox*, 1975 (see [Sorensen, 2008]), used to motivate dialetheism in [Mares, 2004b].
54 As Merleau-Ponty argues; see [Dillon, 1997].
55 Thanks to Aaron Cotnoir for collaboration on this topic in [Weber and Cotnoir, 2015].

1.3.1 The Problem: Symmetry and Connected Space

There is a Great Red Spot on the planet Jupiter.[56] Estimates vary as to how old it is (in the vicinity of 350 years), but all agree it is an anticyclonic vortex, a gigantic and remarkably persistent storm at least twice the size of the Earth.[57] Although the Spot is dynamic, fluctuating in size and shape, it is stable and stark, visible through Earth-based telescopes. What is it that makes the Great Red Spot a distinctive part of the Jovian sky?

The Great Red Spot is a very large and red example of a much more general phenomenon – the existence of material, ordinary objects. What is it that makes the Great Red Spot an *object* at all? This is like the question that we asked, and answered, around set composition, about how the *many* particles become *one* spot (Section 1.1.3.1). For the world is unquestionably full of *stuff*;[58] some of this stuff *composes* so as to make other stuff, called *ordinary objects*;[59] and a natural observation about ordinary objects is that each of them has a *boundary*. Aristotle defined the boundary of an object as "the first thing outside of which no part is to be found, and the first thing inside of which every part is to be found" (*Metaphysics* V, 17, 1022a4–5). As Varzi puts it, "whether sharp or blurry, natural or artificial, for every object there appears to be a boundary that marks it off from the rest of the world" [Varzi, 2004]. A boundary contributes crucially to there being a figure at all, a *thing*, and not merely a disenfranchised mess of unrelated *stuff*.

Boundaries are puzzling, too. The sorites paradox has already shown this: the existence of twilight threatens to, but cannot, mean there is no difference between day and night.[60] Now focus on the dual problem: could there be a difference between day and night *without* the intermediary of twilight? Take the following puzzle: consider the Great Red Spot on Jupiter, and trace a path from a red-colored point inside the Spot to a non-red colored point outside the Spot. What happens as we pass through the boundary of the Spot?

The most obvious answer is that we pass from the last red point to the first non-red point (passing over any question of whether *points* have a color). But on the assumption that space is *dense* (recalling the continuous sorites in Section 1.2.1), there would be an infinite number of points in between. What color are they? Do they belong to the Great Red Spot, or to its complement in the Jovian atmosphere? Either some point both is and is not part of the Great Red Spot, or else neither, leaving a "gap" on the face of Jupiter. It is *arbitrary* simply to assign the boundary to the Spot, say, and leave its complement in the atmosphere unbounded [Varzi, 2004], or vice versa. That would violate the *principle of sufficient reason* (PSR), our Premise 0 from the preface. It is incomplete not to assign the boundary at all; but the remaining alternative is inconsistent, as I will now try to bring out.[61]

56 There is also a giant hexagon on Saturn. Look it up.
57 [Rogers, 1995, pp. 191–196].
58 "Stuff" is a technical term; as in [Miller, 2009], it is contrasted with "thing": there is *a* thing but *some* stuff.
59 Another technical term; see [Thomasson, 2007], also [Koslicki, 2008].
60 Example attributed to Dr. Samuel Johnson.
61 For discussions of similar material, see Varzi [1997]; Casati and Varzi [1999, p. 74]; Dainton [2010, ch. 17]; Hudson [2005, chs. 2, 3]; Priest [2006b, chs. 11, 12, 15]; Arntzenius [2012, ch. 4]; Hellman and Shapiro [2018, 7§4].

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1.3.1.1 Symmetry

As with the sorites paradox, the boundary puzzle does not go away once we bring it mathematical rigor. It becomes more puzzling.

As mentioned in the Introduction, the modern theory of continuity was developed in part by Dedekind [Dedekind, 1901], as a reverse engineering feat around the intermediate value theorem. He wanted to ensure that, given a starting point inside a set of points, e.g., the Great Red Spot, and a finishing point outside it, that a continuous path must *cross* the boundary between them. As for which side the boundary line belongs to, Dedekind dismissed the problem: one or the other, but whichever we like, he said.

