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Nilpotent Orbits and Whittaker Functions for Derived Functor Modules of $\mathrm{Sp}(2,\mathbb{R})$

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Abstract. We study the moderate growth generalized Whittaker functions, associated to a unitary character ψ of a unipotent subgroup, for the non-tempered cohomological representation of $G = \mathrm{Sp}(2,\mathbb{R})$. Through an explicit calculation of a holonomic system which characterizes these functions we observe that their existence is determined by the including relation between the real nilpotent coadjoint G-orbit of ψ in $\mathfrak{g}_{\mathbb{R}}^*$ and the asymptotic support of the cohomological representation.

Introduction

In this paper we study the generalized Whittaker functions for non-tempered Zuckerman's derived functor modules of $G = \mathrm{Sp}(2,\mathbb{R})$ which denotes the real symplectic group of degree 2. Fix a proper parabolic subgroup of G and denote its unipotent radical by N. For a unitary non-degenerate character ψ of N we will consider the reduced generalized Gelfand-Graev representation [Y1] of G. Then we study the space of G-equivariant maps from an irreducible admissible representation, in particular a derived functor module, π of G into this G-module, which we call (N,ψ)-Whittaker embeddings of π. We are especially interested in such an embedding whose image consists of moderate growth functions on G. Conjecturally its existence and properties might be related to a nature of a closed union of nilpotent coadjoint G-orbits in $\mathfrak{g}_{\mathbb{R}}^*$, which was introduced by Barbasch and Vogan [B-V] for a given π as follows. The global distribution character of π lifts to an invariant eigendistribution on a neighborhood of the origin in $\mathfrak{g}_{\mathbb{R}}$ by the exponential map. The lift has an asymptotic expansion near the origin. Then its leading term is given by a linear sum of tempered distributions, and the Fourier transform gives a combination of invariant measures supported on nilpotent coadjoint G-orbits in $\mathfrak{g}_{\mathbb{R}}^*$. We take the union of those orbits and call it *the asymptotic support* of π. We will explain in Section 6 a relation occurring in our several examples between the nilpotent orbit and the existence of a moderate growth Whittaker embedding, which supports a part of conjectures found in [K], [Ma].

Our main technique to investigate the Whittaker embedding for a derived functor module is to use a set of differential operators which acts on the space of smooth sections of a vector bundle on G/K. Here K denotes a maximal compact subgroup of G. Such an operator was introduced by Schmid to realize the discrete series representations of a semisimple Lie group, and then generalized to characterize the derived functor modules by Wong [W], Barchini [Ba]. We will apply these differential opera-

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tors on the image of a Whittaker embedding of a derived functor module $A_{\mathfrak{q}}(\lambda)$, then analyze its kernel space explicitly.

1 The Zuckerman-Vogan Derived Functor Modules $A_{\mathfrak{q}}(\lambda)$

1.1

Let $\mathfrak{g}_{\mathbb{R}}$ be a real semisimple Lie algebra, $\mathfrak{g}$ its complexification. In the following we omit the subscript $\mathbb{R}$ to express the complexification of real algebras. Denote by G a connected real semisimple Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$. We fix a G-invariant non-degenerate quadratic form on $\mathfrak{g}$ by which we identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. Fix a Cartan involution θ, then we have the Cartan decomposition: $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$, where $\mathfrak{k}_{\mathbb{R}}$, or $\mathfrak{p}_{\mathbb{R}}$, is the $+1$, or -1, eigenspace of θ respectively. Denote by K the compact Lie group of G with Lie algebra $\mathfrak{k}_{\mathbb{R}}$. We assume that $\operatorname{rank} G = \operatorname{rank} K$, so there exist the discrete series of G. We fix a compact Cartan subalgebra $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$. Given an elliptic element $X \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}$, we define algebras $\mathfrak{u} = \mathfrak{u}(X)$, $\mathfrak{l} = \mathfrak{l}(X)$ and $\bar{\mathfrak{u}} = \bar{\mathfrak{u}}(X)$ to be the sum of eigenspaces of $\operatorname{ad}(X)$ in $\mathfrak{g}$ with positive, 0, and negative eigenvalues, respectively. Then $\mathfrak{q} = \mathfrak{q}(X) = \mathfrak{l} + \mathfrak{u}$ is said to be a *θ-stable parabolic subalgebra* of $\mathfrak{g}$. The algebra $\mathfrak{l}$, which contains $\mathfrak{t}$, is the Lie algebra of $L_{\mathbb{C}} = Z_{G_{\mathbb{C}}}(X)$, the centralizer of X in $G_{\mathbb{C}}$. It is defined over $\mathbb{R}$, so we can write $\mathfrak{l} = \mathfrak{l}_{\mathbb{R}} \otimes \mathbb{C}$. Denote by Q the θ-stable parabolic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{q}$. The real G-orbit G/L determines an open submanifold in the flag variety $G_{\mathbb{C}}/Q$; thus it has an invariant complex structure. For $\lambda \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^*$ we define a character of L; then we can define a line bundle on $G_{\mathbb{C}}/Q$. Then it is known that the derived functor module $A_{\mathfrak{q}}(\lambda)$ has a geometric realization

$$A_{\mathfrak{q}}(\lambda) \simeq H_{\bar{\partial}}^{\dim_{\mathbb{C}}(\mathfrak{u}\cap\mathfrak{k})}(G/L, \tilde{\mathbb{C}}_{\lambda+2\rho_{\mathfrak{u}}})_K,$$

by the underlying Harish-Chandra module of the Dolbeault cohomology group of G/L, [W]. Here $2\rho_{\mathfrak{u}} = \det(\operatorname{Ad}_L|_{\mathfrak{u}})$ and the holomorphic line bundle $\tilde{\mathbb{C}}_{\lambda+2\rho_{\mathfrak{u}}}$ is defined as the pull back of $G_{\mathbb{C}} \times_Q \mathbb{C}_{\lambda+2\rho_{\mathfrak{u}}}$ on $G_{\mathbb{C}}/Q$. A linear form $\lambda \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^*$ is said to be *good*, if

$$\operatorname{Re}\langle\lambda + \rho_{\mathfrak{u}} - \rho_{\mathfrak{l}}, \alpha\rangle \geq 0 \quad \text{for all } \alpha \in \Delta(\mathfrak{u}, \mathfrak{t}),$$

where $2\rho_{\mathfrak{l}}$ is the sum of positive roots of $\mathfrak{t}$ on $\mathfrak{l}$, and $\Delta(\mathfrak{u}, \mathfrak{t})$ is the set of roots of $\mathfrak{t}$ on $\mathfrak{g}$ whose root vectors are in $\mathfrak{u}$. Define $2\rho_{\mathfrak{u}\cap\mathfrak{p}}$ (resp. $2\rho_{\mathfrak{u}\cap\mathfrak{k}}$) to be the sum of roots of $\mathfrak{t}$ on $\mathfrak{g}$ whose root vectors are in $\mathfrak{u} \cap \mathfrak{p}$ (resp. $\mathfrak{u} \cap \mathfrak{k}$). We call that $\lambda \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^*$ is *integral*, if $\lambda + 2\rho_{\mathfrak{u}\cap\mathfrak{p}}$ determines the highest weight of an irreducible finite dimensional representation of $K_{\mathbb{C}}$.

Let π be an irreducible admissible $(\mathfrak{g}, K_{\mathbb{C}})$-module. We denote by $\operatorname{Ass}(\pi)$ its associated variety [V1]. It is a closed union of nilpotent coadjoint $K_{\mathbb{C}}$-orbits in $\mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$, where $\mathcal{N}^*$ is the set of nilpotent elements of $\mathfrak{g}^*$ and $(\mathfrak{g}/\mathfrak{k})^*$ denotes the set of linear forms on $\mathfrak{g}$ which vanish on $\mathfrak{k}$.

We recall some facts regarding derived functor modules.

Proposition 1.2

(i) *If $\lambda \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^*$ is integral and good, then the Harish-Chandra module $A_{\mathfrak{q}}(\lambda)$ is non-zero, irreducible and infinitesimally unitary.*

(ii) *Suppose λ is integral and good. Then each of the K-types occurring in $A_{\mathfrak{q}}(\lambda)|_K$ has the highest weight*

$$\lambda + 2\rho_{\mathfrak{u}\cap\mathfrak{p}} + \sum_{\beta\in\Delta(\mathfrak{u},\mathfrak{t})} n_\beta \beta, \quad n_\beta \in \mathbb{N}.$$

The K-type with the highest weight $\lambda + 2\rho_{\mathfrak{u}\cap\mathfrak{p}}$ occurs in $A_{\mathfrak{q}}(\lambda)$ with multiplicity one, and is called the minimal K-type of $A_{\mathfrak{q}}(\lambda)$.

(iii) *Suppose λ is integral and good, then the associated variety $\mathrm{Ass}\big(A_{\mathfrak{q}}(\lambda)\big)$, under identification $\mathfrak{g} \simeq \mathfrak{g}^*$, is given by*

$$\mathrm{Ass}\big(A_{\mathfrak{q}}(\lambda)\big) = \mathrm{Ad}(K_{\mathbb{C}})(\bar{\mathfrak{u}} \cap \mathfrak{p}).$$

2 The Symplectic Lie Group and Algebra, and Its Derived Functor Modules

2.1

Let $G = \mathrm{Sp}(2,\mathbb{R})$ be the real symplectic Lie group of degree 2; $G = \left\{ g \in \mathrm{SL}_4(\mathbb{R}) \mid {}^t g J g = J, J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \right\}$. Denote by $\mathfrak{g}_{\mathbb{R}}$ its real Lie algebra, and set $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$. It has a Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ with a Cartan involution $\theta(X) = -{}^t X$ for $X \in \mathfrak{g}_{\mathbb{R}}$; denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its complexification. Here $\mathfrak{k}_{\mathbb{R}}$ is the Lie algebra of a maximal compact subgroup K of G which is isomorphic to $\mathfrak{u}(2)_{\mathbb{R}}$ and explicitly given by $\mathfrak{k}_{\mathbb{R}} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{g}_{\mathbb{R}} \mid A, B \in M_2(\mathbb{R}), A = -{}^t A, B = {}^t B \right\}$. Note that the real ranks coincide for G and K, which is equal to 2. Take a compact Cartan subalgebra $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$, then the set of roots $\Delta(\mathfrak{g},\mathfrak{t})$ of $\mathfrak{t} = \mathfrak{t}_{\mathbb{R}} \otimes \mathbb{C}$ on $\mathfrak{g}$ is of type C_2. It is written as $\Delta(\mathfrak{g},\mathfrak{t}) = \{\pm e_1 \pm e_2, \pm 2e_1, \pm 2e_2\}$ with bases e_1 and e_2 of $\sqrt{-1}\mathfrak{t}_{\mathbb{R}}^*$. Denote by X_α the root vector for $\alpha \in \Delta(\mathfrak{g},\mathfrak{t})$.

