PLANETARY PERTURBATION EQUATIONS BASED ON RELATIVISTIC KEPLERIAN MOTION

Neil Ashby
Department of Physics, Campus Box 390
University of Colorado
Boulder, Colorado, USA 80309

ABSTRACT. Solutions of the geodesic equations for bound test particle motion in a Schwarzschild field are expressed using Jacobian elliptic functions. Keplerian orbital elements are identified and related to a set of canonical constants. Relativistic Lagrange planetary perturbation equations are derived.

1. INTRODUCTION

In this paper Lagrange's planetary equations[1] are derived for perturbed relativistic motion of a test body in a Schwarzschild gravitational field. This will be treated as a problem in Hamiltonian mechanics, with a constraint. First, an exact solution describing an unperturbed test body orbit is expressed using Jacobian elliptic functions. Perturbation theory then leads to a formulation of Lagrange's planetary equations in which the unperturbed orbit includes all the effects of General Relativity, e.g., periastron precession.

It is well known that test body motion in a Schwarzschild field can be expressed in quadratures which lead to elliptic functions. Hagihara[2] has studied such orbits using Weierstrass elliptic functions. We shall express the unperturbed orbit in terms of Jacobian elliptic functions, which are closer in spirit to trigonometric functions with which most are familiar.

2. HAMILTONIAN FORMULATION OF UNPERTURBED PROBLEM

Because the Lagrangian $-m_0 c\int ds$ is homogeneous in the four-velocities, the Hamiltonian of this problem vanishes on the test body trajectory. Also, this motion is subject to the constraint

$$g^{\mu\nu}p_{\mu}p_{\nu} + (m_0 c)^2 = 0 \tag{1}$$

where $p_{\mu} = m_0 c g_{\mu\nu} dx^{\nu}/ds$ are the four-momenta. The Hamiltonian only vanishes "weakly", however, as its derivatives with respect to the

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canonically conjugate momenta and coordinates do not vanish.

A Hamiltonian with constraints can be shown[3] to be "strongly" equal to a linear combination of the constraint equations expressed so that some of the momenta are determined as a function of the remaining independent momenta by solving the equations of constraint. The constraint equation (1) is thus solved for p_0 in terms of p_k (latin indices run from 1 to 3) by writing $\psi(p_k, q^\mu)$ for p_0 in Eq. (1),

$$g^{00}\psi^2 + 2g^{0i}\psi p_i + g^{ij}p_i p_j + (m_0 c)^2 = 0 , \tag{2}$$

from which ψ may be found. Then the Hamiltonian can be written:

$$H = v(s)[\, p_0 + \psi(p_i, q^\mu)\,] , \tag{3}$$

where $v(s)$ is a coefficient to be determined. Positive square roots are chosen in Eq. (3) because $p_0 < 0$. (We use a metric signature $-1,1,1,1$ and other notation as in Weber[4].) In Eq. (3), the scalar parameter s is the proper time measured along the trajectory.

The Hamiltonian equations of motion are:

$$\dot{p}_\mu = - \frac{\partial H_0}{\partial q^\mu} , \qquad \dot{q}^\mu = + \frac{\partial H_0}{\partial p_\mu} . \tag{4}$$

The second of Eqs. (4) above can be shown to give $\dot{q}^\mu p^0 = \dot{q}^0 p^\mu$, where $v = \dot{q}^0$, thus identifying the multiplier v in Eq. (3). Then the first of Eqs. (4) leads directly to the geodesic equations of motion; this is valid for an arbitrary metric. We are interested here in the case of the Schwarzschild metric which is:

$$- ds^2 = -X(dX^0)^2 + dR^2/X + R^2 d\theta^2 + R^2 \sin^2\theta \, d\Phi^2 \tag{5}$$

where $q^\mu = (X^0, R, \theta, \Phi)$, $X \equiv 1 - 2\mu/R$, and $\mu = GM/c^2$. In the spirit of the Hamiltonian formulation, we shall obtain a solution by studying the Hamilton-Jacobi equation for this problem. In the Schwarzschild metric, Eq. (5), the variables X^0 and Φ are cyclic. Therefore p_0 and p_3 are constants of the motion, which we write as follows:

