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THE 3-IRREDUCIBLE PARTIALLY ORDERED SETS

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The *dimension* [**4**] of a partially ordered set (poset) is the minimum number of linear orders whose intersection is the partial ordering of the poset. For a positive integer m, a poset is *m-irreducible* [**10**] if it has dimension m and removal of any element lowers its dimension. By the compactness property of finite dimension, every m-irreducible poset is finite and every poset of dimension $\geqq m$ contains an m-irreducible subposet. Thus, the set of all m-irreducible posets (up to isomorphism) can be characterized as the smallest set $\mathscr{S}$ of posets such that a poset has dimension $\leqq (m-1)$ if and only if it does not contain any poset in $\mathscr{S}$. Henceforth, only 3-irreducible posets are considered.

In this paper, the set of all 3-irreducible posets (up to isomorphism) will be exhibited. The number of 3-irreducible posets with m elements is: $0(m \leqq 5)$, $3(m = 6)$, $21(m = 7)$, $4(\text{even } m \geqq 8)$, or $5(\text{odd } m \geqq 9)$. If a poset and its dual were counted only once, the number would then be: $0(m \leqq 5)$, $2(m = 6)$, $13(m = 7)$, $3(\text{even } m \geqq 8)$, or $4(\text{odd } m \geqq 9)$. Let

$$\mathscr{P} = \{A_n | n \geqq 0\} \cup \{B, B^d, C, C^d, D, D^d\} \cup \{E_n, E_n{}^d, F_n, G_n, H_n | n \geqq 0\},$$

where these posets appear in Figure 1, and P^d denotes the dual of a poset P. The set $\mathscr{P}$ was introduced by I. Rival and the author in [**8**], where every poset in $\mathscr{P}$ was shown to be 3-irreducible. By [**7**, Theorem 6.1], $\mathscr{P}$ is the smallest set of posets such that a *lattice* has dimension $\leqq 2$ if and only if it does not contain any poset in $\mathscr{P}$ as a subposet. Let

$$\mathscr{R} = \mathscr{P} \cup \{CX_1, CX_1{}^d, CX_2, CX_2{}^d, CX_3, CX_3{}^d, EX_1, EX_1{}^d, EX_2, FX_1, FX_1{}^d, FX_2\} \cup \{I_n, I_n{}^d, J_n | n \geqq 0\},$$

where these posets appear in Figure 1. We will prove that $\mathscr{R}$ is the set of all 3-irreducible posets. Using different techniques, W. T. Trotter, Jr. and J. I. Moore, Jr. [**11**] have given an independent proof of this result.

1. The posets in $\mathscr{R}$ are 3-irreducible. We use the term *completion* for what is also called the *completion by cuts* [**3**] or *MacNeille completion*. The completion of a poset P is denoted by $\mathbf{L}(P)$. We will make use of the following five lemmas. Planar lattices are defined in [**3**] and [**7**].

LEMMA 1 (Banaschewski [**2**] or Schmidt [**9**]). *The completion of a poset P is*

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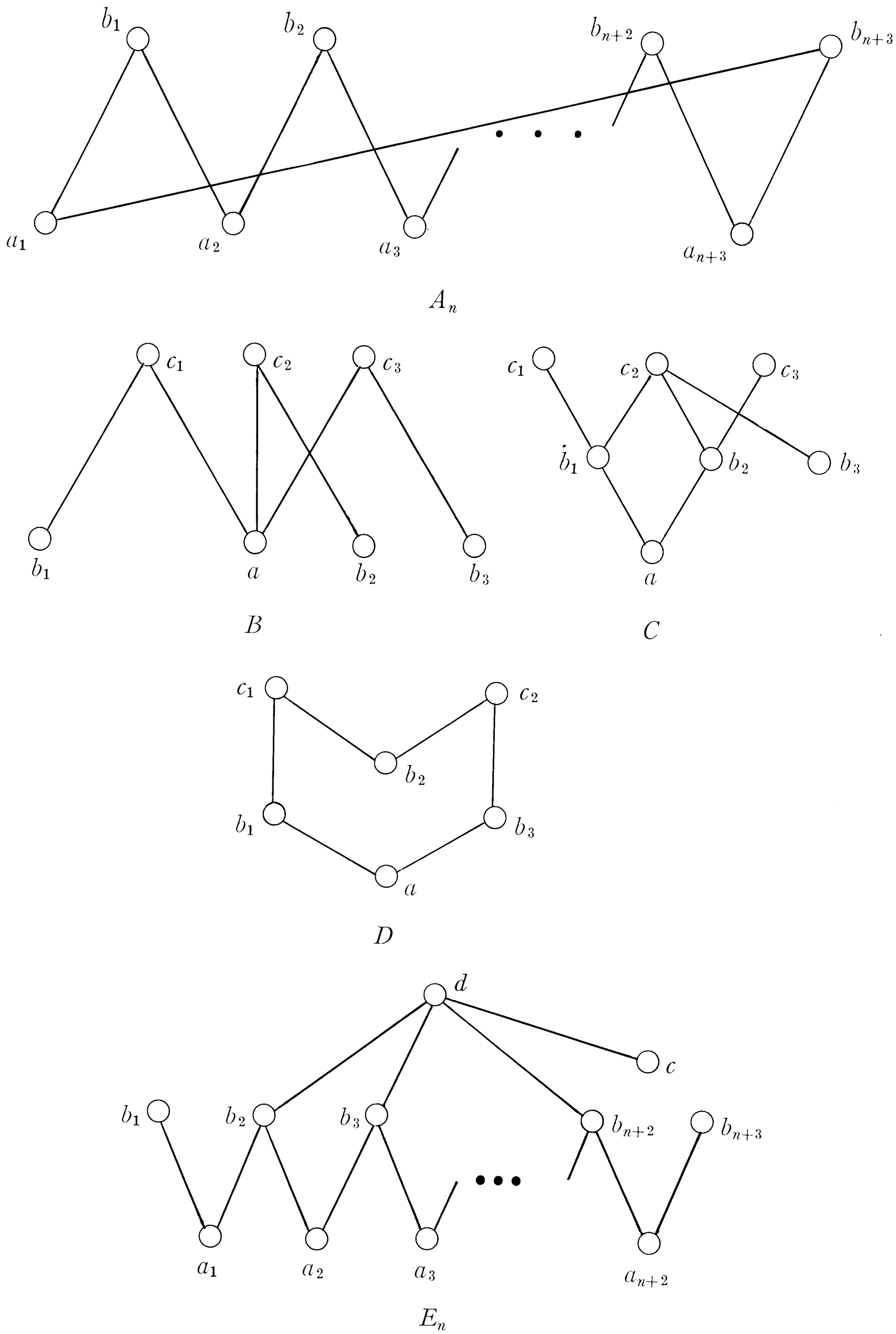