We give a parameterization of $K_{\mathbb{C}}$-conjugacy classes of the θ-stable parabolic subalgebras of $\mathfrak{g}$. This parameterization corresponds to the subsets of $\sqrt{-1}\mathfrak{t}_{\mathbb{R}}^*$ under certain partition given below. Writing $\xi = \xi_1 e_1 + \xi_2 e_2$, we define the subsets Ξ_k, $1 \le k \le 10$, of the linear forms $\sqrt{-1}\mathfrak{t}_{\mathbb{R}}^*$ by

$$\Xi_1 := \{\xi \mid \xi_2 < \xi_1, \xi_1 > 0, \xi_2 > 0\}; \quad \Xi_2 := \{\xi \mid -\xi_1 < \xi_2, \xi_1 > 0, \xi_2 < 0\};$$
$$\Xi_3 := \{\xi \mid \xi_2 < -\xi_1, \xi_1 > 0, \xi_2 < 0\}; \quad \Xi_4 := \{\xi \mid \xi_2 < \xi_1, \xi_1 < 0, \xi_2 < 0\};$$
$$\Xi_5 := \{\xi \mid \xi_1 = \xi_2, \xi_1 > 0\}; \quad \Xi_6 := \{\xi \mid \xi_2 = 0, \xi_1 > 0\};$$
$$\Xi_7 := \{\xi \mid \xi_1 = -\xi_2, \xi_1 > 0\}; \quad \Xi_8 := \{\xi \mid \xi_1 = 0, \xi_2 < 0\};$$
$$\Xi_9 := \{\xi \mid \xi_1 = \xi_2, \xi_1 < 0\}; \quad \Xi_{10} := \{\xi \mid \xi_1 = \xi_2 = 0\}.$$

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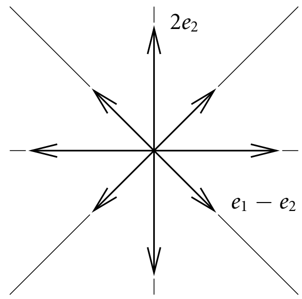

Figure 1: C_2 root system

To each $\xi \in \Xi_k$, we attach a θ-stable parabolic subalgebra $\mathfrak{q}(\xi) = \mathfrak{l}(\xi) + \mathfrak{u}(\xi)$;

$$\text{for } \xi \in \Xi_1, \quad \mathfrak{l}(\xi) = \mathfrak{t},\ \mathfrak{u}(\xi) = \mathbb{C}X_{2e_2} + \mathbb{C}X_{e_1+e_2} + \mathbb{C}X_{2e_1} + \mathbb{C}X_{e_1-e_2};$$
$$\text{for } \xi \in \Xi_2, \quad \mathfrak{l}(\xi) = \mathfrak{t},\ \mathfrak{u}(\xi) = \mathbb{C}X_{e_1+e_2} + \mathbb{C}X_{2e_1} + \mathbb{C}X_{e_1-e_2} + \mathbb{C}X_{-2e_2};$$
$$\text{for } \xi \in \Xi_3, \quad \mathfrak{l}(\xi) = \mathfrak{t},\ \mathfrak{u}(\xi) = \mathbb{C}X_{2e_1} + \mathbb{C}X_{e_1-e_2} + \mathbb{C}X_{-2e_2} + \mathbb{C}X_{-e_1-e_2};$$
$$\text{for } \xi \in \Xi_4, \quad \mathfrak{l}(\xi) = \mathfrak{t},\ \mathfrak{u}(\xi) = \mathbb{C}X_{e_1-e_2} + \mathbb{C}X_{-2e_2} + \mathbb{C}X_{-e_1-e_2} + \mathbb{C}X_{-2e_1};$$
$$\text{for } \xi \in \Xi_5, \quad \mathfrak{l}(\xi) = \mathfrak{t} + \mathbb{C}X_{-e_1+e_2} + \mathbb{C}X_{e_1-e_2},\ \mathfrak{u}(\xi) = \mathbb{C}X_{2e_2} + \mathbb{C}X_{e_1+e_2} + \mathbb{C}X_{2e_1};$$
$$\text{for } \xi \in \Xi_6, \quad \mathfrak{l}(\xi) = \mathfrak{t} + \mathbb{C}X_{2e_2} + \mathbb{C}X_{-2e_2},\ \mathfrak{u}(\xi) = \mathbb{C}X_{e_1+e_2} + \mathbb{C}X_{2e_1} + \mathbb{C}X_{e_1-e_2};$$
$$\text{for } \xi \in \Xi_7, \quad \mathfrak{l}(\xi) = \mathfrak{t} + \mathbb{C}X_{e_1+e_2} + \mathbb{C}X_{-e_1-e_2},\ \mathfrak{u}(\xi) = \mathbb{C}X_{2e_1} + \mathbb{C}X_{e_1-e_2} + \mathbb{C}X_{-2e_2};$$
$$\text{for } \xi \in \Xi_8, \quad \mathfrak{l}(\xi) = \mathfrak{t} + \mathbb{C}X_{2e_1} + \mathbb{C}X_{-2e_1},\ \mathfrak{u}(\xi) = \mathbb{C}X_{e_1-e_2} + \mathbb{C}X_{-2e_2} + \mathbb{C}X_{-e_1-e_2};$$
$$\text{for } \xi \in \Xi_9, \quad \mathfrak{l}(\xi) = \mathfrak{t} + \mathbb{C}X_{e_1-e_2} + \mathbb{C}X_{-e_1+e_2},\ \mathfrak{u}(\xi) = \mathbb{C}X_{-2e_2} + \mathbb{C}X_{-e_1-e_2} + \mathbb{C}X_{-2e_1};$$
$$\text{for } \xi \in \Xi_{10}, \quad \mathfrak{q}(\xi) = \mathfrak{l}(\xi) = \mathfrak{g}.$$

The θ-stable parabolic subalgebra $\mathfrak{q}(\xi)$ depends only on the class Ξ_k which contains ξ; so we will denote by $\mathfrak{q}_k$, $1 \leq k \leq 10$, the $K_{\mathbb{C}}$-conjugacy classes of θ-stable parabolic subalgebras of $\mathfrak{g}$. We say that a derived functor module $A_{\mathfrak{q}}(\lambda)$ is of *the class k*, if $\mathfrak{q} = \mathfrak{q}_k$. Note that a derived functor module of the class 2, or 3, is, indeed, a large discrete series representation, and that the one of the class 10 is the trivial representation, *etc.*

2.2

We will parameterize the associated varieties of the derived functor modules of each class above. The nilpotent $K_{\mathbb{C}}$-orbits in $(\mathfrak{g}/\mathfrak{k})^*$ are indexed by the signed Young tableaux, [C-M], Chapter 9. In our case, $\mathfrak{g} = \mathfrak{sp}(2)$, the corresponding signed Young tableaux are given in Figure 2.

Here the lines connecting two tableaux suggest the closure relations among the linked nilpotent orbits, [D]; more precisely, the closure of the nilpotent orbit indexed by the upper tableau contains the orbits in the lower layer orbit linked to it. Denote by $\mathcal{O}_Y$ the orbit to each signed tableau Y; for example, $\mathcal{O}_{\boxed{+}\boxed{-}\boxed{+}\boxed{-}}$, *etc.* The numbers

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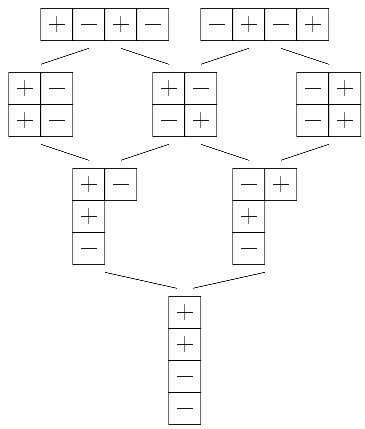

Figure 2: The signed Young tableaux for $\mathfrak{sp}(2)$

$$\operatorname{Ass}\big(A_{\mathfrak{q}_1}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}}\big) = \mathcal{O}_{\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}} \amalg \mathcal{O}_{\begin{smallmatrix} + & - \\ + & \\ - & \end{smallmatrix}} \amalg \mathcal{O}_{\begin{smallmatrix} + \\ + \\ - \\ - \end{smallmatrix}};$$
$$\operatorname{Ass}\big(A_{\mathfrak{q}_2}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} + & - & + & - \end{smallmatrix}}\big);$$
$$\operatorname{Ass}\big(A_{\mathfrak{q}_3}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} - & + & - & + \end{smallmatrix}}\big); \quad \operatorname{Ass}\big(A_{\mathfrak{q}_4}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} - & + \\ - & + \end{smallmatrix}}\big);$$
$$\operatorname{Ass}\big(A_{\mathfrak{q}_5}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}}\big); \quad \operatorname{Ass}\big(A_{\mathfrak{q}_6}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}}\big);$$
$$\operatorname{Ass}\big(A_{\mathfrak{q}_7}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} + & - \\ - & + \end{smallmatrix}}\big); \quad \operatorname{Ass}\big(A_{\mathfrak{q}_8}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} - & + \\ - & + \end{smallmatrix}}\big);$$
$$\operatorname{Ass}\big(A_{\mathfrak{q}_9}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} - & + \\ - & + \end{smallmatrix}}\big); \quad \operatorname{Ass}\big(A_{\mathfrak{q}_{10}}(\lambda)\big) = \operatorname{Cl}\big(\mathcal{O}_{\begin{smallmatrix} + \\ + \\ - \\ - \end{smallmatrix}}\big) = \{0\}.$$

Figure 3: Table of the associated varieties

written in the right side of Figure 2 mean the complex dimensions of the nilpotent $K_{\mathbb{C}}$-orbits $\mathcal{O}_Y$ in the layer; for example, $\dim_{\mathbb{C}} \mathcal{O}_{\begin{smallmatrix} + & - & + & - \end{smallmatrix}} = \dim_{\mathbb{C}} \mathcal{O}_{\begin{smallmatrix} - & + & - & + \end{smallmatrix}} = 4$, *etc.*

The associated varieties $\operatorname{Ass}\big(A_{\mathfrak{q}_k}(\lambda)\big)$ of the derived functor modules of the class k are given in Figure 3, at least when λ is good and integral. In Figure 3, $\operatorname{Cl}\big(\mathcal{O}\big)$ denotes the closure of the orbit $\mathcal{O}$. We remark that it depends only on the class of a θ-stable parabolic subalgebra, but not on each λ.

3 The Generalized Whittaker Functions and Differential Equations

3.1

We use the notation defined in Section 2. Let P be a parabolic subgroup of G and $P = MAN$ its Langlands decomposition, where N is the unipotent radical and M is

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semisimple. Fix a non-degenerate unitary character $\psi\colon N \to \mathbb{C}^*$. The subgroup M of P normalizes N, and acts naturally on the set of characters of N. Define $M(\psi)$ to be the identity component of the subgroup of M which stabilizes the character ψ under this action. As we will see below case by case, it is always abelian in our studies. So taking a unitary character χ of $M(\psi)$, we can determine a character $\eta = \chi \cdot \psi$ of the semi-direct product $R := M(\psi) \ltimes N$ obtained by multiplying both values. Then we consider the representation of G induced from η in C^∞-context:

$$C^\infty\text{-}\operatorname{Ind}_R^G(\eta) := \{f\colon G \to \mathbb{C} \mid \text{smooth}, f(rg) = \eta(r)f(g), (r,g) \in R \times G\},$$

on which G acts by the right translation. It has also a compatible Harish-Chandra module structure. We use the same symbols ψ, χ, and η for the representations of the corresponding Lie algebras $\mathfrak{n}_\mathbb{R}$, $\mathfrak{m}(\psi)_\mathbb{R}$, and $\mathfrak{r}_\mathbb{R} = \mathfrak{m}(\psi)_\mathbb{R} + \mathfrak{n}_\mathbb{R}$.