$$p_3 = m_0 c\, R^2 \sin^2\theta \frac{d\Phi}{ds} \equiv P_3 = \text{constant}; \tag{6}$$

and

$$p_0 = - m_0 c\, X \frac{dX^0}{ds} \equiv - P_1 = \text{constant}. \tag{7}$$

We are interested in solutions for non-circular motion in a plane of orientation described by inclination I, and angle of the line of ascending nodes, Ω, as in Fig. 1. To describe the position of the test body, we introduce an angle w measured in the plane of the orbit from the nodal line. The angular transformations between polar angles θ, Φ, and angles I, Ω, and w can be written

$$\cos w = \sin\theta \cos(\Phi - \Omega); \quad \sin w \cos I = \sin\theta \sin(\Phi - \Omega)$$

$$\sin w \sin I = \cos\theta , \tag{8}$$

and then by differentiation one may show that

$$p_3 = m_0 c\ R^2 \cos I \frac{dw}{ds} \equiv P_2 \cos I\ . \qquad (9)$$

The constants P_i will reappear as separation constants in the solution of the Hamilton-Jacobi equation.

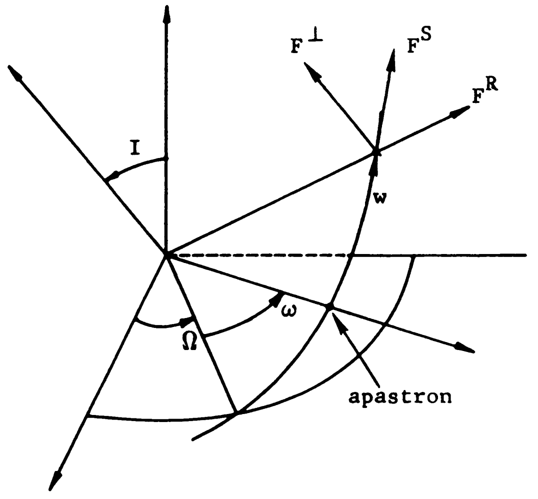

Figure 1. Angles describing unperturbed relativistic trajectory for a bound orbit. The altitude of apastron, the first time the trajectory passes through apastron after initally passing through the ascending node, is denoted by ω. The angle w is measured from the nodal line.

2.1. Hamilton-Jacobi equation

The Hamilton-Jacobi equation is obtained by seeking a transformation function $W(q^\mu, P_k)$ which depends on the old coordinates and the new momenta, and which satisfies

$$dW = p_\mu\ dq^\mu + Q^k\ dP_k\ .$$

Thus $p_\mu = \partial W/\partial q^\mu$; the Hamilton-Jacobi equation is $H(q^\mu, \partial W/\partial q^\mu) = 0$. Since the Hamiltonian, Eq. (3), is obtained by solving the constraint equation it is easy to see the Hamilton-Jacobi equation is equivalent to

$$g^{\mu\nu} \frac{\partial W}{\partial q^\mu} \frac{\partial W}{\partial q^\nu} + (m_0 c)^2 = 0\ . \qquad (10)$$

This form of the equation has been discussed by V. Fock[5].

2.2. Solution of Hamilton-Jacobi equation

The Hamilton-Jacobi equation is solved by separation of variables. Assume $W = W_0(X^0)+W_1(R)+W_2(\Theta)+W_3(\Phi)$. The solution may be obtained by methods similar to those found in textbooks[6], and is

$$W = \pm P_1 X^0 \pm \int_{R_0}^{R} J\, dR/X \pm \int_{\pi/2}^{\Theta} N\, d\Theta \pm P_3 \Phi \ , \tag{11}$$