FIGURE 1. The 3-irreducible posets

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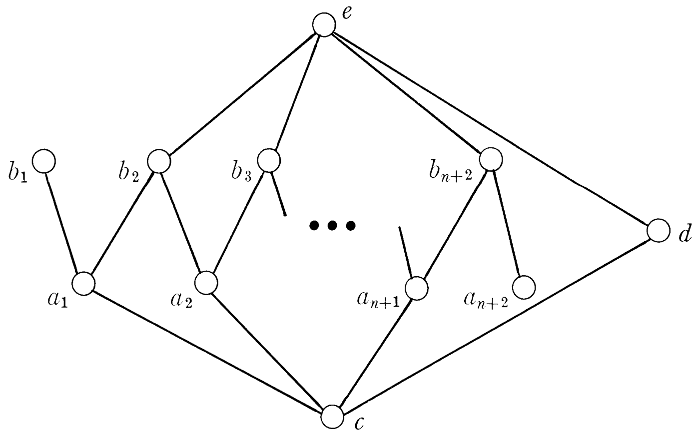
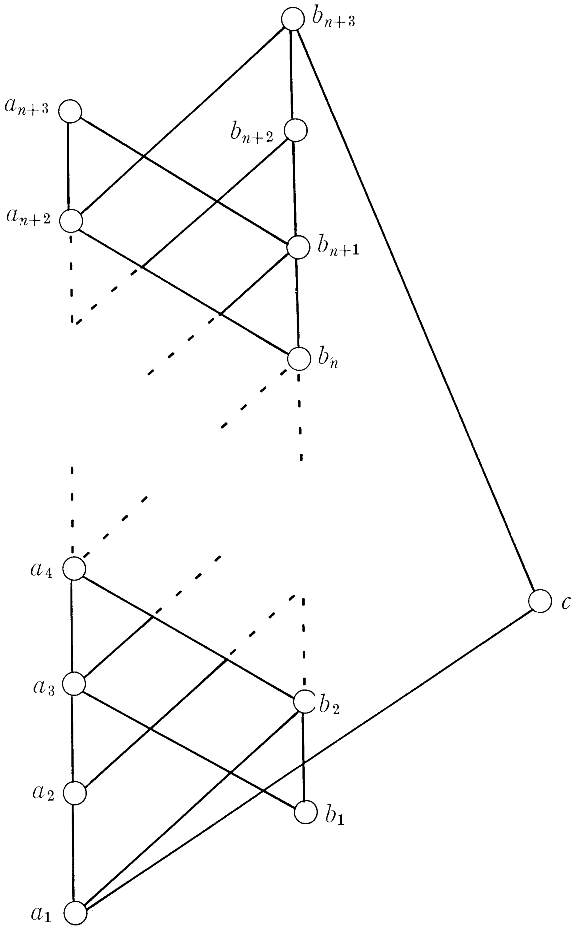


F_n



G_n

FIGURE 1—(*Continued*)

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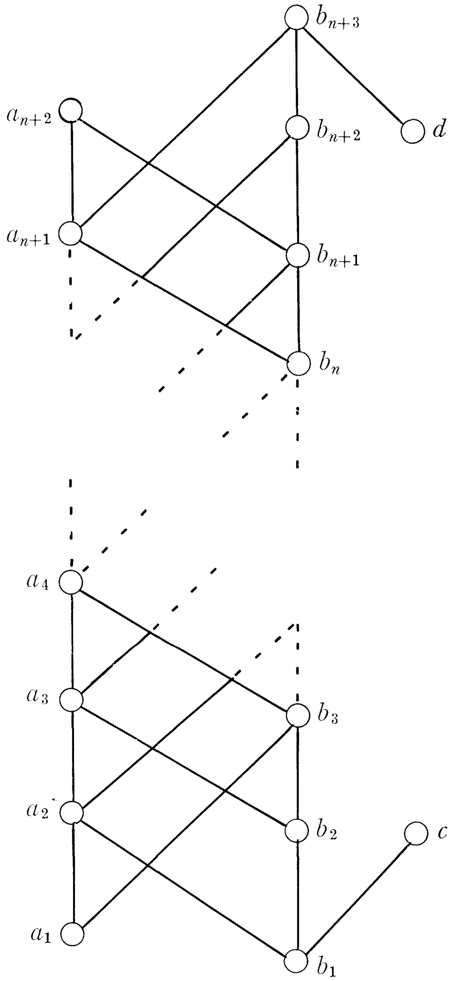
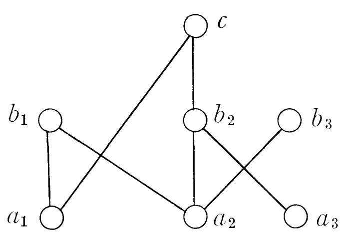


H_n



CX_1

FIGURE 1—(*Continued*)

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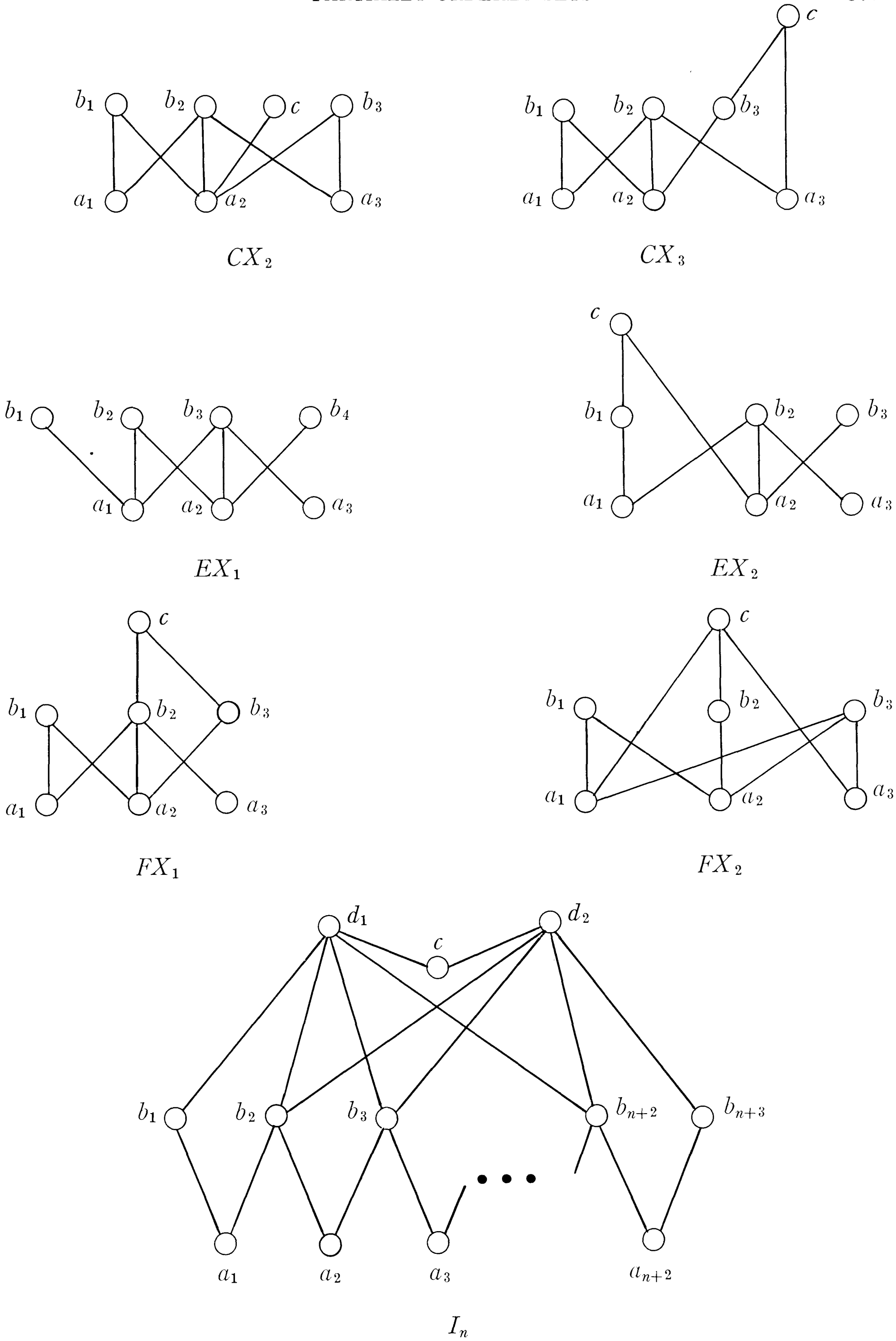

FIGURE 1—(*Continued*)