We recall a definition of Schmid operator. For an irreducible finite dimensional $K_\mathbb{C}$-module (τ, V), we define $C^\infty(G, V)$ the space of V-valued smooth functions φ on G satisfying

$$\varphi(gk) = \tau(k)^{-1}\varphi(g), \quad (r,k) \in R \times K.$$

Take an orthonormal basis $(X_j)_{1 \le j \le \dim \mathfrak{p}_\mathbb{C}}$ of $\mathfrak{p}_\mathbb{C}$ with respect to the restriction of the complexified Killing form on $\mathfrak{g}_\mathbb{C}$. Then we define

$$\nabla\varphi(g) := \sum_{1 \le j \le \dim \mathfrak{p}_\mathbb{C}} R_{X_j}\varphi(g) \otimes \bar{X}_j$$

for $\varphi \in C^\infty(G, V)$, where $R_{X_j}\varphi(g) := \frac{d}{dt}\varphi\big(g \cdot \exp(tX_j^{(1)})\big)\big|_{t=0} + \sqrt{-1}\frac{d}{dt}\varphi\big(g \cdot \exp(tX_j^{(2)})\big)\big|_{t=0}$, with $X_j = X_j^{(1)} + \sqrt{-1}X_j^{(2)}$; $X_j^{(1)}, X_j^{(2)} \in \mathfrak{p}_\mathbb{R}$. It determines an operator $\nabla\colon C^\infty(G, V) \to C^\infty(G \otimes \mathfrak{p}_\mathbb{C})$, where $\mathfrak{p}_\mathbb{C}$ is regarded as a $K_\mathbb{C}$-module by the adjoint representation. We call it Schmid operator.

Every irreducible finite dimensional $K_\mathbb{C}$-module has the highest weight which occurs with multiplicity one and characterizes the $K_\mathbb{C}$-module. As $K_\mathbb{C} \simeq \mathrm{GL}(2, \mathbb{C})$, those highest weights can be parameterized by pairs of integers (ℓ_1, ℓ_2) with $\ell_1 \ge \ell_2$. Denote by τ_{ℓ_1,ℓ_2} the irreducible finite dimensional $K_\mathbb{C}$-module of the highest weight (ℓ_1, ℓ_2) on the space V_{ℓ_1,ℓ_2}. Indeed, we have a realization $\tau_{\ell_1,\ell_2} = \det^{\ell_2} \otimes S^{\ell_1-\ell_2}(\mathbb{C}^2)$, where $S^k(V)$ is the k-th symmetric product of a representation on V. Let us set $d = \ell_1 - \ell_2$. Then we have $\dim_\mathbb{C} V_{\ell_1,\ell_2} = d + 1$. As a $K_\mathbb{C}$-module, $\mathfrak{p}_\mathbb{C}$ is decomposed into irreducible ones $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ which corresponds to the hermitian structure on G/K, where $\mathfrak{p}_+ \simeq V_{(2,0)}$ and $\mathfrak{p}_- \simeq V_{(0,-2)}$, and both are 3 dimensional. This decomposition allows us to write the operator ∇ into a sum $\nabla = \nabla^+ + \nabla^-$, $\nabla^\pm\colon C^\infty(G, V) \to C^\infty(G, V \otimes \mathfrak{p}_\pm)$. We also have the irreducible decomposition of $V \otimes \mathfrak{p}_\pm$ for $V = V_{\ell_2,\ell_2}$ by $V_{\ell_1,\ell_2} \otimes \mathfrak{p}_+ \simeq V_{\ell_1+2,\ell_2} \oplus m_1 V_{\ell_1+1,\ell_2+1} \oplus m_2 V_{\ell_1,\ell_2+2}$, or $V \otimes \mathfrak{p}_- \simeq V_{\ell_1,\ell_2-2} \oplus m_1 V_{\ell_1-1,\ell_2-1} \oplus m_2 V_{\ell_1-2,\ell_2}$, where (m_1, m_2) equals $(1,1)$ except the following cases: $(m_1, m_2) = (0,0)$, if $\ell_1 = \ell_2$; and $= (1,0)$, if $\ell_1 = \ell_2 + 1$. We consider the projection onto each irreducible component; denote, for example, by P^{ℓ_1+2,ℓ_2} the projection onto V_{ℓ_1+2,ℓ_2}.

For a character η of R and a $K_\mathbb{C}$-module (τ, V), denote by $C^\infty(G, \mathbb{C}_\eta \otimes V)$ the space of $\mathbb{C}_\eta \otimes V$-valued functions φ on G satisfying

$$\varphi(rgk) = \eta(r)\tau(k)^{-1}\varphi(g), \quad (r,k) \in R \times K.$$

We use the same explicit realization of τ_{ℓ_1,ℓ_2} as in [O] Section 3, p. 268 (also in [M] Lemma 3.1); we denote by $\{v_j^{\ell_1,\ell_2} \mid 0 \leq j \leq d\}$ the basis of V_{ℓ_1,ℓ_2} in this realization. We will express an element $\varphi(g)$ of $C^\infty(G, \mathbb{C}_\eta \otimes V_{\ell_1,\ell_2})$ by a sum $\varphi(g) = \sum_{j=0}^d b_j(g) v_j^{\ell_1,\ell_2}$ with $\mathbb{C}_\eta$-valued functions b_j on $G_\mathbb{C}$.

3.2

There are 3 conjugacy classes of parabolic subgroups in $G = \mathrm{Sp}(2,\mathbb{R})$; let us denote by P_0 the minimal parabolic, P_1 the Siegel maximal parabolic, and P_2 the other maximal one. In this paper we consider the spaces $C^\infty\text{-}\mathrm{Ind}_R^G(\eta)$ for characters ψ of $N = N_1$ and N_2. For the N_0-case there is a study by Oda [O].

Let $P_1 = M_1 A_1 N_1$ be the Siegel parabolic subgroup of G with abelian unipotent radical $N_1 = \left\{ n(x) = \begin{pmatrix} 1_2 & x \\ 0 & 1_2 \end{pmatrix} \;\middle|\; x = {}^t x \right\}$. A non-degenerate unitary character of N_1 is given by

$$\psi\big(n(x)\big) = \exp\big(2\pi\sqrt{-1}\,\mathrm{tr}(hx)\big)$$

with a nonsingular symmetric matrix $h = \begin{pmatrix} h_1 & h_3 \\ h_3 & h2 \end{pmatrix} \in M_2(\mathbb{R})$. Then the group $M_1(\psi)$ is isomorphic to $\mathrm{SO}(2)$ or $\mathrm{SO}(1,1)$ depending on the signature of h. We fix a generator c of $\mathfrak{m}_1(\psi)_\mathbb{R}$ by $c := \begin{pmatrix} b_\psi & 0 \\ 0 & -{}^t b_\psi \end{pmatrix}$, $b_\psi = h^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Define a maximal split torus A_0 in G by $A_0 = \{a = (a_1, a_2) := \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1 > 0, a_2 > 0\}$. We recall a Cartan-Iwasawa type decomposition of $\mathfrak{g}$; $\mathfrak{g} = \mathrm{Ad}(a^{-1})\big(\mathfrak{m}_1(\psi) + \mathfrak{n}_1\big) + \mathfrak{a}_0 + \mathfrak{k}$, $a \in A_0$. By the restriction map, it enables us to define an inclusion of $C^\infty(G, \mathbb{C}_\eta \otimes V)$ into the space of $\mathbb{C}_\eta \otimes V$-valued functions $C^\infty(A_0, \mathbb{C}_\eta \otimes V)$ on A_0. Then, as in [M] Section 5, we can consider the A_0-radial part $R_\eta(\nabla^\pm_{\ell_1,\ell_2})$: $C^\infty(A_0, \mathbb{C}_\eta \otimes V) \to C^\infty\big(A_0, \mathbb{C}_\eta \otimes (V \otimes \mathfrak{p}_\pm)\big)$ of the operator $\nabla^\pm_{\ell_1,\ell_2} : C^\infty(G, V) \to C^\infty(G, V \otimes \mathfrak{p}_\pm)$ for $V = V_{\ell_1,\ell_2}$.

Now we give explicit formulas of the A_0-radial parts of Schmid operators. In the formulas below, we assume $h_1 \neq 0$, $h_2 \neq 0$, and $h_3 = 0$ for the character ψ of N_1. Concerning with this assumption, see a remark in the end of 3.6.

Proposition 3.3 **([M] Section 5, Proposition 5.3)** *Let us set some symbols:*

$$\partial_i = a_i \frac{\partial}{\partial a_i}, \quad \mathcal{L}_i^\pm = \partial_i \pm 4\pi h_i a_i^2 \quad (i = 1, 2);$$

$$D = h_1 a_1^2 - h_2 a_2^2 \quad \textit{and} \quad S_1 = \chi(c) \frac{h_1 a_1 h_2 a_2}{D},$$

with $c \in \mathfrak{m}_1(\psi)_\mathbb{R}$ defined in 3.2. Consider the operators $\nabla^\pm_{\ell_1,\ell_2} : C^\infty(G, V_{\ell_1,\ell_2}) \to C^\infty(G, V_{\ell_1,\ell_2} \otimes \mathfrak{p}_\pm)$ for an irreducible $K_\mathbb{C}$-module τ_{ℓ_1,ℓ_2}. In the situation of 3.2, then, for the A_0-radial parts $R_\eta(\nabla^\pm_{\ell_1,\ell_2})$ considered on the restriction to A_0 of $\varphi \in C^\infty(G, V_{\ell_1,\ell_2})$ we have the following formulas: writing $\varphi(a) = \sum_{j=0}^d b_j(a) v_j^{\ell_1,\ell_2}$ with $d = \ell_1 - \ell_2$, then *for* $P^{\ell_1+2,\ell_2}R_\eta(\nabla^+_{\ell_1,\ell_2})\varphi(a) = \sum_{j=0}^{d+2}\tilde{b}_j(a)v_j^{\ell_1+2,\ell_2}$, *we have*

$$\begin{aligned}\tilde{b}_j(a) = {} & j(j-1)\Big(\mathcal{L}_1^+ + j - 2 + \ell_2 - 2(d+2-j)\frac{h_2a_2^2}{D}\Big)\,b_{j-2}(a)\\ & - 2j(d+2-j)S_1\cdot b_{j-1}(a)\\ & + (d+1-j)(d+2-j)\Big(\mathcal{L}_2^+ - j + \ell_1 + 2j\frac{h_1a_1^2}{D}\Big)\,b_j(a),\end{aligned}\tag{3.1}$$