where

$$J = [\ P_1{}^2 - X(P_2{}^2/R^2 + (m_0 c)^2)\ ]^{1/2}\ , \tag{12}$$

$$N = [\ P_2{}^2 - P_3{}^2/\sin^2\Theta\ ]^{1/2}\ , \tag{13}$$

and P_i are separation constants. Only three independent separation constants occur in Eq. (11) because of the constraint, that is, because the Hamilton-Jacobi equation is satisfied on the trajectory. The fourth separation constant, $P_0 \equiv \partial W/\partial q^0 = \partial W/\partial X^0$, is $\pm P_1$ by virtue of the Hamilton-Jacobi equation itself. The signs of the square roots in Eq. (11) must be chosen to correspond to the portion of the orbit under consideration. For example, we note that R decreases from the initial point R_0 (apastron). This choice is considerably more convenient when using standard elliptic functions. Also, Θ decreases from $\pi/2$ where the angle Φ has a value equal to the angle of the nodal line. We shall choose P_i to be our new canonical momenta. By the method of solution it is evident that they are constants of the motion.

Then three coordinates which are also canonical constants, and conjugate to the P_i, may be obtained from $Q^i = \partial W/\partial P_i$. For R and Θ decreasing,

$$Q^1 = -X^0 - \int_{R_0}^{R} P_1\, dR\ /\ XJ\ \ ; \tag{14}$$

$$Q^2 = \int_{R_0}^{R} P_2\, dR\ /\ R^2 J - \int_{\pi/2}^{\Theta} P_2\, d\Theta\ /\ N\ ; \tag{15}$$

$$Q^3 = \Phi + \int_{\pi/2}^{\Theta} P_3\, d\Theta\ /\ N \sin^2\Theta\ . \tag{16}$$

Since Q^i are constants of the motion, one may differentiate Eqs. (14-16) with respect to the proper time along the trajectory and thus compute the original canonical momenta p_μ, with the aid of the constraint. It is then found that the separation constants P_i appearing in Eqs. (11-16) are identical with the constants P_i introduced in Eqs. (6), (7), and (9).

The canonical constants Q^i may be interpreted as follows. First, at the initial point of the R integration, where $R = R_0$, $Q^1 = -X^0 = -cT$, where T is the time of first apastron passage. Second, using the angles defined in Eqs. (8) and illustrated in Fig. (1), it can be shown that

$$-\int_{\pi/2}^{\Theta} P_2\, d\Theta\ /\ N = w\ . \tag{17}$$

Therefore when $R = R_0$, Eq. (15) implies $Q^2 = w = \omega$, the altitude of first apastron passage. (Subsequent apastron passages will be at different altitudes because the apastron precesses.) Lastly, when $\Theta = \pi/2$, Eq. (16) gives $Q^3 = \Phi = \Omega$, the nodal angle.

To identify the momenta P_1 and P_2 in terms of Keplerian elements, we factor the quantity J^2, given by Eq. (12), in terms of its three roots in the variable $U = 1/R$ as follows:

$$J^2/P_2{}^2 = 2\mu\left(\frac{1}{2\mu\lambda} - U \right)\left(\frac{1}{a(1-e)} - U \right)\left(U - \frac{1}{a(1+e)} \right)$$

$$\equiv 2\mu(U_1 - U)(U_2 - U)(U - U_3) \qquad (18)$$

where U_i are roots of the equation $J^2 = 0$. The parameters a and e are defined in terms of P_1 and P_2 by Eq. (18). We consider in this paper only the orbits $a(1-e) \leq R \leq a(1+e)$, corresponding classically to bound Keplerian orbits. Relativistic effects arise from the factor $(1/2\mu\lambda - U)$. Identifying the two expressions for J, Eqs. (12) and (18), yields the following expressions for λ, P_1, and P_2 in terms of a and e:

$$\lambda = 1 - 4\mu/p , \qquad (19)$$

$$P_1 = m_0 c[(1-4\mu/p+4\mu^2/ap)/D_4]^{1/2}; \quad P_2 = m_0 c[\mu p/D_4]^{1/2} , \qquad (20)$$

where $p \equiv a(1-e^2)$, and $D_4 \equiv 1-4\mu/p+\mu/a$.