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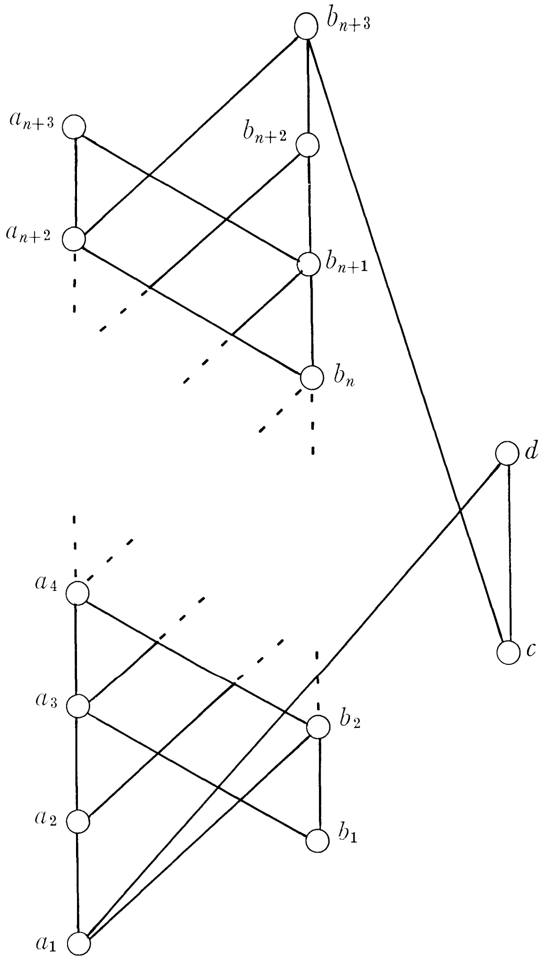


J_n

FIGURE 1—(*Concluded*)

the unique (up to isomorphism) complete lattice L containing P such that every element of L is both a join and a meet of elements of P.

LEMMA 2 (Birkhoff [**3**]). *If a lattice contains a finite poset P, then it also contains the completion of P as a subposet.*

Proof. Associate each element S of the completion of P, considered as a subset of P, with the join of S in the lattice (with the meet of P replacing the empty join).

LEMMA 3 (Baker [**1**]). *A poset and its completion have the same dimension.*

LEMMA 4 (Baker [**1**]). *A finite lattice is planar if and only if it is of dimension* $\leqq 2$.

LEMMA 5 (Hiraguchi [**5**]). *Adding one element to a poset increases its dimension by at most one.*

We now show that every poset in $\mathscr{R}$ is 3-irreducible. This statement was already proved for $\mathscr{P}$ in [**8**], and we will use similar techniques for $\mathscr{R} - \mathscr{P}$. We first verify that the diagrams in Figure 2 are correct. Each poset in Figure 2 is *dismantlable* [**6**], and therefore, a lattice by [**7**, Proposition 2.1]. The correctness of Figure 2 now follows by Lemma 1. The diagrams for the collection $\mathscr{L} = \{\mathbf{L}(P)|P \in \mathscr{P}\}$ are given in [**7**], where each completion is denoted by the corresponding boldface letter (with the same subscript, if any). The completions of CX_1, CX_2 and CX_3 contain $\mathbf{C}^d$ (as a subposet); $\mathbf{L}(CX_3)$ also contains $\mathbf{D}$. $\mathbf{L}(EX_1)$ contains $\mathbf{E}_0$ while $\mathbf{L}(EX_2)$ contains $\mathbf{E}_0$ and $\mathbf{E}_0{}^d$. The completions of FX_1 and FX_2 contain $\mathbf{F}_0$. For the rest of the paper, n will always denote a non-negative integer. The completion of I_n contains $\mathbf{E}_n$, and the completion of J_n contains $\mathbf{G}_n$. Lemma 3 now implies that each poset in $\mathscr{R} - \mathscr{P}$ has dimension $\geqq 3$. (Using [**7**, Proposition 5.3], it can be shown that the completions of the posets in $\mathscr{R} - \mathscr{P}$ do not contain any other lattices in $\mathscr{L}$, and thus, by Lemma 2, contain only the corresponding posets in $\mathscr{P}$.)

It only remains to show that removing any one element from any poset in $\mathscr{R} - \mathscr{P}$ leaves a poset of dimension 2. It will then follow from Lemma 5 that each poset in $\mathscr{R} - \mathscr{P}$ is of dimension 3, and therefore, 3-irreducible. If any element of CX_1 except b_1 is removed from $\mathbf{L}(CX_1)$, a planar lattice is left; thus, by Lemma 4, such a removal from CX_1 leaves a poset of dimension 2. The completion of the poset $CX_1 - \{b_1\}$, obtained by merely adding a zero and one, is obviously planar. Thus, applying Lemma 4, $CX_1 - \{b_1\}$ has dimension 2. The remaining posets in $\mathscr{R} - \mathscr{P}$ are handled similarly. For example, for $I_n(J_n)$, d_1 and d_2 (c and d) play the role that b_1 did for CX_1. Only for J_n must more than three elements be added to form one of the corresponding completions. This completes the proof that all the posets in $\mathscr{R}$ are 3-irreducible.

2. Starting the proof that $\mathscr{R}$ contains every 3-irreducible poset. Let L be a finite lattice, and let $P = \mathbf{P}(L)$, the subposet of *irreducible* elements of L. (An element of a finite lattice is *join-irreducible* (*meet-irreducible*) if it is not the joint (meet) of two incomparable elements; an element is *irreducible* if it is join-irreducible and distinct from 0, or meet-irreducible and distinct from 1.) The remainder of this paper is devoted to proving the following statement:

If L contains a poset in $\mathscr{P}$, then P contains a poset in $\mathscr{R}$.

Let P be a 3-irreducible poset, and let $L = \mathbf{L}(P)$. By Lemma 3, L has dimension 3. Therefore, applying [**7**, Theorem 6.1], L contains a poset in $\mathscr{P}$. Since $\mathbf{P}(L) \subseteq P$ by Lemma 1, the above statement will show that P contains a poset in $\mathscr{R}$; thus, $P \in \mathscr{R}$, completing the proof that $\mathscr{R}$ is the set of all 3-irreducible posets. (Actually, it is easily seen that $\mathbf{P}(\mathbf{L}(P)) = P$ for any m-irreducible poset P.)