for $P^{\ell_1+1,\ell_2+1}R_\eta(\nabla^+_{\ell_1,\ell_2})\varphi(a) = \sum_{j=0}^{d}\tilde{b}_j(a)v_j^{\ell_1+1,\ell_2+1}$, *we have*

$$\begin{aligned}\tilde{b}_j(a) = {} & j\Big(\mathcal{L}_1^+ + j - 1 + \ell_2 - (d-2j)\frac{h_2a_2^2}{D}\Big)\,b_{j-1}(a)\\ & - (d-2j)S_1\cdot b_j(a)\\ & - (d-j)\Big(\mathcal{L}_2^+ - j - 1 + \ell_1 - (d-2j)\frac{h_1a_1^2}{D}\Big)\,b_{j+1}(a),\end{aligned}\tag{3.2}$$

for $P^{\ell_1,\ell_2+2}R_\eta(\nabla^+_{\ell_1,\ell_2})\varphi(a) = \sum_{j=0}^{d-2}\tilde{b}_j(a)v_j^{\ell_1,\ell_2+2}$, *we have*

$$\begin{aligned}\tilde{b}_j(a) = {} & \Big(\mathcal{L}_1^+ + j + \ell_2 + 2(j+1)\frac{h_2a_2^2}{D}\Big)\,b_j(a)\\ & + 2S_1\cdot b_{j+1}(a) + \Big(\mathcal{L}_2^+ - j - 2 + \ell_1 - 2(d-j-1)\frac{h_1a_1^2}{D}\Big)\,b_{j+2}(a),\end{aligned}\tag{3.3}$$

for $P^{\ell_1,\ell_2-2}R_\eta(\nabla^-_{\ell_1,\ell_2})\varphi(a) = \sum_{j=0}^{d+2}\tilde{b}_j(a)v_j^{\ell_1,\ell_2-2}$, *we have*

$$\begin{aligned}\tilde{b}_j(a) = {} & j(j-1)\Big(\mathcal{L}_2^- + j - 2 - \ell_1 + 2(d+2-j)\frac{h_1a_1^2}{D}\Big)\,b_{j-2}(a)\\ & + 2j(d+2-j)S_1\cdot b_{j-1}(a)\\ & + (d+1-j)(d+2-j)\Big(\mathcal{L}_1^- - j - \ell_2 - 2j\frac{h_2a_2^2}{D}\Big)\,b_j(a),\end{aligned}\tag{3.4}$$

for $P^{\ell_1-1,\ell_2-1}R_\eta(\nabla^-_{\ell_1,\ell_2})\varphi(a) = \sum_{j=0}^{d}\tilde{b}_j(a)v_j^{\ell_1-1,\ell_2-1}$, *we have*

$$\begin{aligned}\tilde{b}_j(a) = {} & j\Big(\mathcal{L}_2^- + j - 1 - \ell_1 + (d-2j)\frac{h_1a_1^2}{D}\Big)\,b_{j-1}(a)\\ & + (d-2j)S_1\cdot b_j(a)\\ & - (d-j)\Big(\mathcal{L}_1^- - j - 1 - \ell_2 + (d-2j)\frac{h_2a_2^2}{D}\Big)\,b_{j+1}(a),\end{aligned}\tag{3.5}$$

for $P^{\ell_1-2,\ell_2}R_\eta(\nabla^-_{\ell_1,\ell_2})\varphi(a) = \sum_{j=0}^{d-2}\tilde{b}_j(a)v_j^{\ell_1-2,\ell_2}$, *we have*

$$\begin{aligned}\tilde{b}_j(a) = {} & \Big(\mathcal{L}_2^- + j - \ell_1 - 2(j+1)\frac{h_1a_1^2}{D}\Big)\,b_j(a)\\ & - 2S_1\cdot b_{j+1}(a) + \Big(\mathcal{L}_1^- - j - 2 - \ell_2 + 2(d-j-1)\frac{h_2a_2^2}{D}\Big)\,b_{j+2}(a).\end{aligned}\tag{3.6}$$

Proof We take an orthonormal basis $\{X_j\}$ of $\mathfrak{p}_{\mathbb{C}}$ as in [M], 5.1, p. 250. Then $\nabla^{\pm}_{\ell_1,\ell_2}$ are given by (5.1) and (5.2) of [M], and for the A_0-radial parts we get the formulas (5.4) and (5.5) in [M] Proposition 5.3. We input the expression $\varphi(a) = \sum_{j=0}^{d} b_j(a) v_j^{\ell_1,\ell_2}$ to the formulas. By [M] Lemma 3.1 and Lemmas 3.3, 3.4, and 3.5, we calculate the action of the elements in $\mathfrak{k}$ and the projection onto each irreducible component of $V_{\ell_1,\ell_2} \otimes \mathfrak{p}_{\pm}$. Then we obtain the coefficient $\tilde{b}_j(a)$ for the each base of the components as shown.

3.4

Let $P_2 = M_2 A_2 N_2$ be the other maximal parabolic subgroup of G where

$$M_2 = \left\{ m(a,b,c,d;\varepsilon) = \begin{pmatrix} \varepsilon & & & \\ & a & & b \\ & & \varepsilon & \\ & c & & d \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \varepsilon \in \{\pm 1\} \right\},$$

$$N_2 = \left\{ n(x_1,x_2,z) = \begin{pmatrix} 1 & x_1 & & \\ & 1 & & \\ & & 1 & \\ & & -x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & z & x_2 \\ & 1 & x_2 & \\ & & 1 & \\ & & & 1 \end{pmatrix} \;\middle|\; x_1,x_2,z \in \mathbb{R} \right\}.$$

A non degenerate unitary character ψ of Heisenberg group N_2 is given by

$$\psi\big(n(x_1,x_2,z)\big) = \exp\big(2\pi\sqrt{-1}(h_1x_1 + h_2x_2)\big)$$

with $h_1, h_2 \in \mathbb{R}$, which is trivial on the center $\{n(0,0,z) \in N_2 \mid z \in \mathbb{R}\}$. Considering the action of M_2 on $\hat{N}_2$, we can change ψ into its conjugate which satisfies $h_2 = 0$. In the following discussion we replace ψ with this conjugate in the normal position (see a remark at the end of 3.6), and call it ψ again. Then we get $M_2(\psi) = \{m(1,s,0,1;1) \in M_2 \mid s \in \mathbb{R}\}$. Define $c \in \mathfrak{m}_2(\psi)_{\mathbb{R}}$ by $c = E_{2,4}$ the elementary matrix with the $(2,4)$-coefficient to be 1; then $\mathfrak{m}_2(\psi)_{\mathbb{R}} \simeq \mathbb{R} \cdot c$. We have Iwasawa decomposition of $\mathfrak{g}$ by $\mathfrak{g} = \mathrm{Ad}(a^{-1})(\mathfrak{m}_2(\psi) + \mathfrak{n}_2) + \mathfrak{a}_0 + \mathfrak{k}$, $a \in A_0$. Indeed, we remark that $\mathfrak{m}_2(\psi) + \mathfrak{n}_2$ coincides with the nilpotent radical $\mathfrak{n}_0$ of the minimal parabolic subalgebra of $\mathfrak{g}$; thus the space $C^{\infty}\text{-}\mathrm{Ind}_{M_2(\psi)N_2}^{G}(\eta)$ can be identified with the space considered in [O], p. 261. Then, as in [O] Section 6, we can consider the A_0-radial parts for the Schmid operators.

Proposition 3.5 **([O] Proposition 6.1 and Section 7)** *Let us set some symbols:*

$$\partial_i = a_i \frac{\partial}{\partial a_i}, \quad i = 1,2;\; S_2 = 2\pi\sqrt{-1}h_1 \frac{a_1}{a_2}.$$

Consider the operators $\nabla^{\pm}_{\ell_1,\ell_2} \colon C^{\infty}(G, V_{\ell_1,\ell_2}) \to C^{\infty}(G, V_{\ell_1,\ell_2} \otimes \mathfrak{p}_{\pm})$ *for a* $K_{\mathbb{C}}$*-module. In the situation of 3.4, then, we have the following formulas for the* A_0*-radial parts* $R_{\eta}(\nabla^{\pm}_{\ell_1,\ell_2})$ *of the Schmid operators: writing* $\varphi(a) = \sum_{j=0}^{d} b_j(a) v_j^{\ell_1,\ell_2}$ *with* $d = \ell_1 - \ell_2$ *for* $\varphi \in C^{\infty}(G, V_{\ell_1,\ell_2})$*, then* *for* $P^{\ell_1+2,\ell_2}R_\eta(\nabla^+_{\ell_1,\ell_2})\varphi(a)=\sum_{j=0}^{d+2}\tilde b_j(a)v_j^{\ell_1+2,\ell_2}$, *we have*

$$\begin{aligned}\tilde b_j(a)=\ & j(j-1)\big(\partial_1+\ell_1+d-j+2\big)\,b_{j-2}(a)+2j(d+2-j)S_2\cdot b_{j-1}(a)\\ &+(d+1-j)(d+2-j)\big(\partial_2-2\sqrt{-1}\chi(c)a_2^2+\ell_1-j\big)\,b_j(a),\end{aligned}\tag{3.7}$$

for $P^{\ell_1+1,\ell_2+1}R_\eta(\nabla^+_{\ell_1,\ell_2})\varphi(a)=\sum_{j=0}^{d}\tilde b_j(a)v_j^{\ell_1+1,\ell_2+1}$, *we have*

$$\begin{aligned}\tilde b_j(a)=\ & j\big(\partial_1+\ell_1-j-1\big)\,b_{j-1}(a)+(d-2j)S_2\cdot b_j(a)\\ &-(d-j)\big(\partial_2-2\sqrt{-1}\chi(c)a_2^2+\ell_1-j-1\big)\,b_{j+1}(a),\end{aligned}\tag{3.8}$$

for $P^{\ell_1,\ell_2+2}R_\eta(\nabla^+_{\ell_1,\ell_2})\varphi(a)=\sum_{j=0}^{d-2}\tilde b_j(a)v_j^{\ell_1,\ell_2+2}$, *we have*

$$\begin{aligned}\tilde b_j(a)=\ & \big(\partial_1+\ell_1-d-j-2\big)\,b_j(a)-2S_2\cdot b_{j+1}(a)\\ &+\big(\partial_2-2\sqrt{-1}\chi(c)a_2^2+\ell_1-j-2\big)\,b_{j+2}(a),\end{aligned}\tag{3.9}$$

for $P^{\ell_1,\ell_2-2}R_\eta(\nabla^-_{\ell_1,\ell_2})\varphi(a)=\sum_{j=0}^{d+2}\tilde b_j(a)v_j^{\ell_1,\ell_2-2}$, *we have*

$$\begin{aligned}\tilde b_j(a)=\ & j(j-1)\big(\partial_2+2\sqrt{-1}\chi(c)a_2^2-\ell_1+j-2\big)\,b_{j-2}(a)\\ &-2j(d+2-j)S_2\cdot b_{j-1}(a)\\ &+(d+1-j)(d+2-j)\big(\partial_1-\ell_1+d+j\big)\,b_j(a),\end{aligned}\tag{3.10}$$

for $P^{\ell_1-1,\ell_2-1}R_\eta(\nabla^-_{\ell_1,\ell_2})\varphi(a)=\sum_{j=0}^{d}\tilde b_j(a)v_j^{\ell_1-1,\ell_2-1}$, *we have*

$$\begin{aligned}\tilde b_j(a)=\ & j\big(\partial_2+2\sqrt{-1}\chi(c)a_2^2-\ell_1+j-1\big)\,b_{j-1}(a)-(d-2j)S_2\cdot b_j(a)\\ &-(d-j)\big(\partial_1-\ell_1+j-1\big)\,b_{j+1}(a),\end{aligned}\tag{3.11}$$

for $P^{\ell_1-2,\ell_2}R_\eta(\nabla^-_{\ell_1,\ell_2})\varphi(a)=\sum_{j=0}^{d-2}\tilde b_j(a)v_j^{\ell_1-2,\ell_2}$, *we have*

$$\begin{aligned}\tilde b_j(a)=\ & \big(\partial_2+2\sqrt{-1}\chi(c)a_2^2-\ell_1+j\big)\,b_j(a)+2S_2\cdot b_{j+1}(a)\\ &+\big(\partial_1-\ell_1-d+j\big)\,b_{j+2}(a).\end{aligned}\tag{3.12}$$