The expression (15), using (17), may then be written in the form

$$\int_{U_3}^{U} dU \; [(U_1-U)(U_2-U)(U-U_3)]^{1/2} = \surd(2\mu)(w-\omega) , \qquad (21)$$

where we choose $\omega = w$ ($= Q^2$) at $U = U_3 = 1/a(1+e) = 1/R_0$. To reduce the integral in Eq. (21) to a standard elliptic integral, we make the following change of variable:

$$U = U_3 + (U_2-U_3)\, sn^2(\surd D_6(w-\omega)/2;m) , \qquad (22)$$

where

$$D_6 \equiv 1-2\mu(3-e)/p \; ; \; m \equiv 4\mu e/pD_6 = (U_2-U_3)/(U_1-U_3) . \qquad (23)$$

Then Eq. (21) is satisfied. Thus for the radial variable R:

$$R = p/(1 - e + 2e\, sn^2 (\surd D_6(w-\omega)/2;m)). \qquad (24)$$

The nonrelativistic limit can be recovered by letting $m \to 0$, $D_6 \to 1$, whereupon we obtain $R = p/(1 - e \cos (w-\omega))$. The negative sign in this equation arises because $w = \omega$, the altitude of apastron, at $R = a(1+e)$.

An application of this result is an expression for the precession of apastron (or periastron), correct to all orders in e and μ/a. The elliptic sn function increases from zero at $w = \omega$ to a value of unity at periastron, where the argument is given by $\surd D_6(w-\omega)/2 = K(m)$, where K is the complete elliptic integral of the first kind. The change in w during the motion from apastron to periastron is therefore $\Delta w = 2K(m)/\sqrt{D_6}$. For comparison with a perturbation theory result, we specialize to the case $\mu/a \ll 1$ by using the following approximate expression for the elliptic integral[6]: $K(m) = (\pi/2)[1+m/4+ 9m^2/64+\dots]$. The apastron precession per revolution is $2\Delta w - 2\pi$ and is approximately

$$2\Delta w - 2\pi = 2\pi \left[\frac{3\mu}{p} + \frac{\mu^2}{p^2}\left(\frac{27}{2} - \frac{15}{16}e^2\right) + \dots\right]. \tag{25}$$

The first term on the right side of the above equation is the usual result[4]; the additional term is the leading higher-order correction.

Summarizing the main results of the section, the canonically conjugate constants are:

$$Q^1 = -cT \;; \quad P_1 = m_0 c\left[\frac{1 - 4\mu/p + 4\mu^2/ap}{1 - 4\mu/p + \mu/a}\right]^{1/2} \;;$$

$$Q^2 = \omega \;; \quad P_2 = m_0 c[\mu p/(1 - 4\mu/p + \mu/a)]^{1/2} \;; \tag{26}$$

$$Q^3 = \Omega \;; \quad P_3 = P_2 \cos I \,.$$

It is also of interest to discuss the explicit integrations of Eqs. (16) and (14). One finds no new information from Eq. (16) as the integration yields relations consistent with Eqs. (8). The integration of Eq. (14) can be expressed in intermediate form by

$$P_2(X^0 - cT)/P_1 = [2R_0{}^2/\sqrt{D_6}]\left[\int_0^u \frac{du}{Q^2} + B_0\int_0^u \frac{du}{Q} + B_0{}^2\int_0^u \frac{du}{1 - B_0 Q}\right] \tag{27}$$

where $B_0 = 2\mu/R_0$, $n=2e/(1-e)$, $Q=1+n\,\mathrm{sn}^2(u;m)$, and $u = \sqrt{D_6}(w-\omega)/2$. The integrals are given in the Appendix.

It is also useful to obtain a relation from which s can be calculated in terms of w. Using Eq. (24) in (9),

$$\frac{p^2 du}{(1 - e + 2e\,\mathrm{sn}^2)^2} = \frac{1}{2} P_2\sqrt{D_6}ds \,. \tag{28}$$

The result is

$$[\mu D_6/a^3 D_4]^{1/2} \quad (s - s_0) = \frac{4e^2(1 + e)^{1/2}}{(n + m)(1 - e)^{1/2}} \left[\frac{\mathrm{sn}\;\mathrm{cn}\;\mathrm{dn}}{Q} \right. \tag{29}$$

$$\left. - (n+m)u/n^2 + E(u)/n + (1/e+2m/n+3m/n^2)\Pi(-n;u|m)\right] .$$

This is the equation which in the classical case determines the eccentric anomaly in terms of the mean anomaly.