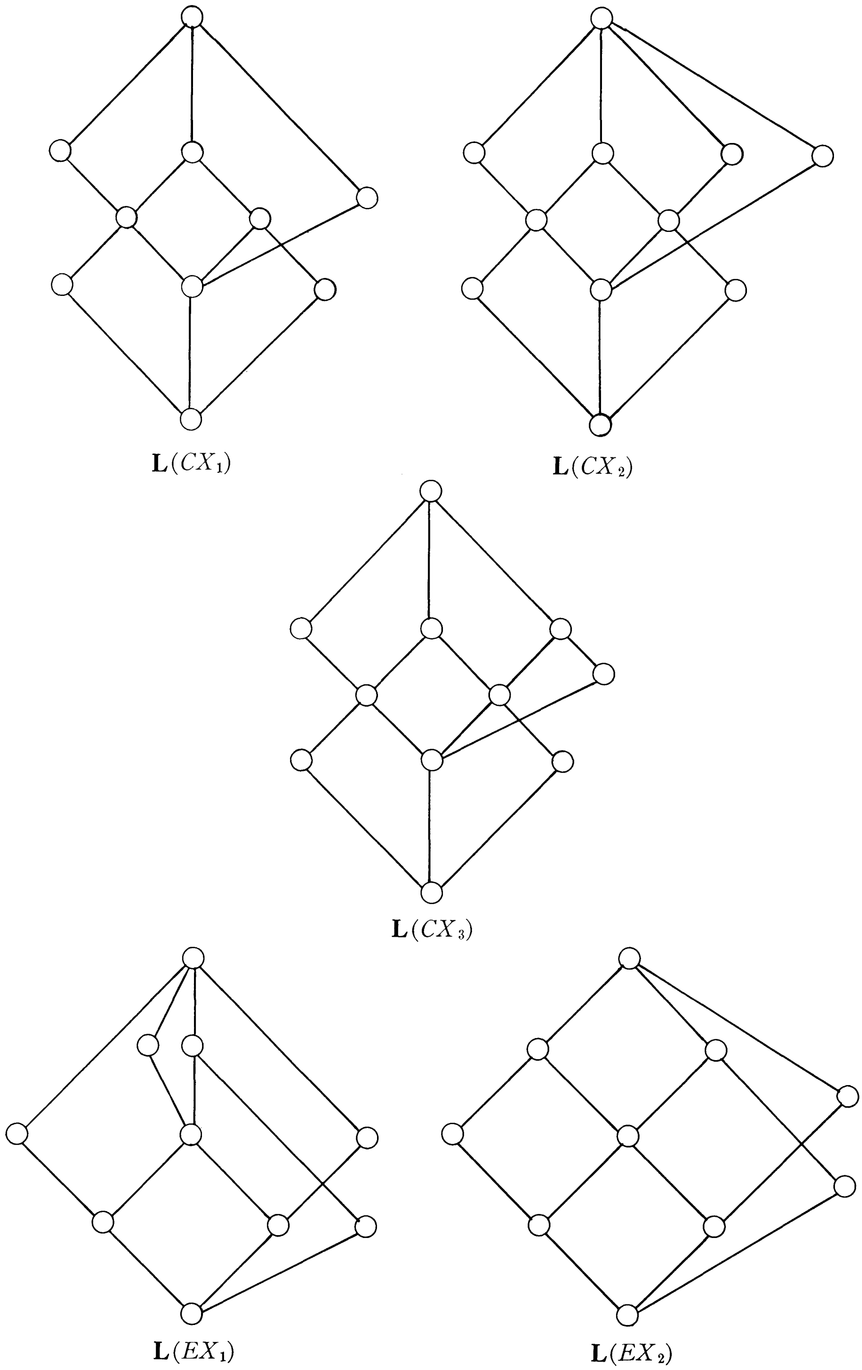
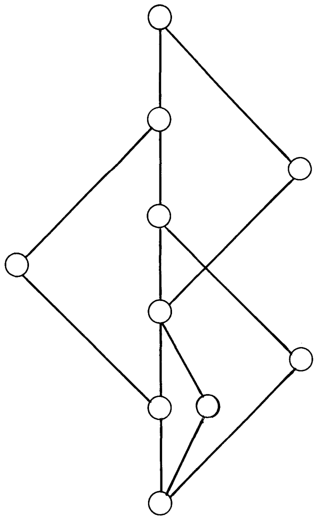
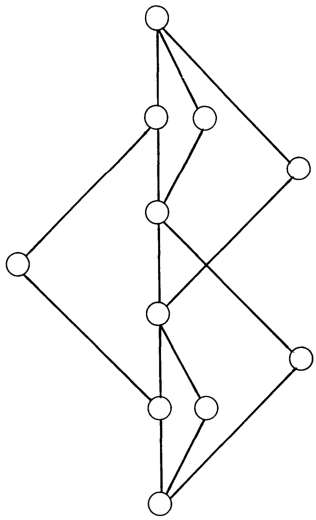


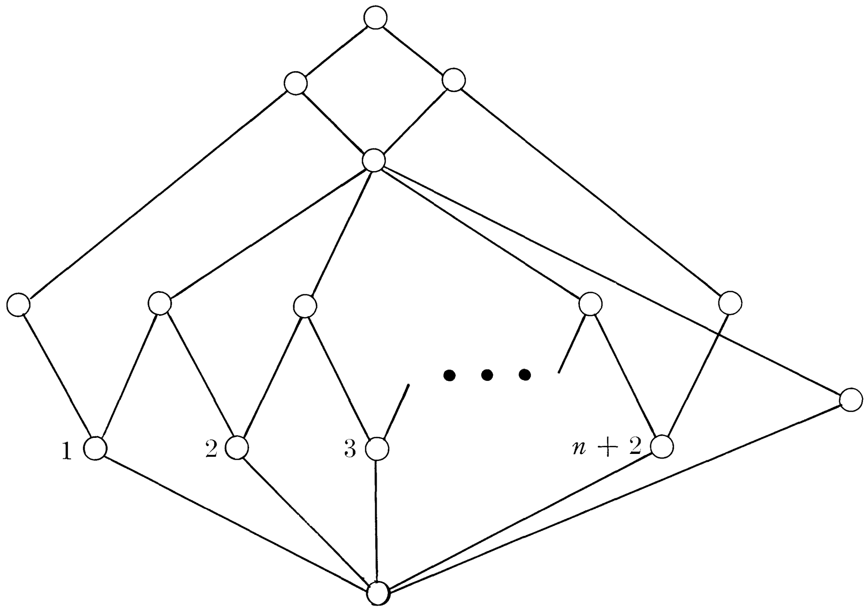
FIGURE 2. Completions of the posets in $\mathscr{R} - \mathscr{P}$



$\mathbf{L}(FX_1)$



$\mathbf{L}(FX_2)$



$\mathbf{L}(I_n)$

FIGURE 2—(*Continued*)

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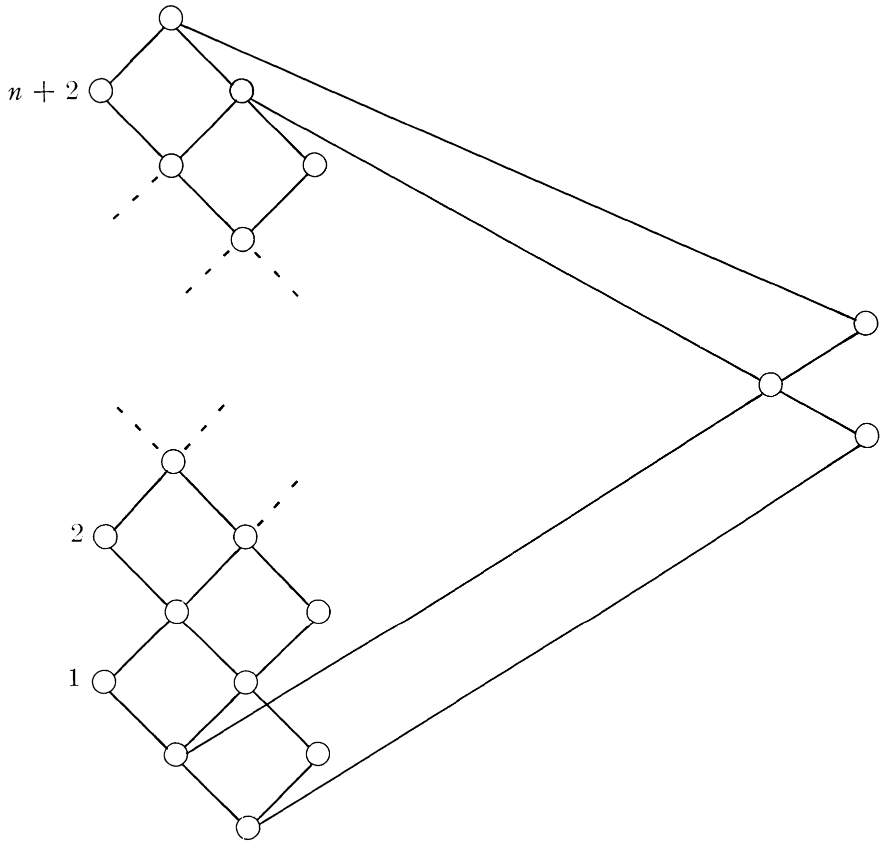

$\mathbf{L}(J_n)$

FIGURE 2—(*Concluded*)

For each $Q \in \mathscr{P}$, we will show that if L contains Q, then P contains a poset isomorphic to some R in $\mathscr{R}$. By duality, it is enough to let Q be a poset in Figure 1. We will consider separately each case where Q is denoted by one of the letters A to H (possibly subscripted). In any case involving a subscript n, we will assume that the smallest n was chosen such that L contains Q or Q^d. Unless stated otherwise, we also assume that all previously considered cases and their duals do not occur. These are our "standard assumptions".