Proof We start the formulas (i) and (ii) in [O] Proposition 6.1. And we input the expression $\varphi(a)=\sum_{j=0}^{d}b_j(a)v_j^{\ell_1,\ell_2}$ into them. Here $\{v_j^{\ell_1,\ell_2}\mid 0\le j\le d=\ell_1-\ell_2\}$ is the basis of V_{ℓ_1,ℓ_2} in the realization of τ_{ℓ_1,ℓ_2} given in [O] Section 3, p. 268. Then, by [O] Lemmas 3.1, 3.2, and 3.3, we calculate the action of elements in $\mathfrak{k}$ and the projection onto each irreducible component of $V_{\ell_1,\ell_2}\otimes\mathfrak{p}_\pm$. The coefficients $\tilde b_j(a)$ for the bases of each component are given as in our statement.

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3.6

Let $(\pi, \mathcal{H}_\pi)$ be an irreducible admissible Hilbert representation of G. Denote by $\mathcal{H}_{\pi,K}$ all K-finite vectors which determines a Harish-Chandra module. We will investigate the following space of equivariant realizations:

$$\mathrm{Wh}_\eta(\pi) := \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})}\big(\mathcal{H}_{\pi,K}, C^\infty\text{-}\,\mathrm{Ind}_R^G(\eta)_K\big).$$

We call it the space of (N,ψ)-Whittaker embedding of π with the character χ on $M(\psi)$. Fixing a K-morphism $\iota\colon V_{\ell_1,\ell_2} \hookrightarrow \mathcal{H}_{\pi,K}$ for a K-type τ_{ℓ_1,ℓ_2} occurring in $\mathcal{H}_{\pi,K}$, we consider the restriction $\iota^*\Phi = \Phi \circ \iota$ of a functional Φ in $\mathrm{Wh}_\eta(\pi)$. We can define a $\mathbb{C}_\eta \otimes V^*_{\ell_1,\ell_2}$-valued function φ_{ℓ_1,ℓ_2} on G such that

$$\iota^*\Phi(v)(g) = \langle v, \varphi_{\ell_1,\ell_2}(g)\rangle$$

for any $v \in V_{\ell_1,\ell_2}$, $g \in G$, and the canonical dual pairing $\langle\, ,\, \rangle$ on $V_{\ell_1,\ell_2} \times (\mathbb{C}_\eta \otimes V^*_{\ell_1,\ell_2})$ valued in $\mathbb{C}_\eta$, where $(\tau^*_{\ell_1,\ell_2}, V^*_{\ell_1,\ell_2})$ is the contragredient of τ_{ℓ_1,ℓ_2}. Then φ_{ℓ_1,ℓ_2} belongs to $C^\infty(G, \mathbb{C}_\eta \otimes V^*_{\ell_1,\ell_2})$. We will investigate this vector valued function φ_{ℓ_1,ℓ_2} for an (N,ψ)-Whittaker embedding. Theorem 10.1 and Corollary 10.2 of [Ba] and [Y2] Section 1 tell us that it can be characterized by a solution of a set of differential equations, which is given by use of Schmid operators, when π is a derived functor module $A_\mathfrak{q}(\lambda)$ with λ in good range. We will give the differential equations separately in each situation below.

Here we have a remark about the twisting of (N,ψ)-Whittaker embedding Φ by elements in M. For $\Phi \in \mathrm{Wh}_\eta(\pi)$ and $m \in M$, define a new functional ${}^m\Phi$ by ${}^m\Phi(v)(g) := \Phi(v)(mg)$ for $v \in \mathcal{H}_{\pi,K}, g \in G$. Then it satisfies the property:

$${}^m\Phi(v)(r'g) = \eta(mr'm^{-1})\,{}^m\Phi(v)(g)$$

for $v \in \mathcal{H}_{\pi,K}$ and $(r', g) \in (m^{-1}R\cdot m) \times G$. So the functional ${}^m\Phi$ belongs to $\mathrm{Wh}_{{}^m\eta}(\pi)$ for the character ${}^m\eta(r') = \eta(mr'm^{-1})$ on $m^{-1}R\cdot m$. This process allows us to change a non-degenerate unitary character ψ of N into a suitable standard form, which we have done in 3.2, 3.4.

4 The Derived Functor Modules of the Class 7

4.1

We consider a derived functor module $A_{\mathfrak{q}_7}(\lambda)$ of the class 7 defined in 2.1, with an integral λ in the good range. The associated variety of it is the closure of $\mathcal{O}_{\begin{smallmatrix}+&-\\-&+\end{smallmatrix}}$, whose complex dimension is 3. So it is known that $A_{\mathfrak{q}_7}(\lambda)$ has no (N_0,ψ)-Whittaker embedding with respect to a non-degenerate character ψ of N_0 (non generic in the usual sense), [V2], [Ko].

Write $\lambda + \rho_{\mathfrak{u}_7} = \lambda_1 e_1 + \lambda_2 e_2 \in \sqrt{-1}\mathfrak{t}^*_\mathbb{R}$, then we have $(\lambda_1, \lambda_2) = (\frac{m}{2}, -\frac{m}{2})$, where m is a positive odd integer. We assume that $m \geq 3$, so λ is in the good range. The module has the minimal K-type $\tau_{\ell,-\ell}$ with $\ell = \frac{m+1}{2}$, which is of $2\ell + 1$ dimension. Each

K-type of $A_{\mathfrak{q}_7}(\lambda)$ has the highest weight given by $(\ell+2n_1, -\ell-2n_2)$ with nonnegative integers n_1 and n_2.

To begin with, we study the (N_1, ψ)-Whittaker embedding for the derived functor module $A_{\mathfrak{q}_7}(\lambda)$. Our knowledge on its K-type decomposition tells us that the function defined in 3.6, $\varphi_{\ell,-\ell}(g) \in C^\infty(G, \mathbb{C}_\eta \otimes V_{\ell,-\ell})$, for the minimal K-type should satisfy the following equations:

$$P^{\ell+1,-\ell+1}\nabla^+_{\ell,-\ell}\varphi_{\ell,-\ell}(g) = 0, \quad P^{\ell,-\ell+2}\nabla^+_{\ell,-\ell}\varphi_{\ell,-\ell}(g) = 0,$$
$$P^{\ell-1,-\ell-1}\nabla^-_{\ell,-\ell}\varphi_{\ell,-\ell}(g) = 0, \quad P^{\ell-2,-\ell}\nabla^-_{\ell,-\ell}\varphi_{\ell,-\ell}(g) = 0.$$

By Proposition 3.3, these are explicitly given for the restriction of $\varphi_{\ell,-\ell}$ to A_0; $\varphi_{\ell,-\ell}(a) = \sum_{j=0}^{2\ell} b_j(a) v_j^{\ell,-\ell}$, by

for $0 \le j \le 2\ell$,

(4.1a)
$$j\Big(\mathcal{L}_1^+ + j - 1 - \ell - 2(\ell-j)\frac{h_2 a_2^2}{D}\Big) b_{j-1}(a) - 2(\ell-j)S_1 \cdot b_j(a)$$
$$- (2\ell - j)\Big(\mathcal{L}_2^+ - j - 1 + \ell - 2(\ell-j)\frac{h_1 a_1^2}{D}\Big) b_{j+1}(a) = 0,$$

for $1 \le j \le 2\ell - 1$,

(4.1b)
$$\Big(\mathcal{L}_1^+ + j - 1 - \ell + 2j\frac{h_2 a_2^2}{D}\Big) b_{j-1}(a) + 2S_1 \cdot b_j(a)$$
$$+ \Big(\mathcal{L}_2^+ - j - 1 + \ell - 2(2\ell - j)\frac{h_1 a_1^2}{D}\Big) b_{j+1}(a) = 0,$$

for $0 \le j \le 2\ell$,

(4.1c)
$$j\Big(\mathcal{L}_2^- + j - 1 - \ell + 2(\ell-j)\frac{h_1 a_1^2}{D}\Big) b_{j-1}(a) + 2(\ell-j)S_1 \cdot b_j(a)$$
$$- (2\ell - j)\Big(\mathcal{L}_1^- - j - 1 + \ell + 2(\ell-j)\frac{h_2 a_2^2}{D}\Big) b_{j+1}(a) = 0,$$

for $1 \le j \le 2\ell - 1$,

(4.1d)
$$\Big(\mathcal{L}_2^- + j - 1 - \ell - 2j\frac{h_1 a_1^2}{D}\Big) b_{j-1}(a) - 2S_1 \cdot b_j(a)$$
$$+ \Big(\mathcal{L}_1^- - j - 1 + \ell + 2(2\ell - j)\frac{h_2 a_2^2}{D}\Big) b_{j+1}(a) = 0.$$

Here we have the first observation.