> If now any separation of the system R into two classes A_1, A_2 is given which possesses only *this* characteristic property that every number a_1 in A_1 is less than every number a_2 in A_2, then for brevity we shall call such a separation a *cut* [Schnitt] and designate it by (A_1, A_2). We can then say that every rational number a produces one cut or, strictly speaking, two cuts, which, however, we shall not look upon as essentially different [Dedekind, 1901, § IV, emphasis in the original text.]

By definition, a *closed* object includes its boundary. The boundary of an *open* object belongs to its complement. So Dedekind's solution is that a boundary splits a (continuous) space into two parts, where one part will be open and the other closed. The same goes for dividing a one-dimensional line, as we saw (in the Introduction): the center of the line is an indivisible geometric point, so any division of it leaves one side open and the other side closed.

Dedekind's solution is ingenious, but it solves a mathematical problem by creating a metaphysical one. Given no other facts about the space, *why* do the open/closed properties fall where they do? Neither macroscopic analysis nor mathematical argument settle the matter, since physically the object loses its definition at very small scales, and logically, the properties of being open and closed may be interchanged without any rational difference.[62] The principle of sufficient reason[63] motivates the following constraint:

Symmetry: Objects that otherwise have no relevant difference between them have no relevant difference between their boundaries.

From Symmetry follows a corollary: if there is an object with a closed proper part, then, absent any further compelling reason otherwise, the relative complement of that part is closed too, if no reason can be given why one side should get the boundary rather than the other. Note carefully, however, that Symmetry does *not* say that *every* division of an object is symmetric. That would be silly. Nor does it say that all objects are closed. Symmetry says that there is *some* way to divide an object without creating an imbalance. Symmetry is violated when *every* way of dividing objects forces an arbitrary difference between its otherwise indifferent parts.

62 [Pratt-Harmon, 2007, p. 14], although see Chapter 9.

63 "The principle of sufficient reason, on the strength of which we hold that no fact can ever be true or existent, no statement correct, unless there is a sufficient reason why things are as they are and not otherwise – even if in most cases we can't know what the reason is" [Leibniz, 1714, p. 32]. For more on the PSR, and its role in Spinoza, see [Della Rocca, 2008].

To put SYMMETRY the other way around, then, if there *is* a difference between the boundaries of two objects a, b, (which we will write $\partial(a), \partial(b)$ (a precise definition of ∂ is given in Section 9.2.1)), a principled and explicable difference that persists across all cases, then there is an explicable difference between the objects after all. This is itself a continuity intuition (cf. Section 8.3.2), and closely related to tolerance – roughly, if a is very similar to b, then $\partial(a)$ is very similar to $\partial(b)$. Where one object is open and another is closed, there should be some reason as to why; to ask which is which about Jupiter and its Spot is to ask "an embarrassing question" [Casati and Varzi, 1999, p. 87].

The puzzle is not merely physical or mathematical. An object may be *open and closed* (even classically, where this is sometimes called being "clopen"), but two open-and-closed objects *cannot*, on the classical account, touch (Section 1.3.1.2). What metaphysical sense can we make of the classical account of bounded objects in connected space? Are there mysterious metaphysical forces such that some objects (ones that are both closed) simply must repel one another?[64] Or, if not mysterious forces, then primitive mysterious *differences*, a "preordained asymmetry"? Nothing at this level of generality accounts for some objects being open and others closed. And the level of generality is important: we are not trying to understand why some objects are round, and others square, or why some objects are flowers and others are trees; we want to know why for the continuum, and so for any continuous object, symmetric division always fails. The reason must be something about objects qua objects – not a case-by-case inspection of each situation, but something about the pure geometry of how things sit together in the world, just by virtue of their being things. No matter how serendipitous one thinks the conferring of structural properties has been, differences at this level of generality call out for an explanation. The SYMMETRY problem is the postulation of a distinction without a difference.

1.3.1.2 Connectivity

The bite of the boundary puzzle comes from the assumption that space is connected. The key fact is just this:

Separation: A space is *separated* iff there is an exclusive and exhaustive division of the space into two closed parts.