The elements of Q are named as in Figure 1. The elements of the poset isomorphic to R will be given in the order determined by the labelling assigned to R or R^d in Figure 1 in the following manner: the alphabetical order predominates and then the numerical order on any subscripts is considered. For example, a poset isomorphic to B or B^d would be given in the order

$\{a, b_1, b_2, b_3, c_1, c_2, c_3\}$.

3. L **contains** A_n. For $1 \leqq i \leqq n+3$, let a_i' be a join-irreducible element of L such that $a_i' \leqq a_i$ but $a_i' \nleqq b_{i+1}$. (Subscripts are taken modulo $n+3$). If $a_1' \leqq b_k$ for some $k \geqq 3$, then taking the least such k, $\{a_1', a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ would be isomorphic to A_{k-3}. By the minimality of n, $k = n+3$. By symmetry, $a_i' \leqq b_j$ only if j is $i-1$ or i $(1 \leqq i \leqq n+3)$, and thus $\{a_1', a_2', \ldots, a_{n+3}', b_1, b_2, \ldots, b_{n+3}\} \cong A_n$. Meet-irreducible elements b_1' $(1 \leqq i \leqq n+3)$ can now be defined dually to yield a subposet of P that is isomorphic to A_n. We recall that a *crown* is a poset isomorphic to A_n for some $n \geqq 0$. We have actually proved

PROPOSITION. †) *If a finite lattice L contains a crown, then there is a crown in* $\mathbf{P}(L)$.

In the above proposition, $\mathbf{P}(L)$ need not contain crowns of the same size as L does. (Consider the lattice of subsets of a four-element set.) Using [**6**, Theorem 3.1], we obtain the

COROLLARY. *If a finite poset contains no crowns, then its completion is dismantlable.*

According to our standard assumptions, we henceforth assume that L contains no crowns. Since L does not contain A_0, L has breadth 2 [**6**, Lemma 3.4]. Therefore, any element $a \in L - (P \cup \{0, 1\})$ can be written as $a = x_1 \vee x_2 = y_1 \wedge y_2$ for suitable $x_1, x_2, y_1, y_2 \in P$. We note that none of the posets in $\mathscr{R}$ contain a zero or one. From now on, x and y (with or without subscripts) will always denote elements of P.

In each of the remaining cases where L contains the poset Q in $\mathscr{P}$, we now outline the procedure that will be followed. Each element a of Q is considered in turn. If $a \in P$, we proceed to the next element. If $a \notin P$, then it can be written as $a = x_1 \vee x_2 = y_1 \wedge y_2$. (In the sequel, whenever we write $a = x \vee x_2$, the conditions on x will ensure that $x \notin Q$, and dually.) Often, we can simply replace a in Q by one of x_1, x_2, y_1 or y_2 to obtain a poset Q' isomorphic to Q. Otherwise, we will show there is a subposet R of $(Q - \{a\}) \cup \{x_1, x_2, y_1, y_2\}$ that is in $\mathscr{R}$. This procedure is repeated with Q' or R replacing Q until we obtain a subposet of P that is in $\mathscr{R}$. The poset isomorphic to R will be given in the order determined by the labelling of R or R^d in Figure 1. When R replaces Q, its elements are named as in Figure 1.

We recall that a *down-down fence* [**7**] is a poset $\{a_1, a_2, \ldots, a_{n+2}, b_1, b_2, \ldots, b_{n+1}\}$ in which $a_i < b_i$ and $a_{i+1} < b_i$ $(1 \leqq i \leqq n+1)$ are the only comparabilities. (For any E_n or F_n, the subposet consisting of all the a_i's and b_j's is a fence.) The following two lemmas will be repeatedly applied in the sequel.

LEMMA 6. *If $\{a_1, a_2, \ldots, a_{n+2}, b_1, b_2, \ldots, b_{n+1}\}$ is a down-down fence in a poset L containing no crowns, and $y > a_1$ but $y \ngtr a_2$, then $y \ngtr a_i$ for all $i \geqq 2$.*

†This proposition and its corollary were obtained jointly with I. Rival.

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Consequently, y is incomparable with every element of the fence except a_1 and possibly b_1.

Proof. If $y > a_i$ for some $i \geqq 3$, then choosing the least such i, $A_{i-3} \cong \{a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_{i-1}, y\}$.

LEMMA 7. *Let $S = \{a_1, a_2, \ldots, a_{n+2}, b_1, b_2, \ldots, b_{n+1}\}$ be a down-down fence in a lattice L that contains no crowns. If $c = y_1 \wedge y_2$ in L and c is incomparable with every element of S, then y_1 or y_2 is incomparable with every element of S.*

Proof. If the statement of the lemma were false, we could assume there were integers i and j with $1 \leqq i < j \leqq n + 2$ such that $y_1 > a_i$, $y_2 > a_j$, $y_1 \ngtr a_k$ whenever $i < k \leqq j$, and $y_2 \ngtr a_l$ whenever $i \leqq l < j$. Then $A_{j-i-1} \cong \{a_i, a_{i+1}, \ldots, a_j, c, b_i, b_{i+1}, \ldots, b_{j-1}, y_2, y_1\}$, contrary to assumption.

4. L contains D.

(i) $a \notin P$.† If $a = x \vee x_2$ with $x \nless b_2$, then x can replace a. Therefore, we can now assume that $a \in P$.

(ii) $b_2 \notin P$. Let $b_2 = x_1 \vee x_2$. Applying the dual of Lemma 7 to $S = \{b_1, b_3, a\}$ shows that x_1 or x_2 can replace b_2. We can now assume that a, $b_2 \in P$.

(iii) $c_1 \notin P$. If $c_1 = y \wedge y_2$ with $y \ngtr b_2$, then y can replace c_1. Therefore, $a, b_2, c_1, c_2 \in P$.

(iv) $b_1 \notin P$. Let $b_1 = x \vee x_2 = y \wedge y_2$ with $x \nless c_2$ and $y \ngtr b_2$. Then, $y \parallel b_3$ by Lemma 6. If $x > a$ ($y < c_1$), then $x(y)$ could replace b_1. Therefore, we can assume that $x \parallel a$ and $y \parallel c_1$. Then, $CX_3 \cong \{x, a, b_2, y, c_1, b_3, c_2\}$, where every element, except possibly b_3, is in P.

(v) *L contains CX_3 with only $b_3 \notin P$.* Let $b_3 = y \wedge y_2$ with $y \ngtr a_3$. Then, $y \parallel a_1$ by Lemma 6 applied to the fence $\{b_3, a_3, a_1, c, b_2\}$. If $y < c$, then y could replace b_3; therefore, $y \parallel c$. Then, $CX_2 \cong \{a_1, a_2, a_3, b_1, b_2, c, y\}$, where every element is in P.

In summary, we have shown that if L contains D (and contains no crowns), then P contains D, CX_2 or CX_3.

5. L contains C.

(i) $a \notin P$. If $a = x \vee x_2$ with $x \nless b_3$, then x can replace a.

(ii) $b_3 \notin P$. If $b_3 = x_1 \vee x_2$, then x_1 or x_2 can replace b_3 by the dual of Lemma 7 applied to $\{c_1, c_3, a\}$.

†In each paragraph, the original assumption of the heading will be shown *not* to hold. In other words, we show that the element being considered can be assumed in the sequel to be an element of P.

(iii) $c_1 \notin P$. If $c_1 = y_1 \wedge y_2$, then y_1 or y_2 can replace c_1 by Lemma 7 applied to $\{b_2, b_3, c_2\}$.