Lemma 4.2 *Assume that $\{b_j(a) \mid 0 \le j \le 2\ell\}$ forms a solution of the system of equations above. Then we have that $b_{2k+1}(a_1, a_2) = 0$ for $0 \le k \le \ell - 1$. Moreover, for $\mathrm{Wh}_{\chi\cdot\psi}\big(A_{\mathfrak{q}_7}(\lambda)\big) \neq \{0\}$ it is necessary that $\chi = 0$ for the linear form χ on $\mathfrak{m}_1(\psi)_{\mathbb{R}}$.*

Proof The equations (4.1a) and (4.1b) are equivalent to the followings:

for $0 \le j \le 2\ell - 1$,

$$(4.2a)\quad j\frac{h_2 a_2^2}{D} b_{j-1}(a) + S_1 \cdot b_j(a) + \left(\mathcal{L}_2^+ + \ell - j - 1 - (2\ell - j)\frac{h_1 a_1^2}{D}\right) b_{j+1}(a) = 0,$$

for $1 \le j \le 2\ell$,

(4.2b)
$$\left(\mathcal{L}_1^+ - \ell + j - 1 + j\frac{h_2 a_2^2}{D}\right) b_{j-1}(a) + S_1 \cdot b_j(a) - (2\ell - j)\frac{h_1 a_1^2}{D} b_{j+1}(a) = 0.$$

The equations (4.1c) and (4.1d) are also equivalent to

for $0 \le j \le 2\ell - 1$,

(4.2c)
$$j\frac{h_1 a_1^2}{D} b_{j-1}(a) + S_1 \cdot b_j(a) - \left(\mathcal{L}_1^- + \ell - j - 1 + (2\ell - j)\frac{h_2 a_2^2}{D}\right) b_{j+1}(a) = 0,$$

for $1 \le j \le 2\ell$,

(4.2d)
$$\left(\mathcal{L}_2^- - \ell + j - 1 - j\frac{h_1 a_1^2}{D}\right) b_{j-1}(a) - S_1 \cdot b_j(a) + (2\ell - j)\frac{h_2 a_2^2}{D} b_{j+1}(a) = 0.$$

Then we can find the following $\mathbb{C}(a_1, a_2)$-linear relations among $b_j(a)$'s; from (4.2a) and (4.2d) one has

for $1 \le j \le 2\ell - 1$,

$$(4.3a)\qquad (j-1)\frac{h_2 a_2^2}{D} b_{j-2}(a) + S_1 \cdot b_{j-1}(a) + \left(8\pi h_2 a_2^2 - 2(\ell - j)\frac{h_2 a_2^2}{D}\right) b_j(a) + S_1 \cdot b_{j+1}(a) - (2\ell - j - 1)\frac{h_2 a_2^2}{D} b_{j+2}(a) = 0,$$

and the others are obtained from the equations (4.2b) and (4.2c);

for $1 \le j \le 2\ell - 1$,

$$(4.3b)\qquad (j-1)\frac{h_1 a_1^2}{D} b_{j-2}(a) + S_1 \cdot b_{j-1}(a) + \left(8\pi h_1 a_1^2 - 2(\ell - j)\frac{h_1 a_1^2}{D}\right) b_j(a) + S_1 \cdot b_{j+1}(a) - (2\ell - j - 1)\frac{h_1 a_1^2}{D} b_{j+2}(a) = 0.$$

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Taking the differences of (4.3a) and (4.3b), we get for $1 \le j \le 2\ell - 1$,

$$(j-1)b_{j-2}(a) + \big(8\pi D - 2(\ell - j)\big)\, b_j(a) - (2\ell - j - 1)b_{j+2}(a) = 0. \tag{4.4}$$

Note that the indexes $j-2$, j, and $j+2$ occurring in each equation of (4.4) have the common parity; hence we can separate the equations of (4.4) into 2 families; the first one consists of the equations for the functions $\{b_{2k}(a) \mid 0 \le k \le \ell\}$ with even indexes, and the other consisting of the equations for $\{b_{2k+1}(a) \mid 0 \le k \le \ell - 1\}$ with odd indexes. Solving the latter set of equations for odd indexed ones, which consists of independent equations, we can conclude easily that the first assertion in lemma holds.

So we can set $b_j(a) = 0$ for all odd j's in (4.2a), (4.2b), (4.2c) and (4.2d). Then we obtain $S_1 \cdot b_{2k}(a) = 0$ for $0 \le k \le \ell$, which implies the second assertion. This completes the proof of lemma.

Set $b_j(a) = \big(\sqrt{|h_1|}a_1\sqrt{|h_2|}a_2\big)^{\ell+1} c_j(a)$ for $0 \le j \le 2\ell$ and introduce new variables:

$$x = 2\pi(h_1 a_1^2 + h_2 a_2^2), \quad y = 2\pi(h_1 a_1^2 - h_2 a_2^2).$$

By Lemma 4.2, the equations (4.2a), (4.2b), (4.2c), and (4.2d) give us the following equations:

for $1 \le k \le \ell$,

$$(2k-1)c_{2k-2}(x,y) + \left(2y\Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\Big) + 2y - 2\ell + 2k - 1\right) c_{2k}(x,y) = 0, \tag{4.5a}$$

for $1 \le k \le \ell$,

$$\left(2y\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) + 2y + 2k - 1\right) c_{2k-2}(x,y) - (2\ell - 2k + 1)c_{2k}(x,y) = 0, \tag{4.5b}$$

for $1 \le k \le \ell$,

$$(2k-1)c_{2k-2}(x,y) - \left(2y\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) - 2y - 2\ell + 2k - 1\right) c_{2k}(x,y) = 0, \tag{4.5c}$$

for $1 \le k \le \ell$,

$$\left(2y\Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\Big) - 2y - 2k + 1\right) c_{2k-2}(x,y) + (2\ell - 2k + 1)c_{2k}(x,y) = 0. \tag{4.5d}$$

Comparing the equations in (4.5a) with the ones in (4.5c), or those in (4.5b) with the ones in (4.5d), we find that

$$\frac{\partial}{\partial x} c_{2k}(x,y) = 0$$

for $0 \le k \le \ell$. Therefore we obtain

Proposition 4.3 *For a solution of the set of equations above, $c_{2k}(x,y)$ does not depend on the variable x for all $0 \le k \le \ell$.*

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So we may consider $c_{2k}(y) = c_{2k}(x, y)$ and study the ordinary differential and difference equations of one variable y. The equations (4.5a), (4.5b), (4.5c), and (4.5d) are, then, reduced to the followings:

for $1 \le k \le \ell$,

$$(2k-1)c_{2k-2}(y) - \Big(2y\frac{d}{dy} - 2y + 2\ell - 2k + 1\Big)\, c_{2k}(y) = 0, \tag{4.6a}$$

for $1 \le k \le \ell$,

$$\Big(2y\frac{d}{dy} + 2y + 2k - 1\Big)\, c_{2k-2}(y) - (2\ell - 2k + 1)c_{2k}(y) = 0. \tag{4.6b}$$

For each k, $0 \le k \le \ell$, we make a pair of equations in (4.6a) and (4.6b) that contain the functions $c_{2k-2}(y)$ and $c_{2k}(y)$, or $c_{2k}(y)$ and $c_{2k+2}(y)$. Then we find the following equation of second degree for a single $c_{2k}(y)$, for $0 \le k \le \ell$,

$$\Big(y^2\frac{d^2}{dy^2} + (\ell+1)y\frac{d}{dy} + (\ell - 2k)y - y^2\Big)\, c_{2k}(y) = 0. \tag{4.7}$$

During this process to obtain (4.7), we can also check that the integrable condition is satisfied for the set of equations (4.6a) and (4.6b). Indeed, we can find 2 pairs in (4.6a) and (4.6b) to give equations for $c_{2k}(y)$, $1 \le k \le \ell - 1$ of second degree, and the results become exactly the same one for both of the pairs; it is the equation (4.7) for $c_{2k}(y)$.

Now we separate our computation into 2 cases depending on the parity of ℓ; the case that ℓ is even, or ℓ is odd.

4.4 The Even Case

Consider the equation (4.7) for $2k = \ell$:

$$\Big(y^2\frac{d^2}{dy^2} + (\ell+1)y\frac{d}{dy} - y^2\Big)\, c_\ell(y) = 0.$$

It is holonomic of rank 2 and has solutions:

$$c_\ell(y) = \beta_1 \cdot y^{-\ell/2}K_{\frac{\ell}{2}}(y) + \beta_2 \cdot y^{-\ell/2}I_{\frac{\ell}{2}}(y), \tag{4.8}$$

where β_1, β_2 are constants, and $K_\nu(z)$, $I_\nu(z)$ are the modified Bessel functions, [M-O-S], p. 66, and p. 69, 3.2. All the other $c_{2k}(y)$, $0 \le k \le \ell$, are determined recursively through the equations (4.6a) and (4.6b). Then the following recurrence relations between the modified Bessel functions tells us that each $c_{2k}(y)$ can be written as a $\mathbb{C}[y, y^{-1}]$-linear combination of the modified Bessel functions:

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -2(\nu/z)K_\nu(z), \quad K_{\nu-1}(z) + K_{\nu+1}(z) = -2\big(dK_\nu(z)/dz\big),$$
$$I_{\nu-1}(z) - I_{\nu+1}(z) = 2(\nu/z)I_\nu(z), \quad I_{\nu-1}(z) + I_{\nu+1}(z) = 2\big(dI_\nu(z)/dz\big),$$

[M-O-S], p. 67, 3.1.1.

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4.5 The Odd Case

Consider the equations (4.6a) and (4.6b) for $2k = \ell + 1$:

$$\ell c_{\ell-1}(y) - \left(2y\frac{d}{dy} - 2y + \ell\right) c_{\ell+1}(y) = 0,$$

$$\left(2y\frac{d}{dy} + 2y + \ell\right) c_{\ell-1}(y) - \ell c_{\ell+1}(y) = 0.$$

It is holonomic of rank 2, and the solutions are given by

$$c_{\ell\pm1}(y) = y^{-\frac{\ell-1}{2}}\left\{\beta_1 \cdot \left(K_{\frac{\ell-1}{2}}(y) \mp K_{\frac{\ell+1}{2}}(y)\right) + \beta_2 \cdot \left(I_{\frac{\ell-1}{2}}(y) \pm I_{\frac{\ell+1}{2}}(y)\right)\right\}, \tag{4.9}$$

where β_1, β_2 are constants. Similarly as in the even case, the other functions $c_{2k}(y)$, $0 \leq k \leq \ell$, are given recursively through the equations (4.6a) and (4.6b), and each of them can be written as a $\mathbb{C}[y, y^{-1}]$-linear sum of the modified Bessel functions in the variable y.

Theorem 4.6 *Consider the set of difference-differential equations (4.1a), (4.1b), (4.1c), and (4.1d), and make $b_{2k+1}(a) = 0$ for $0 \leq k \leq \ell - 1$ after Lemma 4.2. Then it determines a holonomic system of rank 2. There exists a unique solution, up to a scalar multiple, satisfying the conditions (i) and (ii) below if and only if the character ψ of N_1 corresponds to an indefinite quadratic form; that is $h_1h_2 < 0$. The conditions on $b_{2k}(a)$, $0 \leq k \leq \ell$, are the followings:*

(i) *all $b_{2k}(a)$ are holomorphic on A_0, and*
(ii) *all $b_{2k}(a)$ decay rapidly when $a_1, a_2 \to +\infty$.*

The unique solution with the properties (i) and (ii) is obtained recursively through the equations (4.6a,b) from the following function(s):

if $h_1 > 0$ and $h_2 < 0$, and ℓ is even, then

$$b_\ell(a) = \frac{\left(\sqrt{|h_1h_2|}a_1a_2\right)^{\ell+1}}{\left(2\pi(h_1a_1^2 - h_2a_2^2)\right)^{\frac{\ell}{2}}} K_{\frac{\ell}{2}}\left(2\pi(h_1a_1^2 - h_2a_2^2)\right), \quad \textit{or} \tag{E}$$

if $h_1 > 0$ and $h_2 < 0$, and ℓ is odd, then

$$b_{\ell\mp1}(a) = \frac{\left(\sqrt{|h_1h_2|}a_1a_2\right)^{\ell+1}}{\left(2\pi(h_1a_1^2 - h_2a_2^2)\right)^{\frac{\ell-1}{2}}} \left(K_{\frac{\ell-1}{2}}\left(2\pi(h_1a_1^2 - h_2a_2^2)\right) \pm K_{\frac{\ell+1}{2}}\left(2\pi(h_1a_1^2 - h_2a_2^2)\right)\right). \tag{O}$$

Proof As we obtained in 4.4 and 4.5, all the solutions can be written as products of $\left(\sqrt{|h_1h_2|}a_1a_2\right)^{\ell+1}$ and $\mathbb{C}[y, y^{-1}]$-linear sums of the modified Bessel functions. We recall that that the function $I_\nu(z)$ rapidly increases as $|z| \to +\infty$ and $z \in \mathbb{R}$, [M-O-S]

p. 139. If $h_1h_2 > 0$, then the variable $y = h_1a_1^2 - h_2a_2^2$ admits zeros in A_0. Then, in the linear sum, the terms containing a K-Bessel function have poles of finite degree along these zeros; thus such terms cannot appear in the solution under the condition (i). On the other hand the terms containing $I_\nu(z)$ increases rapidly, which violates the condition (ii). Therefore we have no solution satisfying (i), (ii), if $h_1h_2 > 0$.