Spaces that are not separable are *connected*:

Connection: A space is *connected* iff every exhaustive division of the space into two closed parts is not exclusive (has a nonempty overlap).

Since the basics of topology tell us that the complement of a closed set is open,[65] both parts of the partition in a separation are open, too. Connectivity is enmeshed with the concepts of open and closed sets: if a space is connected, then according to classical topology following

[64] "A certain class of objects are unaccountably differential to one another – always just managing to step out of one another's way – while they bang heedlessly into the members of another class of objects. Surely repulsive forces would *have* to be posited to explain such behavior" Zimmerman [1996, p. 12]. See Sider [2000, p. 587] for a reply.

[65] Kelley [1955, p. 40].

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Dedekind, it cannot be that it divides symmetrically. That would just be a separation.[66] What this shows is that a space that satisfies the SYMMETRY intuition cannot, apparently, be connected. And this is itself a problem, because the space of naive experience *is* connected. There are no rips or tears.

In classical topology, the definition given in SEPARATION is equivalent to another: that *the only sets in connected space that are both open and closed are the entirety of the space, and the empty set*.[67] Why? How does this rather recondite restatement follow from intuitive claims about space? Well, consider an apparently connected space like Jupiter, and some nonempty proper part of it, like the Great Red Spot, G. The boundary of G, intuitively, is made up of all the points that are "extremely close" to both G and its complement $\overline{G}$. Suppose G is both open and closed. Then:

(1) If G includes its boundary $\partial(G)$, if it is closed, then all the points in $\partial(G)$ are points in G.
(2) If G is open, then all the points in $\partial(G)$ are points in $\overline{G}$.
(3) Ergo if G is both open and closed, then for any $x \in \partial(G)$,

$$x \in G \qquad \text{and} \qquad x \in \overline{G}$$

which is a contradiction, $x \in G$ and $x \notin G$.
(4) By classical reductio, then, there is no $x \in \partial(G)$, and the boundary is *empty*, $\partial(G) = \varnothing$.
(5) But an empty boundary between G and its host space means that G is *separated* from the host space, and that the overall surface is *disconnected*.

Therefore, if the Great Red Spot is really a nonempty proper subspace of Jupiter's atmospheric surface, *and* it is both open and closed, then there is a rip in the fabric of the space, and so the space is not connected. Thus a connected space has no nonempty subspaces that are both open and closed.

The existence of bounded, discrete entities grates against the connectivity of space (cf. Section 9.2.1). The problem aligns closely with the sorites, as discussed in Section 1.2, with SYMMETRY for TOLERANCE and SEPARATION for the line-drawing negation of TOLERANCE. If objects differ, they must inevitably begin to differ somewhere, but any exact point where the difference begins looks to "cut" our lived-world impossibly.

Reasoning has led us into a corner once again. On the one hand, objects that are touching share a boundary; space is connected. On the other hand, objects do not arbitrarily differ; boundaries are not *random*; and so *both* objects in the connection can still have their boundary as a part. This breaks the space into two subobjects that are both open and closed. If a subobject in space is both open and closed, then the space is not connected – but there *are* subobjects in space that are both open and closed, and yet space *is* connected! The underlying principles here – intuitions about the topology of our lived experience – are

[66] Cf. [Paul, 2006].
[67] [Kelley, 1955, p. 53; Hocking and Young, 1961, p. 15].

perhaps not as axiomatically compelling as naive set comprehension, but they are robust, as we're about to check by considering if there are other options.

1.3.2 Options

The previous section has already detailed the orthodox cause and solution to the problem, for example defended in [Bolzano, 1851]: a boundary may be part of an object *or* its complement, but not both. Other approaches all endeavor to explain how a boundary never arbitrarily belongs to an object over its complement, always assuming the connectivity of space. The increasingly familiar options of incompleteness and inconsistency return; but now, unlike in the case of abstracta such as propositions or sets, the implications are literally palpable.