3. PERTURBATION THEORY

Now we consider the independent variables $p_k = p_k(Q^i, P_i, s)$, $q^k = q^k(Q^i, P_i, s)$ along the unperturbed orbit, as functions of the constants of the motion Q^i, P_i and the proper time s along the trajectory. The unperturbed Hamiltonian is denoted by H_0. If the motion is perturbed by non-gravitational forces or by additional gravitational forces from other bodies, we imagine that this is accounted for by letting Q^i and P_i determine an "osculating orbit" in the usual sense. The perturbations will cause Q^i and P_i to change slowly. Then we write the equations of motion as

$$\frac{\partial p_k}{\partial s} + \sum_i \frac{\partial p_k}{\partial Q^i}\dot{Q}^i + \sum_j \frac{\partial p_k}{\partial P_j}\dot{P}_j = -\frac{\partial H_0}{\partial q^k} + F_k \tag{30}$$

$$\frac{\partial q^k}{\partial s} + \sum_i \frac{\partial q^k}{\partial Q^i}\dot{Q}^i + \sum_j \frac{\partial q^k}{\partial P_j}\dot{P}_j = \frac{\partial H_0}{\partial p_k} + G^k \tag{31}$$

where F_k and G_k are used here to denote the generalized perturbing forces. We shall neglect perturbing forces that may contribute to the right side of Eq. (31). The total Hamiltonian is written $H = H_0 + H_1$; H_0 is the Hamiltonian discussed above, appropriate for the description of unperturbed motion in the Schwarzschild field. Then $F_k = -\partial H_1/\partial q^k$ and we neglect G^k in the following discussion. The osculating orbit satisfies Eqs. (4), thus:

$$\frac{\partial p_k}{\partial Q^i}\dot{Q}^i + \frac{\partial p_k}{\partial P_j}\dot{P}_j = F_k \; ; \qquad \frac{\partial q^k}{\partial Q^i}\dot{Q}^i + \frac{\partial q^k}{\partial P_j}\dot{P}_j = 0 \, . \tag{32}$$

By taking appropriate linear combinations of Eqs. (32), the equations can be recast using Lagrange brackets

$$\sum [Q^\ell, Q^i]\dot{Q}^i + \sum [Q^\ell, P_j]P_j = \sum \frac{\partial q^k}{\partial Q^\ell} F_k \; ; \tag{33}$$

$$\sum [P_\ell, Q^i]\dot{Q}^i + \sum [P_\ell, P_j]\dot{P}_j = \sum \frac{\partial q^k}{\partial P_\ell} F_k \, . \tag{34}$$

Using the well-known properties of Lagrange brackets, which can be shown to apply in the present case,

$$[Q^\ell, Q^i] = [P_\ell, P_i] = 0 \; ; \quad [Q^\ell, P_i] = \delta^\ell_i \, , \tag{35}$$

the equations reduce to:

$$\dot{Q}^\ell = -\sum_k \frac{\partial q^k}{\partial P_\ell} F_k \; ; \qquad \dot{P}_\ell = \sum_k \frac{\partial q^k}{\partial Q^\ell} F_k \, . \tag{36}$$

These equations are the general form of the Lagrange planetary equations, neglecting momentum dependence in the perturbing Hamiltonian H_1. The remainder of the calculation consists in evaluating the partial derivatives on the right sides of Eqs. (39) and (40), using the solutions to the equations of motion as expressed by the quadratures, Eqs. (14-16). In this process the variable s is kept constant, since the s-dependence is contained in the motion of the osculating orbit.

3.1. Equations of motion for Ω, I

We shall discuss this case in detail to illustrate the procedure. It is necessary to compute all the derivatives of R, Θ, Φ with respect to P_ℓ and Q^ℓ. The q^k (that is, R, Θ, and Φ) appear explicitly in Eqs. (14-16) so the process of computation is straightforward.