If $h_1h_2 < 0$, we can see that only the terms containing $K_\nu(z)$ satisfy both of the conditions, because $y = h_1a_1^2 - h_2a_2^2$ cannot be zero on A_0, and $K_\nu(y)$ has good asymptotic behavior as a_1 and $a_2 \to +\infty$, if $h_1 > 0$ and $h_2 < 0$, [M-O-S] p. 139. If $h_1 < 0$ and $h_2 > 0$, then, rewriting (4.6a) and (4.6b) for $-y$, we can discuss again similarly as above. These together give our assertions.

4.7

In the next place, we investigate the (N_2, ψ)-Whittaker embedding of $A_{\mathfrak{q}_7}(\lambda)$. The K-type decomposition of the module tells us that the following equations should be satisfied for the function $\varphi_{\ell,-\ell}(g) \in C^\infty(G, \mathbb{C}_\eta \otimes V_{\ell,-\ell})$ defined in 3.6 with respect to the minimal K-type $\tau_{\ell,-\ell}$:

for $0 \le j \le 2\ell$,

$$\text{(4.10a)} \qquad j\big(\partial_1 + \ell - j - 1\big)\, b_{j-1}(a) + 2(\ell - j)S_2 \cdot b_j(a) \\ - (2\ell - j)\big(\partial_2 - 2\sqrt{-1}\chi(c)a_2^2 + \ell - j - 1\big)\, b_{j+1}(a) = 0,$$

for $1 \le j \le 2\ell$,

$$\text{(4.10b)} \qquad \big(\partial_1 - \ell - j - 1\big)\, b_{j-1}(a) - 2S_2 \cdot b_j(a) \\ + \big(\partial_2 - 2\sqrt{-1}\chi(c)a_2^2 + \ell - j - 1\big)\, b_{j+1}(a) = 0,$$

for $0 \le j \le 2\ell$,

$$\text{(4.10c)} \qquad j\big(\partial_2 + 2\sqrt{-1}\chi(c)a_2^2 - \ell + j - 1\big)\, b_{j-1}(a) \\ - 2(\ell - j)S_2 \cdot b_j(a) - (2\ell - j)(\partial_1 - \ell + j - 1)b_{j+1}(a) = 0,$$

for $1 \le j \le 2\ell$,

$$\text{(4.10d)} \qquad \big(\partial_2 + 2\sqrt{-1}\chi(c)a_2^2 - \ell + j - 1\big)\, b_{j-1}(a) + 2S_2 \cdot b_j(a) \\ + (\partial_1 - 3\ell + j - 1)b_{j+1}(a) = 0.$$

Since the Gelfand-Kirillov dimension of $A_{\mathfrak{q}_7}(\lambda)$ is not of maximal, there cannot exist any non-zero embedding under consideration, if both χ and ψ are non-trivial; [Ko], [V]. This fact can be observed, of course, directly also in our setting:

Lemma 4.8 *To obtain a non-zero functional in* $\mathrm{Wh}_{\chi\cdot\psi}\big(A_{\mathfrak{q}_7}(\lambda)\big)$ *it is necessary that* $\chi = 0$ *for the linear form* χ *on* $m_2(\psi)_{\mathbb{R}}$. *For any solution* $\{b_j(a) \mid 0 \le j \le 2\ell\}$ *of (4.10a), (4.10b), (4.10c), and (4.10d), we also have linear relations* $b_{j-1}(a) + b_{j+1}(a) = 0$ *for* $1 \le j \le 2\ell - 1$.

Proof The equations (4.10a) and (4.10b) are equivalent to

for $0 \le j \le 2\ell - 1$,

$$jb_{j-1}(a) + S_2 b_j(a) - \left(\partial_2 - 2\sqrt{-1}\chi(c)a_2^2 + \ell - j - 1\right) b_{j+1}(a) = 0, \tag{4.11a}$$

for $1 \le j \le 2\ell$,

$$(\partial_1 - \ell - 1)b_{j-1}(a) - S_2 b_j(a) = 0. \tag{4.11b}$$

Also the equations (4.10c) and (4.10d) are equivalent to

for $0 \le j \le 2\ell - 1$,

$$\left(\partial_2 + 2\sqrt{-1}\chi(c)a_2^2 - \ell + j - 1\right) b_{j-1}(a) + S_2 b_j(a) - (2\ell - 1)b_{j+1}(a) = 0, \tag{4.11c}$$

for $1 \le j \le 2\ell$,

$$S_2 b_j(a) + (\partial_1 - \ell - 1)b_{j+1}(a) = 0. \tag{4.11d}$$

We can find $\mathbb{C}(a_1, a_2)$-linear relations among $b_j(a)$'s by pairing (4.11a) and (4.11c), which are given by

for $1 \le j \le 2\ell - 1$,

(4.12a)
$$\begin{aligned}(j-1)b_{j-2}(a) &+ S_2 b_{j-1}(a)\\ &- 2\left(2\sqrt{-1}\chi(c)a_2^2 + \ell - j\right) b_j(a) + S_2 b_{j+1}(a) - (2\ell - j - 1)b_{j+2}(a) = 0.\end{aligned}$$

Similarly the equations (4.11b) and (4.11d) yield

for $1 \le j \le 2\ell - 1$,

$$S_2\left(b_{j-1}(a) + b_{j+1}(a)\right) = 0. \tag{4.12b}$$

As we have assumed that $h_1 \neq 0$ in the definition, (4.12b) implies that $b_{j-1}(a) + b_{j+1}(a) = 0$ for $1 \le j \le 2\ell - 1$. Then (4.12a) and these relations give us

$$1 \le j \le 2\ell - 1, \quad 4\sqrt{-1}\chi(c)a_2^2 \cdot b_j(a) = 0. \tag{4.13}$$

If $\chi \neq 0$, then it tells us $b_j(a) = 0$ for $1 \le j \le 2\ell - 1$. But it provides us only with the trivial solution. Thus χ should be zero and the proof is completed.

Set $b_j(a) = a_1^{\ell+1}a_2^{-\ell+1}c_j(a)$ for $0 \le j \le 2\ell$. Here we introduce the new variables:

$$x = a_1 a_2, \quad y = a_1/a_2,$$

and consider the functions $c_j(x, y) = c_j(a_1, a_2)$ for $0 \le j \le 2\ell$.

Lemma 4.9 *A solution $\{c_j(x, y) \mid 0 \le j \le 2\ell\}$ satisfying the set of equations does not depend on the variable x.*

Proof By Lemma 4.8, our equations are reduced to the followings:

for $1 \leq j \leq 2\ell$,

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) c_{j-1}(x,y) + 2\pi\sqrt{-1}h_1 y \cdot c_j(x,y) = 0, \tag{4.14a}$$

for $1 \leq j \leq 2\ell$,

$$2\pi\sqrt{-1}h_1 y \cdot c_{j-1}(x,y) - \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) c_j(x,y) = 0, \tag{4.14b}$$

for $1 \leq j \leq 2\ell$,

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) c_{j-1}(x,y) - 2\pi\sqrt{-1}h_1 y \cdot c_j(x,y) = 0, \tag{4.14c}$$

for $1 \leq j \leq 2\ell$,

$$2\pi\sqrt{-1}h_1 y \cdot c_{j-1}(x,y) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) c_j(x,y) = 0. \tag{4.14d}$$

By pairing (4.14a) and (4.14c), or (4.14b) and (4.14d), we find

$$\frac{\partial}{\partial x} c_j(x,y) = 0$$

for $0 \leq j \leq 2\ell$. This implies our assertion.

After all we obtain the following difference-differential equations:

for $1 \leq j \leq 2\ell$,

$$\frac{d}{dy} c_{j-1}(y) - 2\pi\sqrt{-1}h_1 \cdot c_j(y) = 0, \tag{4.15a}$$

$1 \leq j \leq 2\ell$,

$$2\pi\sqrt{-1}h_1 \cdot c_{j-1}(y) + \frac{d}{dy} c_j(x,y) = 0. \tag{4.15b}$$

Finally we conclude

Theorem 4.10 *Let $A_{\mathfrak{q}_7}(\lambda)$ be a derived functor module of the class 7 with an integral λ in the good range. Then the set of difference differential equations (4.10a), (4.10b), (4.10c), and (4.10d) defines a holonomic system of rank 2, and the following $b_j(a)$, $0 \leq j \leq 2\ell$, give its solution:*

$$b_j(a) = \left(\sqrt{-1}\right)^j a_1 a_2 \left(\frac{a_1}{a_2}\right)^{\ell} \left((-1)^j \beta_1 e^{2\pi h_1 (a_1/a_2)} + \beta_2 e^{-2\pi h_1 (a_1/a_2)}\right).$$

Here β_1 and β_2 are constants which do not depend on j. If $\beta_1 = 0$, $\beta_2 \neq 0$ and $h_1 > 0$ (resp. $\beta_1 \neq 0$, $\beta_2 = 0$ and $h_1 < 0$), this solution decays rapidly as $a_1/a_2 \to +\infty$, and decays as $a_2 \to +\infty$, since $\ell \geq 2$.

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5 The Derived Functor Modules of the Class 6

5.1

Next we study a derived functor module $A_{\mathfrak{q}_6}(\lambda)$ of the class 6 with an integral λ in the good range. It is known by works of Kostant and Vogan that there is no (N_0, ψ)-Whittaker embedding of this module, so we study N_1, N_2 cases. We have $\lambda + \rho_{\mathfrak{u}_6} = \ell e_1 + 0e_2$ with a non-negative integer ℓ, and λ is good, if $\ell \geq 2$. The minimal K-type of $A_{\mathfrak{q}_6}(\lambda)$ is $(\tau_{\ell,1}, V_{\ell,1})$. Each K-type occurring in the module has the highest weight $(\ell + 2n_1 + n_2, 1 + n_2)$ with non-negative integers n_1 and n_2.