1.3.2.1 Underlap

Analogous to the nihilist view of vagueness, or the nihilist view of mereology, is the so-called *eliminativist* response to this problem: boundaries do not exist. At least, not as parts of objects: boundaries are either abstractions (equivalence classes) of convergent series of nested bodies, the boundary of the space-time receptacle of that object, or some other ersatz replacement concept.[68]

Eliminativism automatically avoids assigning boundaries arbitrarily, but only because it does not assign boundaries at all. Since we are interested in explaining boundaries, not explaining them away, this way of (vacuously) accommodating SYMMETRY is a nonstarter. It may be a *problem* (paradoxical, even) to face liar sentences, Russell sets, vague predicates, and boundaries, but that does not excuse us from doing so.

Analogous to the "indeterminateness" intuition from vagueness, it is natural enough to think that a boundary may be neither part of an object nor part of its complement.[69] And in analogy to the paracomplete/gappy approaches to the sorites paradox, proposing an "underlap" between objects faces basic revenge problems. For what separates the Great Red Spot from its complement? A boundary exists between them, neither part of the Spot nor its complement – which leads to a new question: since the boundary and the Spot are disjoint, what separates the boundary from the Spot itself? What separates the boundary from the Spot's complement? Presumably, by parity of reasoning, there must be a *new* boundary between the Spot and its boundary. If we reapply the underlap solution, *this* boundary is part of neither the Spot, nor the boundary of the Spot, and we're off on a regress of infinitely many distinct gaps between any two objects in contact. To the extent that one boundary needs explaining, multiplying new boundaries only multiplies the problem. This is very similar to the chase of cutoffs in higher-order vagueness (Section 1.2.2.2);[70] this is paradigmatic revenge.

[68] The most notable eliminativist is Whitehead; see [Clarke, 1985; Hellman and Shapiro, 2018].
[69] Varzi [Varzi, 1997, p. 27] attributes this view to Leonardo.
[70] Thanks to Marcus Rossberg here.

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1.3.2.2 Overlap

In what is known as the *coincidence* view, there are two coinciding boundaries: one that is part of an object, and one that is part of its complement.[71] These two boundaries occupy the same space. The theory claims that all entities whatsoever have boundaries as parts as a matter of metaphysical necessity [Smith, 1997, p. 18]:

> . . . the possibility of a coincidence of boundaries is essential to the concept of what is continuous *[Brentano, 1988, p. 5].*

On this view, all entities are *closed*. In this way, there is no arbitrary choice between two adjacent objects as to which is to be open and which closed. Both must be closed, each including their own boundary.[72]

The coincidence view approaches, but does not embrace, acceptance of contradiction. Even if the boundaries coincide, they are distinct. The view then suffers from basic revenge. For, on this view, there are many otherwise indistinguishable boundaries, all colocated and doing the same work. While SYMMETRY is nominally respected in that there is no arbitrary choice between bounded and unbounded objects, there still is a choice as to which object a given boundary belongs. The Great Red Spot has a boundary b_1 and so does its complement b_2. But given the coincidence of b_1 and b_2, what makes it the case that it is b_1 and not b_2 that is the Spot's boundary? The two boundaries occupy exactly the same set of points; to mark any difference at all beyond two names, it must be that boundaries are essentially "directional," intensional. A given boundary is always directed toward the (only) entity that it bounds. (A point in the interior of a solid sphere is a boundary in *every* direction.) But there is no explanation for a given boundary's direction. Boundaries are attached to their objects in an essentially arbitrary way. So the violation of SYMMETRY is very basic: there is a distinction between two boundaries, recorded by directionality, but no *difference* between them. This is little better than the original problem, which was explaining why a boundary went one way rather than the other. This is revenge.

1.3.3 Toward a Paraconsistent Topology

The coincidence view is approaching a simpler *glutty* view: a nonempty boundary may simply be part of *both* an object and its complement. Points very near both the Great Red Spot and its complement are both parts of the Spot and not. It is perhaps Brentano's coincidence view that best shows how our competing intuitions are irreconcilable – if consistency must be maintained. Objects in contact overlap, if they are to touch symmetrically. Overlap means that the objects share some points, either their entire boundaries or some relevant portions thereof; when an object touches its complement, there will be points both on and

[71] See [Chisholm, 1984; Brentano, 1988]; cf. [Smith, 1997].