(iv) $c_2 \notin P$. If $c_2 = y_1 \wedge y_2$, then y_1 or y_2 can replace c_2 since, otherwise, we could assume that $y_1 > c_1$, $y_1 \ngtr c_3$, $y_2 > c_3$, and $y_2 \ngtr c_1$, Then, $D \cong \{a, c_1, b_3, c_3, y_1, y_2\}$, contrary to assumption. Therefore, $a, b_3, c_1, c_2, c_3 \in P$.

(v) $b_1 \notin P$. Let $b_1 = x \vee x_2$ with $x \nless c_3$. If $x > a$, then x could replace b_1; therefore, $x \parallel a$. If $x < b_3$, then $D^d \cong \{c_2, b_3, c_1, b_2, x, a\}$; therefore, $x \parallel b_3$. Then, $CX_1{}^d \cong \{c_1, c_2, c_3, x, b_2, b_3, a\}$, where every element, except possibly b_2, is in P.

(vi) *L contains CX_1 with only $b_2 \notin P$.* Let $b_2 = y \wedge y_2$ with $y \ngtr a_1$. If $y < c$, then y could replace b_2; therefore, $y \parallel c$. If $y \ngtr b_3$, then $CX_2 \cong \{a_1, a_2, a_3, b_1, c, y, b_3\}$ while if $y > b_3$, then $CX_3 \cong \{a_1, a_2, a_3, b_1, c, b_3, y\}$; in both cases, every element is in P.

In summary, P must contain C, $CX_1{}^d$, $CX_2{}^d$, or $CX_3{}^d$.

6. L contains B.

(i) $b_1 \notin P$. If $b_1 = x_1 \vee x_2$, then x_1 or x_2 can replace b_1 by the dual of Lemma 7 applied to $\{c_2, c_3, a\}$.

(ii) $c_1 \notin P$. Let $c_1 = y_1 \wedge y_2$. If neither y_1 nor y_2 can replace c_1, we can assume that $y_1 > b_2$, $y_1 \ngtr b_3$, $y_2 > b_3$, and $y_2 \ngtr b_2$. If $y_1 > c_2$, then $C^d \cong \{y_1, c_1, c_2, c_3, b_1, a, b_2\}$, contrary to assumption. Therefore, $y_1 \parallel c_2$ and $y_2 \parallel c_3$. Then, $CX_2 \cong \{b_1, a, b_2, y_2, y_1, c_2, c_3\}$, which implies that L contains C^d by Lemma 2. Therefore, $b_1, b_2, b_3, c_1, c_2, c_3 \in P$.

(iii) $a \notin P$. Let $a = x_1 \vee x_2$. If neither x_1 nor x_2 can replace a, we can assume that $x_1 < b_1$, $x_1 \nless b_2$, $x_2 < b_2$, and $x_2 \nless b_1$. If $x_1 < b_3$, then $D \cong \{x_1, b_1, x_2, b_3, c_1, c_3\}$; therefore, $x_1 \parallel b_3$ and $x_2 \parallel b_3$. Then, $EX_2 \cong \{x_1, x_2, b_3, b_1, c_3, b_2, c_1\}$, with every element in P.

7. L contains F_n. In addition to our standard assumptions, we assume that L does not contain E_m or $E_m{}^d$ whenever $0 \leqq m < n$.

(i) $c \notin P$. If $c = x \vee x_2$ with $x \nless a_{n+2}$, then x can replace c.

(ii) $d \notin P$. If $d = y \wedge y_2$, we can assume by Lemma 7 that y is incomparable with every a_i and b_i $(1 \leqq i \leqq n + 2)$. If $y \nless e$, then $C \cong \{c, a_1, d, a_{n+2}, b_1, e, y\}$. Thus, $y < e$ and y can replace d.

(iii) $b_1 \notin P$. Let $b_1 = y \wedge y_2$ with $y \ngtr a_2$. If $y > d$, then $D \cong \{a_1, b_1, d, b_2, y, e\}$; therefore, $y \parallel d$. Thus, by Lemma 6, y can replace b_1. Therefore, $a_{n+2}, b_1, c, d, e \in P$.

(iv) $b_i \notin P$ $(2 \leqq i \leqq n + 2)$. Let $b_i = y \wedge y_2$ with $y \ngtr d$. Let $j(k)$ be the least (greatest) value of l such that $y > a_l$. If $y < e$, then $F_m \cong \{a_1, \ldots, a_j,$

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$a_k, \ldots, a_{n+2}, b_1, \ldots, b_j, y, b_{k+1}, \ldots, b_{n+2}, c, d, e\}$ with $m = n - k + j + 1$. (Note that $m \geqq 0$.) By the minimality of n, $m = n$, and consequently, $j = i - 1$ and $k = i$. This means that this is the original F_n with b_i replaced by y; therefore, we can assume that $y \parallel e$. If $j > 1$, then $E_{j-2} \cong \{a_1, \ldots, a_j, b_1, \ldots, b_j, y, d, e\}$, which is contrary to assumption since $j \leqq n + 1$. If $k < n + 2$, then $F_m \cong \{a_k, \ldots, a_{n+2}, y, b_{k+1}, \ldots, b_{n+2}, c, d, e\}$ with $m = n - k + 1$. Since $k \geqq 2$, this would contradict the minimality of n. Therefore, $j = 1$ and $k = n + 2$. If $y > b_1$, then $D \cong \{c, b_1, a_{n+2}, d, y, e\}$. Therefore, $y \parallel b_1$ and $FX_1{}^d = \{y, e, b_1, a_{n+2}, a_1, d, c\}$, where every element, except possibly a_1, is in P.

(v) *L contains FX_1 with only $b_2 \notin P$.* Let $b_2 = y \wedge y_2$ with $y \not> b_3$. Since y could replace b_2 if $y < c$, we can assume that $y \parallel c$. If $y > b_1$, then $D \cong \{a_2, b_1, a_3, b_3, y, c\}$. Therefore, $y \parallel b_1$ and $FX_2 \cong \{a_1, a_2, a_3, b_1, b_3, y, c\}$, where every element is in P.

8. L **contains** E_n**.** The standard assumptions apply except that we assume that L does not contain F_m only when $m = 0$. (Note that cases 7 and 8 cover all situations where L contains some E_n, $E_n{}^d$ or F_n.)

(i) *$c \notin P$.* If $c = x_1 \vee x_2$, it follows by the dual of Lemma 7 that x_1 or x_2 can replace c.

(ii) *$a_1 \notin P$.* Let $a_1 = x_1 \vee x_2$. If neither x_1 nor x_2 can replace a_1, we can assume by the dual of Lemma 6 that $x_1 < c$, $x_1 \not< b_3$, $x_2 < b_3$, and $x_2 \not< c$. Then, $D^d \cong \{d, c, b_1, b_3, x_1, x_2\}$.

(iii) *$a_i \notin P$ $(2 \leqq i \leqq n + 1)$.* Let $a_i = x \vee x_2$. Suppose that neither x nor x_2 can replace a_i. If $x_2 < c$, $x_2 \not< b_{i-1}$, and $x_2 \not< b_{i+2}$, then $E_0{}^d \cong \{b_i, b_{i+1}, a_{i-1}, a_i, a_{i+1}, c, x_2\}$, contradicting that $n \geqq 1$. Consequently, we can assume that $x \not< c$ and $x \not< b_{i-1}$. If j is the greatest value of k such that $x < b_k$, then $m = n - j + i + 1 \geqq 0$ and, by the dual of Lemma 6, $E_m \cong \{a_i, \ldots, a_{i-1}, x, a_j, \ldots, a_{n+2}, b_1, \ldots, b_i, b_j, \ldots, b_{n+3}, c, d\}$. By the minimality of n, $j = i + 1$ and this is the original E_n with a_i replaced by x. Therefore, we can assume that $a_i \in P$ for $1 \leqq i \leqq n + 2$.