Consider first differentiation with respect to P_3. From Eq. (14), if Q^i, X^0, P_1 and P_2 are kept fixed while P_3 changes, R cannot change so $\partial R/\partial P_3 = 0$. Differentiation of Eqs. (15) and (16) and use of the angular transformations, Eqs. (8), yield the expressions:

$$\frac{\partial\Theta}{\partial P_3} = \cos I \sin w/[\, P_2 \sin I \sin\Theta\,] \; ; \tag{37}$$

$$\frac{\partial\Phi}{\partial P_3} = \sin w \cos w\, /[\, P_2 \sin^2\Theta\,]\, . \tag{38}$$

The Lagrange planetary equation for Ω is then:

$$-\dot{\Omega} = \frac{\cos I \sin w}{P_2 \sin I \sin\Theta} F_\Theta + \frac{\sin w \cos w}{P_2 \sin^2\Theta} F_\Phi \, . \tag{39}$$

This can be expressed in a more standard from by noting that the F_k are generalized forces. Let us introduce rectangular force components in the R, Θ, and Φ directions by defining:

$$F^R = XF_R\, , \; F^\Theta = F_\Theta/R\, , \quad F^\Phi = F_\Phi/\, R\sin\Theta\, , \tag{40}$$

and denoting the component of the force which is out of the orbital plane by:

$$F^\perp \equiv -\,[\, \cos w \sin I\, F^\Phi - \cos I\, F^\Theta\,]\, /\, \sin\Theta\, . \tag{41}$$

Then

$$\dot{\Omega} = R \sin w\, F^\perp\, /\, P_2 \sin I\, . \tag{42}$$

This is similar to the classical Lagrange planetary equation for Ω, except that the function R (given by Eq. (24)) depends on a Jacobian elliptic function and P_2 contains relativistic corrections.

To find the rate of change of I, we use the relation between P_2 and P_3 given by Eq. (9), to show:

$$\dot{I} = (\dot{P}_2 \cos I - \dot{P}_3\,)/\, P_2 \sin I\, . \tag{43}$$

Thus rates of change of P_2 and P_3 are both needed. First, from Eqs. (15) and (17) it can be seen that $\partial R/\partial Q^3$ and $\partial w/\partial Q^3$ are proportional. Eq. (14) implies $\partial R/\partial Q^3 = 0$. The angular transformations (8) then give $\partial\Theta/\partial Q^3 = 0$, $\partial\Phi/\partial Q^3 = 1$. Thus

$$\dot{P}_3 = F_\Phi\, . \tag{44}$$

Similarly, it is straightforward to show that:

$$\dot{P}_2 = \;= R\ F^S \tag{45}$$

where F^S is the component of the perturbing force normal to the radius in the orbital plane, defined by:

$$F^S = [\ -\sin I \cos w\ F_\Theta + \cos I\ F_\Phi / \sin\Theta\ ] / R \sin\Theta\ . \tag{46}$$

Combining the above then gives the other member of this pair of planetary equations:

$$\dot{I} = R \cos w\ F^{\perp}\ / P_2\ . \tag{47}$$

3.2. Equations of motion for a and e (P_1 and P_2)

The two quantities P_1 and P_2 depend only on a and e and it is natural to consider them together. Differentiating each of the Eqs. (15-17) in turn with respect to Q^1 yields the following relationships among the derivatives of the coordinates with respect to Q^1:

$$0 = \frac{\partial\Phi}{\partial Q^1} + \frac{P_3}{N \sin^2\Theta}\frac{\partial\Theta}{\partial Q^1}\ ;$$

$$0 = \frac{P_2}{R^2 J}\frac{\partial R}{\partial Q^1} - \frac{P_2}{N}\frac{\partial\Theta}{\partial Q^1}\ ;$$

$$-\ 1 = \frac{P_1}{XJ}\frac{\partial R}{\partial Q^1}\ .$$

These may be solved for the derivatives $\partial q^k/\partial Q^1$ and substituted into Eq. (36) to yield:

$$\dot{P}_1 = JF^R\ /\ P_1 - XP_2F^S/RP_1\ , \tag{48}$$

Eqs. (45) and (48) are thus the planetary equations for P_1 and P_2. Since a and e can be determined by Eqs. (26) in terms of P_1 and P_2 if necessary, it may be easier to use the equations for P_1 and P_2 rather than equations for the rates of change of a and of e. From Eqs. (26) the following expressions for these quantities may be derived:

$$\dot{a} = DP_2[4e\surd D_6 \text{ sn cn dn } F^R/p - 2XF^S/R]/\Delta - B[2P_2RF^S]/\Delta \tag{49}$$

$$\dot{e} = -CP_2[4e\surd D_6 \text{ sn cn dn } F^R/p - 2XF^S/R]/\Delta + A[2P_2RF^S]/\Delta \tag{50}$$

where the arguments of the elliptic functions are suppressed but are given by $u = \surd D_6(w-\omega)/2$. The constants A, B, C, D, and Δ which have not previously been defined are given by:

$$A = \partial P_1{}^2/\partial a = \mu(m_0c)^2[1-8\mu/p+16\mu^2/p^2-4\mu^2/ap]/a^2D_4{}^2\ ;$$

$$B = \partial P_1{}^2/\partial e = (m_0c)^2[8\mu^3e/ap^2]/D_4{}^2\ ;$$

$$C = \partial P_2^2/\partial a = \mu(1-e^2)(m_0 c)^2(1-8\mu/p+2\mu/a)/D_4^2 \; ; \quad (51)$$

$$D = \partial P_2^2/\partial e = -2\mu a e(m_0 c)^2[1-8\mu/p+\mu/a]/D_4^2 \; .$$

$$\Delta \equiv AD-BC = -2P_2^2 D_6^2 e(m_0 c)^2\mu(1-m)/apD_4^2 \; . \quad (52)$$

Except for the complexity of these constants, the results are very similar to the classical results. For example, in the expression for the rate of change of a, the coefficient of the radial perturbing force F^R, the only variable part arises from the product of the elliptic functions. In the non-relativistic limit this product becomes sin(w-ω), as in the classical expression.

3.3. Equations of motion for cT and ω (Q^1 and Q^2)

The planetary equations for this pair of variables are the most complicated to obtain, because they involve differentiation with respect to P_1 and P_2 of the canonical expressions Eqs. (14-16). There are two ways to proceed here. One can actually perform the integrations required as indicated in these equations, and then subsequently differentiate the resulting integrals with respect to Q^1 and Q^2 to obtain the required partial derivatives for substitution into Eq. (36). Alternatively, one can differentiate Eqs. (14-16) directly with respect to P_1 and P_2. The latter process is more efficient, except that interchange of the order of integration and differentiation introduces an apparent singularity at the lower limit in the R integrals, because the dependence of R_0 on P_1 and P_2 must be taken into account. However these are spurious singularities and can be shown to cancel out exactly because R = R_0 is a root of the equation $J^2 = 0$.

Only the results of the calculations will be given here. As in the case of the classical Lagrange planetary equations dω/ds can be expressed as a linear combination of contributions due to the perturbing forces in three orthogonal directions. The result is:

$$e\dot{\omega} = -p\,\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}\,N_{scd}F^R/P_2D_6 + (Rp/P_2\surd D_6^3)\left(\frac{p}{2e}\left[\frac{X}{R^2} - \frac{X_0}{R_0^2}\right]N_{scd} - \frac{2\mu(m_0c)^2}{P_2^2}N_{cd}\right)F^S - eR\cot I\sin w\, F^{\perp}/P_2 \; . \quad (53)$$

Lastly, one finds as in the nonrelativistic case that the perturbing force normal to the osculating plane does not contribute to the equation for T, so:

$$c\dot{T} = \frac{p\,\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}}{eP_1D_6}\left(N_{scd} + \frac{(m_0cp)^2}{P_2^2(1-e)^2}N_{QQscd}\right)F^R + \frac{p^2}{2P_2e^2\surd(D_6^3)}\left(\frac{RP_1}{P_2^2}N_{scd} - \frac{X}{RP_1}\left[N_{scd} + \frac{(m_0cp)^2}{P_2^2(1-e)^2}N_{QQscd}\right]\right)F^S \quad (54)$$

Quantities such as N_{scd} appearing in the above results compactly denote integrals of certain combinations of Jacobian elliptic functions, and are defined and evaluated in the Appendix.