[72] Concomitantly, there are a great deal of coincident objects: "Each point within the interior of a two- or three-dimensional continuum is in fact an infinite (and as it were maximally compressed) collection of distinct but coincident points . . . (Not for nothing were the scholastic philosophers exercised by the question as to how many zero-dimensional beings might be fitted onto the head of a pin)" [Smith, 1997, p. 10].

not on an object, calling for a paraconsistent treatment. Being disjoint (no overlap) may hold simultaneously with overlap. Thereby, SYMMETRY can be satisfied simultaneously with CONNECTION.

That's the thought, in any case. The project of understanding boundaries (at this level of generality) comes down to a need for an account of proper, nonempty subspaces of connected space that may themselves be both open and closed. And, it should be added, the theory that vindicates this still must bear enough resemblance to ordinary topology to be modeling the same concepts. It is left to Chapter 9 to see whether enough precision can be given to this naive idea. For the moment, let's circle back to make the connections between this paradox and the others clearer, by looking at how to express the sorites paradox in purely topological terms.

1.3.3.1 A Topological Sorites

As with the continuous sorites, the sorites paradox does not require a digital or discrete presentation; in fact, it does not even require an ordering relation. Or so a *toplogical sorites* purports to show.[73]

A function f is *locally constant* iff for each $x \in X$ there is a "neighborhood" of nearby points such that the restriction of f to that neighborhood is constant, all taking the same value. A *globally constant* function always takes the same value, without restriction to a neighborhood. It is a classical lemma that *if f is locally constant in a connected space, then f is globally constant.*

Let X be connected, and f a function from X to some set Y. Consider some region A of X. Now the range of f, Y, can be thought of as the pair of truth values $\{0, 1\}$, in which case the *characteristic function* σ of the set A is defined thus:

$$\sigma_A(x) = \begin{cases} 1 & \textit{if} \quad x \in A, \\ 0 & \textit{if} \quad x \notin A, \end{cases}$$

where $A = \{x : \varphi(x)\}$, the analogous $\sigma_\varphi(x) = 1$ if $\varphi(x)$ and 0 otherwise. But if f is locally constant on A, so that $\sigma_\varphi(x) = 1$ for every $x \in A$, then by the preceding lemma, f is globally constant: $\sigma_\varphi(x) = 1$ for every $x \in X$.

In keeping with the proposed definition of vagueness offered at the end of Section 1.2.1, say that a predicate is *vague* iff something satisfies it, and its characteristic function is locally constant – that's TOLERANCE – but not globally constant. As ever, in connected space, vagueness threatens to trigger a slippery slope avalanche: either nothing satisfies a vague predicate, or everything does.

Let X be connected and A be a subset of X, with A the extension of a vague φ. Then by the previous reasoning about locally constant functions, either $A = X$ or $A = \emptyset$. This gives us a *topological inductive* version of the sorites:

[73] This is a completely classical argument. As such, I only sketch the argument; see [Weber and Colyvan, 2010] for unpacking, or just [Jänich, 1984, p. 14], where I got it from. For criticism of this sorites construction, see [Rizza, 2013].

Some member of X is φ.
X is a connected space, and φ is vague.
$\therefore$ Every member of X is φ.

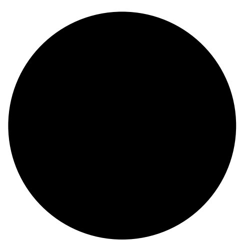
Contrapositively, the result is a *topological line-drawing* sorites: for some vague φ,

Some things in X are φ.
Some are not.
$\therefore$ X is separated.

The boundary of the φs in X is empty, on pain of contradiction. The usual sorites paradox is a special case of the boundary problem in space. The glutty interpretation of the line-drawing form, following from the preceding, is that X is only "separated" in the sense that φ has an *inconsistent boundary*.