(iv) *$b_1 \notin P$.* Let $b_1 = y \wedge y_2$ with $y \not> a_2$. If $y > c$, then $D \cong \{a_1, b_1, c, b_2, y, d\}$. Therefore, $y \parallel c$ and y can replace b_1 by Lemma 6.

(v) *$b_i \notin P$ $(2 \leqq i \leqq n + 2)$.* Let $b_i = y \wedge y_2$ with $y \not> c$. Let $j(k)$ be the least (greatest) value of l such that $y > a_l$. If $y < d$, then we can show that y can replace b_i similarly as in case 7 (iv); therefore, $y \parallel d$. If $j > 1$, then $E_{j-2} \cong \{a_1, \ldots, a_j, b_1, \ldots, b_j, y, c, d\}$; therefore, $j = 1$ and $k = n + 2$. If $y > b_1$ and $y > b_{n+3}$, then $D^d \cong \{y, b_1, d, b_{n+3}, a_1, a_{n+2}\}$. Thus, we can assume that either $y \parallel b_1$, $y \parallel b_{n+3}$ and $EX_1 \cong \{a_1, a_{n+2}, c, b_1, y, d, b_{n+3}\}$, or $y > b_1$, $y \parallel b_{n+3}$ and $EX_2 \cong \{a_1, a_{n+2}, c, b_1, d, b_{n+3}, y\}$. In both cases, every element, except possibly d, is in P.

(vi) *L contains EX_1 with only $b_3 \notin P$.* Let $b_3 = y_1 \wedge y_2$ and suppose that neither y_1 nor y_2 can replace b_3. By symmetry, $y_1 > b_1$, and therefore, $y_2 \parallel b_1$. We first assume that $y_1 \not> b_2$. If $y_2 > b_2$, then $D \cong \{a_1, b_1, a_3, b_2, y_1, y_2\}$. Thus, $y_2 \parallel b_2$ and $FX_2 \cong \{a_2, a_1, a_3, b_2, b_1, y_2, y_1\}$, which implies that L contains F_0 by Lemma 2. We can now assume that $y_1 > b_2$; consequently $y_2 \not> b_2$, and thus, $y_2 > b_4$. If y_1 and y_2 are interchanged, and b_1 and b_4 are interchanged, we return to the case that was considered first.

(vii) *L contains EX_2 with only $b_2 \notin P$.* Let $b_2 = y_1 \wedge y_2$ and suppose that neither y_1 nor y_2 can replace b_2. We can assume that $y_1 > b_1$, $y_1 \not> b_3$, $y_2 > b_3$, and $y_2 \not> c$. If $y_1 > c$, then $D \cong \{a_2, c, a_3, b_3, y_1, y_2\}$. Thus, $y_1 \parallel c$ and $FX_2 \cong \{a_1, a_2, a_3, c, b_3, y_1, y_2\}$, a contradiction. Therefore, we can assume that all the elements of E_n, except possibly d, are in P.

(viii) *$d \notin P$.* If $d = y_1 \wedge y_2$ and neither y_1 nor y_2 can replace d, then we can assume that $y_1 > b_1$, $y_1 \not> b_{n+3}$, $y_2 > b_{n+3}$, and $y_2 \not> b_1$. Then, $I_n \cong \{a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+3}, c, y_1, y_2\}$, where every element is in P.

9. L contains G_n. In addition to the standard assumptions, we assume that L does not contain H_m whenever $0 \leqq m < n$.

(i) $a_1 \notin P$. If $a_1 = x \vee x_2$ with $x \not< b_1$, then x can replace a_1.

(ii) $b_1 \notin P$. If $b_1 = x_1 \vee x_2$, then by the dual of Lemma 7 applied to $\{a_2, c, a_1\}$, x_1 or x_2 can replace b_1. Therefore, $a_1, a_{n+3}, b_1, b_{n+3} \in P$.

(iii) $a_i \notin P$ $(2 \leqq i \leqq n + 2)$. Let $a_i = x \vee x_2$ with $x \not< b_i$. If $x < c$, then $D^d \cong \{b_{n+3}, c, a_{n+3}, b_{n+2}, x, b_{n+1}\}$ when $i = n + 2$, and $G_{n-i+1} \cong \{x, a_{i+1}, \ldots, a_{n+3}, b_i, \ldots, b_{n+3}, c\}$ when $i \leqq n + 1$. Therefore, $x \parallel c$. If $x \not> a_1$ and $x \not> b_1$, then $CX_1 \cong \{b_1, a_1, x, b_2, a_2, c, a_3\}$ when $i = 2$, and $H_{i-3} \cong \{b_1, \ldots, b_{i-1}, a_1, \ldots, a_i, c, x\}$ when $i \geqq 3$. Therefore, $x > a_1$ or $x > b_1$; consequently, one of the following two cases must occur:

(a) there is j $(1 \leqq j \leqq i - 1)$ such that $x > a_j$, $x > b_{j-1}$ (if $j > 1$) and $x \not> b_j$; or

(b) there is k $(1 \leqq k \leqq i - 2)$ such that $x > b_k$, $x \not> b_{k+1}$ and $x \not> a_{k+1}$.

If (a) holds, then $G_{n-i+j+1} \cong \{a_1, \ldots, a_j, x, a_{i+1}, \ldots, a_{n+3}, b_1, \ldots, b_j, b_i, \ldots, b_{n+3}, c\}$. By the minimality of n, $j = i - 1$ and this is the original G_n with a_i replaced by x. If case (b) occurs with $k = i - 2$, then $D \cong \{b_{i-2}, x, a_{i-1}, b_{i-1}, a_i, b_i\}$. Otherwise, case (b) occurs with $k \leqq i - 3$ and $G_{i-k-3} \cong \{b_k, \ldots, b_{i-1}, a_{k+1}, \ldots, a_i, x\}$. Therefore, all the elements of G_n, except possibly c, are in P.

(iv) $c \notin P$. Let $c = x \vee x_2 = y \wedge y_2$ with $x \not< a_{n+3}$ and $y \not> b_1$. Let $j(k)$ be the greatest (least) value of l such that $y > a_l (x < b_l)$; then, $j = 1$ or 2, and $k = n + 2$ or $n + 3$. If $x > a_2$, then $D^d \cong \{b_3, x, a_3, b_2, a_2, b_1\}$ when $k = 3$, and $G_{k-4} \cong \{a_2, \ldots, a_k, b_2, \ldots, b_k, x\}$ when $k \geqq 4$. Therefore, $x \parallel a$, and

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dually, $y \parallel b_{n+2}$. Similarly, if $x > a_1$, then $D \cong \{a_1, a_2, b_1, x, a_3, b_2\}$ when $k = 2$, and $G_{k-3} \cong \{a_1, \ldots, a_k, b_1, \ldots, b_k, x\}$ when $k \geqq 3$. Therefore, $x \parallel a_i$ and $y \parallel b_i$ $(1 \leqq i \leqq n + 3)$. If $k = j = 2$, then $A_0 \cong \{a_2, b_1, x, a_3, b_2, y\}$. If $k = j + 1$, then $CX_3 \cong \{x, a_j, b_j, y, b_{j+1}, a_{j+1}, a_{j+2}\}$. We can therefore assume that $k \geqq j + 2$ and thus, $J_{k-j-2} \cong \{a_j, \ldots, a_k, b_j, \ldots, b_k, x, y\}$, where every element is in P.

10. L contains H_n. The standard assumptions apply except that we assume that L does not contain G_m only when $0 \leqq m < n$.