4. MOMENTUM-DEPENDENT PERTURBING FUNCTION

To complete the theory one must also allow for the possibility that an additional term G^k, arising from derivatives of the perturbing function with respect to momentum, may contribute to the right-hand sides of Eqs. (33) and (34). Then Eqs. (36) would have additional terms on the right which are linear combinations of G^k with coefficients formed from partial derivatives of the momenta p_k, with respect to the canonical constants. The relations needed, expressing p_k in terms of these constants, can be obtained from $p_k = \partial W/\partial q^k$, and are:

$$p_R = \pm J/X \; ; \qquad p_\Theta = \mp N \; ; \qquad p_\Phi = P_3 \; . \tag{55}$$

To use these it is also necessary to express Θ and R in terms of P_i and Q^i. We have however already calculated derivatives of R, Θ, and Φ with respect to the canonical constants in deriving the above generalizations of the Lagrange planetary equations. A second stage of generalization including momentum-dependent perturbing forces can therefore be supplied.

5. CONCLUSION

Although there are probably no new applications of the theory presented here to motion of solar system bodies, that cannot be as easily calculated using classical perturbation theory, it is interesting that a treatment of this problem yields to the application of Hamiltonian methods. Potential applications to relativistic systems may be found in astronomy. For example one may wish to investigate the effect on the orbital elements of the gravitational radiation reaction force on a small body emitting such radiation as it orbits a massive companion.

APPENDIX. EVALUATION OF INTEGRALS

The notation used is as follows:

$$n = 2e/(1-e) \; ; \quad m = 4\mu e/pD_6 \; ; \quad m_1 = 1-m \; ;$$

$$Q = 1 + n\,\mathrm{sn}^2(u,m) \; ; \quad u = \sqrt{D_6}(w-\omega)/2 \quad .$$

E(u) is the elliptic integral of the second kind; Π is the elliptic integral of the third kind. Then in the following the arguments (u;m) of the Jacobian elliptic functions sn, cn, and dn are suppressed. The limits on all integrals are 0 and u.

$$N_d(u) \equiv \int du/\mathrm{dn}^2 = [E(u) - m\,\mathrm{sn}\,\mathrm{cn}/\mathrm{dn}]/m_1;$$

$$N_c(u) \equiv \int du/cn^2 = [-E(u) + m_1 u + sn\ dn\ /cn]/m_1;$$

$$N_s(u) \equiv \int du[1/sn^2 - 1/u^2] - 1/u = -E(u) + u - cn\ dn\ /sn\ ;$$

$$N_{cd}(u) \equiv \int du/(cn^2\ dn^2) = [N_c - mN_d]/m_1\ ;$$

$$N_{scd}(u) = [N_c - m^2 N_d]/m_1 + N_s\ ;$$

$$N_Q \equiv \int du/Q = \Pi(-n;u|m)\ ;$$

$$N_{Qd} \equiv \int du/[Q\ dn^2] = [nN_Q + mN_d]/(n+m)\ ;$$

$$N_{QQ} \equiv \int du/Q^2 = [n^2\ sn\ cn\ dn\ /Q + nE(u) - (n+m)u - (n^2+2n(1+m)+3m)N_Q]/[2(n+1)(n+m)]\ ;$$

$$N_{QQd} \equiv \int du/Q^2 dn^2 = [n(n+m)N_{QQ} + nmN_Q + m^2 N_d]/(n+m)^2\ ;$$

$$N_{QQscd} = -[n^2(n+1)N_{QQd} + n^2(n+2)N_{Qd} + n(n+2)N_{cd}]/(n+1)^2 + N_{scd}$$

$$\int du/(1-B_0 Q) = \Pi(nB_0/(1-B_0);u|m)/(1-B_0)\ .$$

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