* * *

With the topological sorites in view, let's see how it helps with the more usual sorites, and paradoxes in general. Here is *not* how the boundary puzzle was presented: given this object,



then is its boundary black or white? This form of the question is static, third-personal. It has all the needed components for the puzzle, but does not feel puzzling. No, rather the question is posed: *as you cross the threshold, what do you see*? It takes longer acquaintance with the situation to generate the relevant phenomenology of paradox (unbelievable!) with a static object, as the topological version of the sorites attempts to do. The sorites paradox feels most acute when it is dynamic, for example in the form of a "forced march" (... and is this man bald? What about this man? And this ... ?).[74] But topologically, no paths are required. The underlying structure of the space that makes a problem for the march is already there. Our first personal experience is how we come to learn about facts, but we (perhaps to our disappointment) eventually begin to learn that the facts don't care whether we interact with them or not. Marching has nothing to do with it.

I do not think that, if only we could speak or think about these problems clearly enough, they would be resolved. I think the history shows that the more clearly we think about these problems, the more irascible they become. Absurdity arises whether or not we the

[74] See [Horgan, 1994].

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humans bring our demands to bear on the world; we are not so powerful that we can make the otherwise-sanguine world absurd. We notice the paradox at the penumbra because it is there to be noticed – we are here *in the presence* of an absurdity. So, while our participation in the process makes the problem vivid, whatever is driving revenge *would already be there even if no one ever tried to solve the paradoxes*.

If there were only paradoxes to do with sets or semantics, that would be quite a lot – but the effect would remain intangible, abstruse.[75] The paradoxes could be compartmentalized away, as a quirk about human cognition, representation, language, or the like. Similarly, if there were only paradoxes to do with our responses to vague stimuli, that would be quite a lot, but not the job of pure metaphysics or mathematics to answer. The paradoxes go beyond subjective and mental experiences. I think the topological case indicates that the problem can be generated with no more than some points in space.

Just as logic is not a solution to the paradoxes (Section 0.2.3), the paradoxes are not about *us*. To put it colorfully, the paradoxes would be paradoxical even to God, and "God has no need of any arguments, even good ones" [Meyer, 1976, p. 94]. If that is right, our attention can focus at the right level to be trying to understand the paradoxes: what it is about the *world* as such, about objects and their boundaries, that makes there be paradoxes?

1.4 Conclusion

This chapter has presented several interconnected paradoxes. They are a problem for anyone with basic convictions about *comprehension*, *continuity*, and *composition*. Start with something obviously true (set comprehension; tolerance; the existence of boundaries in connected space), and you find a contradiction – but any attempt to evade the contradiction leads to revenge, because what you started with was obviously *true*.

One thing these paradoxes all have in common is that they are *resilient*. This is the revenge phenomenon, and a tell-tale sign of a genuine paradox. Revenge is what drives, I think, an acceptance-based approach of dialetheic paraconsistency (though not without its own costs; see Section 3.3.2, Chapter 4, and Chapter 10). The paradoxes are not solvable. Simple immunity to revenge is not a virtue just on the basis of scoring points against competing views; immunity to revenge shows that a position has got to grips with the real problem, while those that evade and are stung, evade and are stung, again and again, have not. Any solution does not merely fail, but recapitulates the original paradox.

Tappenden points out the striking fact that to state what is *wrong* with these paradoxes is just to *state* the paradoxes. In the case of the liar, we have a sentence ℓ for which

(*): ℓ is true iff ℓ is not true.

Sentence (*) "seems to capture one feature of the sentence ℓ in virtue of which ℓ is paradoxical," says Tappenden (he uses "k" instead of "ℓ"); he explains [Tappenden, 2002, p. 552]:

[75] Semantic dialetheism is discussed in [Mares, 2004b] and defended by Beall [Beall, 2009]. From another direction, *fictionalism* about dialetheism is discussed in [Kroon, 2004].

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Imagine that you are explaining the liar to someone who does not catch on right away. You might well say: "Here is what is funny about ℓ. If ℓ is true then it is not true, and if ℓ is not true then it is true." You have apparently just contradicted yourself by uttering a variation on (*). But (*) seems to be the right thing to say in the situation. And does it not get across precisely what is odd about the liar?