(i) $a_1 \notin P$. If $a_1 = x \vee x_2$, then by the dual of Lemma 7, we can assume that $x \parallel b_2$ and $x \parallel c$. If $x < d$, then $D^d \cong \{b_3, d, a_2, b_2, x, b_1\}$ when $n = 0$, and $G_{n-1} \cong \{x, a_2, \ldots, a_{n+2}, b_2, \ldots, b_{n+3}, d\}$ when $n \geqq 1$. Therefore, $x \parallel d$ and x can replace a_1.

(ii) $b_1 \notin P$. Let $b_1 = x_1 \vee x_2$. If neither x_1 nor x_2 can replace b_1, we can assume that $x_1 < a_1$, $x_1 \nless d$, $x_2 < d$, and $x_2 \nless a_1$. Then, $D^d \cong \{b_{n+3}, a_1, c, d, x_1, x_2\}$. Therefore, $a_1, a_{n+2}, b_1, b_{n+3} \in P$.

(iii) $c \notin P$. Let $c = y \wedge y_2$. Applying Lemma 7 to $\{a_1, b_2, b_3\}$, we can assume that $y \parallel a_1$ and $y \parallel b_2$. If $y > d$, then $D \cong \{b_1, b_2, d, c, b_{n+3}, y\}$. Therefore, $y \parallel d$ and y can repalce c.

(iv) $a_i \notin P$ $(2 \leqq i \leqq n + 1)$. By duality, we can assume that $i \leqq \frac{1}{2}(n + 3)$. Let $a_i = x \vee x_2$ with $x \nless b_{i+1}$. If $x < c$ and $x > b_1$, then $D \cong \{b_1, x, a_1, b_2, a_2, b_3\}$ when $i = 2$, and $G_{i-3} \cong \{b_1, \ldots, b_i, a_1, \ldots, a_i, x\}$ when $i \geqq 3$. If $x < c$ and $x \parallel b_1$, then $CX_3 \cong \{x, b_1, a_1, c, a_2, b_2, b_3\}$ when $i = 2$, and $J_{i-3} \cong \{b_1, \ldots, b_i, a_1, \ldots, a_i, x, c\}$ when $i \geqq 3$. Since the latter case implies that L contains G_{i-3}, we conclude that $x \parallel c$. If $x < d$, then $D^d \cong \{b_4, d, a_3, b_3, x, b_2\}$ when $i = 2$ and $n = 1$; otherwise, $i \leqq n$ and $G_{n-i} \cong \{x, a_{i+1}, \ldots, a_{n+2}, b_{i+1}, \ldots, b_{n+3}, d\}$. Therefore, $x \parallel d$. If $x \ngtr b_1$, then

$$H_{n-i+1} \cong \{x, a_{i+1}, \ldots, a_{n+2}, b_1, b_{i+1}, \ldots, b_{n+3}, c, d\}.$$

Therefore, $x > b_1$ and one of the following two cases must occur:
(a) there is j $(1 \leqq j \leqq i - 1)$ such that $x > a_j$, $x > b_j$ and $x \ngtr b_{j+1}$; or
(b) there is k $(1 \leqq k \leqq i - 1)$ such that $x > b_k$, $x \ngtr b_{k+1}$ and $x \ngtr a_k$.
If (a) holds, then

$$H_{n-i+j+1} \cong \{a_1, \ldots, a_j, x, a_{i+1}, \ldots, a_{n+2}, b_1, \ldots, b_{j+1}, b_{i+1}, \ldots, b_{n+3}, c, d\}.$$

By the minimality of n, $j = i - 1$ and this is the original H_n with a_i replaced by x. If case (b) occurs, then $D \cong \{b_{i-1}, x, a_{i-1}, b_i, a_i, b_{i+1}\}$ when $k = i - 1$, and $G_{i-k-2} \cong \{b_k, \ldots, b_i, a_k, \ldots, a_i, x\}$ when $k \leqq i - 2$.

(iv) $b_i \notin P$ $(2 \leqq i \leqq n + 2)$. By duality, we can assume that $i \leqq \frac{1}{2}n + 2$. Let $b_i = x \vee x_2$ with $x \nless a_i$. If $x < c$ and $x < d$, then $CX_3{}^d \cong \{a_2, b_3, c,$

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$a_1, b_1, d, x\}$ when $n = 0$ and $i = 2$, and $D^d \cong \{b_{n+3}, a_2, c, d, b_1, x\}$ when $n \geqq 1$. If $x < c$ and $x \parallel d$, then for $n = 0$, $CX_1{}^d \cong \{a_2, b_3, c, a_1, x, d, b_1\}$ when $x > b_1$, and $CX_2{}^d \cong \{a_2, b_3, c, a_1, b_1, x, d\}$ when $x \parallel b_1$; for $n \geqq 1$, $i \leqq n + 1$ and $H_{n-i+1} \cong \{a_i, \ldots, a_{n+2}, x, b_{i+1}, \ldots, b_{n+3}, c, d\}$. Therefore, $x \parallel c$. If $x \not> a_1$ and $x \not> b_1$, then $CX_1 \cong \{a_1, b_1, x, a_2, b_2, c, b_3\}$ when $i = 2$, and $H_{i-3} \cong \{a_1, \ldots, a_{i-1}, b_1, \ldots, b_i, c, x\}$ when $i \geqq 3$. Therefore, $x > a_1$ or $x > b_1$; consequently, $x \parallel d$ and one of the following two cases must occur:

(a) there is j $(1 \leqq j \leqq i - 1)$ such that $x > a_{j-1}$ (if $j > 1$), $x \not> a_j$ and $x > b_j$; or

(b) there is k $(1 \leqq k \leqq i - 2)$ such that $x > a_k$, $x \not> a_{k+1}$ and $x \not> b_{k+1}$.

If (a) holds, then

$$H_{n-i+j+1} \cong \{a_1, \ldots, a_j, a_i, \ldots, a_{n+2}, b_1, \ldots, b_j, x, b_{i+1}, \ldots, b_{n+3}, c, d\}.$$

This must be the original H_n with b_i replaced by x. If case (b) occurs, then $D \cong \{a_{i-2}, a_{i-1}, b_{i-1}, x, a_i, b_i\}$ when $k = i - 2$, and $G_{i-k-3} \cong \{a_k, \ldots, a_{i-1}, b_{k+1}, \ldots, b_i, x\}$ when $k \leqq i - 3$. With this contradiction, the proof of the final case is complete.

References

1. K. A. Baker, *Dimension, join-independence, and breadth in partially ordered sets*, 1961 (unpublished).

2. B. Banaschewski, *Hüllensysteme und Erweiterungen von Quasi-Ordnungen*, Z. Math. Logik Grundlagen Math. *2* (1956), 117–130.

3. G. Birkhoff, *Lattice theory*, 3rd. ed. (American Mathematical Society, Providence, R.I., 1967).

4. B. Dushnik and E. W. Miller, *Partially ordered sets*, Amer. J. Math. *63* (1941), 600–610.

5. T. Hiraguchi, *On the dimension of partially ordered sets*, Sci. Reports Kanazawa Univ. *1* (1951), 77–94.

6. D. Kelly and I. Rival, *Crowns, fences and dismantlable lattices*, Can. J. Math. *26* (1974), 1257–1271.

7. ——— *Planar lattices*, Can. J. Math. *27* (1975), 636–665.

8. ——— *Certain partially ordered sets of dimension three*, J. Combinatorial Theory Ser. A *18* (1975), 239–242.

9. J. Schmidt, *Zur Kennzeichnung der Dedekind-MacNeilleschen Hülle einer geordneten Menge*, Arch. Math. *7* (1956), 241–249.

10. W. T. Trotter, Jr., *Irreducible posets with large height exist*, J. Combinatorial Theory Ser. A *17* (1974), 337–344.

11. W. T. Trotter, Jr., and J. I. Moore, Jr., *Characterization problems for graphs, partially ordered sets, lattices, and families of sets*, Discrete Math. (to appear).

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