This is an intuitive way of explaining revenge. An attempt to solve the problem – or even articulate it – triggers the problem itself! As with the liar, so the sorites.[76] Tappenden describes a parallel scenario [Tappenden, 2002, p. 553]: you explain that "if any two samples are observationally indistinguishable to you, then one looks red to you if and only if the other looks red to you." Just like (*), this "seems like just the thing to say [to] correct the other person's mistakes . . . But if we are attempting to say something true, we are failing." And spatial boundary paradoxes suggest a common structure underlying these puzzles: that there is inconsistency at the edges of things.

In the last two chapters, I have tried, in effect, to suggest that this is the case for all the relevant paradoxes: simply to describe them (ordinals are sets of proceeding ordinals; some people are babies and some are not; closed objects in connected space can touch) is to find ourselves led into contradiction. And then, I've suggested ([Priest, 2002a] argues the point at length) that if we try to find our way out of the contraction, by denying the data in one fashion or another, that leads into contradiction *again*. In abstract, the structure of revenge is as follows:

Step 1 – there is a paradox, at the boundary between two (exclusive/exhaustive) categories.
Step 2 – solve the paradox by introducing a new, third category.
Step 3 – a new (is it?) paradox arises at the two new boundaries.

The pattern of the paradoxes is a particular sort of valid argument that ends in a particular sort of contradiction. The argument is one in which *denial of a premise* leads to a regress, or more euphemistically, a "hierarchical" picture (iterative sets, higher-order vagueness, gappy boundaries) where the hierarchy in some direct sense then affirms the very premise being denied; cf. [Priest, 2007].

In emphasizing revenge, I am *not* endorsing a principle like the following: "If two paradoxes respond the same way to treatment, they are the same type of paradox." That is not correct.[77] Rather, the problem in both the liar and the sorites (and by extension, the Russell paradox) is that there is something on a boundary where it shouldn't be – between truth and falsity, high and not high, the Russell set and its complement. We are compelled to admit that it really is there, because attempts to remove the bad overlapper lead to more boundaries with more overlappers, and not as a one-off accident but *no matter what*, a

[76] The thought that the sorites and the paradoxes of self-reference have something important to do with each other is also found in [Dummett, 1978, ch. 12; McGee, 1990; Sorensen, 1994, p. 53; Field, 2008, p. 106]. More recently, prominent approaches to paradox using substructural logic ([Zardini, 2011; Ripley, 2013]) are developed to treat both the liar and the sorites.

[77] See [Colyvan, 2008b].

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structural necessity.[78] The standard consistent/incomplete responses lead to the same end. They lead me to a paraconsistent approach – not as a last resort, but as first step toward something new.

* * *

In this chapter, I have set the scene by making the following case. Naive set theory is true. Therefore, there are true contradictions – paradigmatically, the liar and Russell paradox, among more sophisticated others. Naive sets provide a simple account of multitudes grasped together as a singular entitity. In the case of material, ordinary objects, this cohesion is particularly vivid when looking at the boundaries of things, and points on the boundary. The sorites paradox is the sharp edge of the boundary paradox, occurring on spaces where we find it particularly noticeable that there are boundary points that seem both part and not part of an object, multiple "first" cutoff points. Any response to these paradoxes leads, because of revenge, either explicitly or implicitly to a contradiction.

I've made this case at full speed, and don't pretend to have given proper consideration to all the arguments. We've seen some paradoxes, and I have broadly recommended dialetheic paraconsistency as a promising response. We could keep weighing up the options in the abstract, but to understand the glutty option properly, I think we need to see what it can actually do, in terms of making sense of the paradoxes. That's what I plan to pursue here. The question now, then, is: how is the dialethetic approach to proceed?

[78] Compare, for example, your plan to live a life of unfettered adventure, and your plan to settle down securely with a long-term partner; see [Slote, 2011], echoing Kierkegaard's injunction in *Either/Or* that whether you marry or do not marry, "you will regret it either way." : "Marry and you will be unhappy; do not marry, and you will be unhappy."

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