First we study the (N_1, ψ)-Whittaker embedding of the module $A_{\mathfrak{q}_6}(\lambda)$. The K-type decomposition tell us that the function $\varphi_{-1,-\ell}(g) \in C^\infty(G, \mathbb{C}_\eta \otimes V_{-1,-\ell})$ defined in 3.6 with respect to the minimal K-type should satisfy the following equations:

$$P^{1,-\ell}\nabla^+_{-1,-\ell}\varphi_{-1,-\ell}(g) = 0, \quad P^{0,-\ell+1}\nabla^+_{-1,-\ell}\varphi_{-1,-\ell}(g) = 0,$$
$$P^{-1,-\ell+2}\nabla^+_{-1,-\ell}\varphi_{-1,-\ell}(g) = 0, \quad P^{-3,-\ell}\nabla^-_{-1,-\ell}\varphi_{-1,-\ell}(g) = 0.$$

By Proposition 3.3, these are given explicitly for the restriction $\varphi_{-1,-\ell}|_{A_0}$ on A_0; $\varphi_{-1,-\ell}(a) = \sum_{j=0}^{\ell-1} b_j(a)v_j^{-1,-\ell}$,

for $0 \leq j \leq \ell + 1$,

(5.1a)
$$j(j-1)\Big(\mathcal{L}_1^+ + j - 2 - \ell - 2(\ell + 1 - j)\frac{h_2a_2^2}{D}\Big)b_{j-2}(a) - 2j(\ell + 1 - j)S_1 \cdot b_{j-1}(a)$$
$$+ (\ell - j)(\ell + 1 - j)\Big(\mathcal{L}_2^+ - j - 1 + 2j\frac{h_1a_1^2}{D}\Big)b_j(a) = 0,$$

for $0 \leq j \leq \ell - 1$,

(5.1b)
$$j\Big(\mathcal{L}_1^+ + j - 1 - \ell - (\ell - 1 - 2j)\frac{h_2a_2^2}{D}\Big)b_{j-1}(a) - (\ell - 1 - 2j)S_1 \cdot b_j(a)$$
$$- (\ell - 1 - j)\Big(\mathcal{L}_2^+ - j - 2 - (\ell - 1 - 2j)\frac{h_1a_1^2}{D}\Big)b_{j+1}(a) = 0,$$

for $0 \leq j \leq \ell - 3$,

(5.1c)
$$\Big(\mathcal{L}_1^+ + j - \ell + 2(j+1)\frac{h_2a_2^2}{D}\Big)b_j(a) + 2S_1 \cdot b_{j+1}(a)$$
$$+ \Big(\mathcal{L}_2^+ - j - 3 + \ell - 2(\ell - 2 - j)\frac{h_1a_1^2}{D}\Big)b_{j+2}(a) = 0,$$

for $0 \le j \le \ell - 3$,

(5.1d)
$$\Big(\mathcal{L}_2^- + j + 1 - 2(j+1)\frac{h_1 a_1^2}{D}\Big) b_j(a) - 2S_1 \cdot b_{j+1}(a) + \Big(\mathcal{L}_1^- - j - 2 + \ell + 2(\ell - 2 - j)\frac{h_2 a_2^2}{D}\Big) b_{j+2}(a) = 0.$$

Theorem 5.2 *Consider the set of equations (5.1a), (5.1b), (5.1c), and (5.1d). It has non-trivial one dimensional solutions if and only if*

$$\chi(c)^2 h_1 h_2 = -(\ell - 1)^2$$

is satisfied for the linear form χ on $\mathfrak{m}_1(\psi)$. If this condition is satisfied the solution is given by

$$b_j(a) = c_j \times \big(\sqrt{|h_1|}a_1\big)^{\ell-j}\big(\sqrt{|h_2|}a_2\big)^{j+1} e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)},$$

for $0 \le j \le \ell - 1$. Here c_j are the constants explicitly given by $c_j = \big(\sqrt{-1}\big)^j$ for $0 \le j \le \ell - 1$, if $\chi(c)\sqrt{h_1 h_2} = \sqrt{-1}(\ell - 1)$, or $= \big(\sqrt{-1}\big)^{\ell-1-j}$, if $\chi(c)\sqrt{h_1 h_2} = -\sqrt{-1}(\ell - 1)$. It decays rapidly as $a_1, a_2 \to +\infty$ if and only if $h_1 > 0$ and $h_2 > 0$.

Proof We reduce the difference-differential equations in 5.1. The equations (5.1a) and (5.1b) are equivalent to

for $0 \le j \le \ell$,

(5.2a)
$$j(j-1)\frac{h_2 a_2^2}{D} b_{j-2}(a) + jS_1 \cdot b_{j-1}(a) - (\ell - j)\Big(\mathcal{L}_2^+ - 1 + j\frac{h_2 a_2^2}{D}\Big) b_j(a) = 0,$$

for $1 \le j \le \ell + 1$,

(5.2b)
$$(j-1)\Big(\mathcal{L}_1^+ - 1 - (\ell + 1 - j)\frac{h_1 a_1^2}{D}\Big) b_{j-2}(a) - (\ell + 1 - j)S_1 \cdot b_{j-1}(a) + (\ell - j)(\ell + 1 - j)\frac{h_1 a_1^2}{D} b_j(a) = 0.$$

Also the equations (5.1b) and (5.1c) are equivalent to

for $0 \le j \le \ell - 2$,

(5.3a)
$$j\frac{h_2 a_2^2}{D} b_{j-1}(a) + S_1 \cdot b_j(a) + \Big(\mathcal{L}_2^+ - \ell - 1 - (\ell - 1 - j)\frac{h_2 a_2^2}{D}\Big) b_{j+1}(a) = 0,$$

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for $1 \leq j \leq \ell - 1$,

$$\text{(5.3b)} \quad \Big(\mathcal{L}_1^+ - \ell - 1 + j\frac{h_1 a_1^2}{D}\Big) b_{j-1}(a) + S_1 \cdot b_j(a) - (\ell - 1 - j)\frac{h_1 a_1^2}{D} b_{j+1}(a) = 0.$$

Considering the equations (5.1c) and (5.1d), we get

for $1 \leq j \leq \ell - 2$,

$$\text{(5.4)} \qquad (\mathcal{L}_1^+ + \mathcal{L}_2^- - \ell - 1)b_{j-1}(a) + (\mathcal{L}_1^- + \mathcal{L}_2^+ - \ell - 1)b_{j+1}(a) = 0.$$

Then the following sets of difference-differential equations are equivalent to each other:

$$\begin{Bmatrix} (5.1\text{a}), & (5.1\text{b}), \\ (5.1\text{c}), & (5.1\text{d}) \end{Bmatrix} \approx \begin{Bmatrix} (5.2\text{a}), & (5.2\text{b}), & (5.3\text{a}), \\ (5.3\text{b}), & (5.4) & \end{Bmatrix}.$$

Paring the equations in (5.2a) and (5.3a), or (5.2b) and (5.3b), we get

for $0 \leq j \leq \ell - 1$,

$$\text{(5.5)} \qquad (\mathcal{L}_2^+ - j - 1)b_j(a) = 0, \quad (\mathcal{L}_1^+ - \ell + j)b_j(a) = 0.$$

So, if we put $b_j(a) = \big(\sqrt{|h_1|}a_1\big)^{\ell - j}\big(\sqrt{|h_2|}a_2\big)^{j+1} e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)} c_j(a)$, $0 \leq j \leq \ell - 1$, then (5.5) implies that all $c_j(a)$ are constants. Write the constants $c_j = c_j(a)$, $0 \leq j \leq \ell - 1$. Then (5.2a), (5.2b), (5.3a), (5.3b), and (5.4) give a set of linear relations among c_j, $0 \leq j \leq \ell - 1$, which has a non-zero solution, if and only if $\chi(c)^2 h_1 h_2 = -(\ell - 1)^2$.

5.3

Let us consider the (N_2, ψ)-Whittaker embedding of $A_{\mathfrak{q}_6}(\lambda)$. By the K-type decomposition of the module and Proposition 3.5 we obtain the following set of equations for the function $\varphi_{-1,-\ell}(g)$ defined in 3.6 with respect to the minimal K-type:

for $0 \leq j \leq \ell + 1$,

(5.6a)

$$\begin{aligned} &j(j-1)(\partial_1 + \ell - j)b_{j-2}(a) + 2j(\ell + 1 - j)S_2 \cdot b_{j-1}(a) \\ &\qquad + (\ell - j)(\ell + 1 - j)\big(\partial_2 - 2\sqrt{-1}\chi(c)a_2^2 - j - 1\big) b_j(a) = 0, \end{aligned}$$

for $0 \leq j \leq \ell - 1$,

$$\text{(5.6b)} \quad \begin{aligned} &j(\partial_1 - j - 2)b_{j-1}(a) + (\ell - 1 - 2j)S_2 \cdot b_j(a) \\ &\qquad - (\ell - 1 - j)\big(\partial_2 - 2\sqrt{-1}\chi(c)a_2^2 - j - 2\big) b_{j+1}(a) = 0, \end{aligned}$$

for $0 \leq j \leq \ell - 3$,

$$\text{(5.6c)} \quad \begin{aligned} &(\partial_1 - \ell - j - 2)b_j(a) - 2S_2 \cdot b_{j+1}(a) \\ &\qquad + \big(\partial_2 - 2\sqrt{-1}\chi(c)a_2^2 - j - 3\big) b_{j+2}(a) = 0, \end{aligned}$$

for $0 \le j \le \ell - 3$,

(5.6d)
$$\left(\partial_2 + 2\sqrt{-1}\chi(c)a_2^2 + j + 1\right) b_j(a) + 2S_2 \cdot b_{j+1}(a) + (\partial_1 - \ell + j + 2)b_{j+2}(a) = 0.$$

Theorem 5.4 *The above set of difference-differential equations has only the trivial solution: $b_j(a) = 0$ for $0 \le j \le \ell - 1$. Hence $A_{\mathfrak{q}_6}(\lambda)$ has no non-trivial (N_2, ψ)-Whittaker embedding for a non-degenerate ($h_1 \neq 0$) unitary character ψ of N_2.*

Proof The equations (5.6a) and (5.6b) are equivalent to

for $0 \le j \le \ell$,

(5.7a)
$$j(j-1)b_{j-2}(a) + jS_2 b_{j-1}(a) + (\ell - j)\left(\partial_2 - 2\sqrt{-1}\chi(c)a_2^2 - j - 1\right) b_j(a) = 0,$$

for $1 \le j \le \ell + 1$,

(5.7b)
$$(j-1)(\partial_1 - 1)b_{j-2}(a) + (\ell + 1 - j)S_2 b_{j-1}(a) = 0.$$

Also (5.6b) and (5.6c) yield

for $0 \le j \le \ell - 2$,

(5.8a)
$$jb_{j-1}(a) + S_2 b_j(a) - \left(\partial_2 - 2\sqrt{-1}\chi(c)a_2^2 - j - 2\right) b_{j+1}(a) = 0,$$

for $1 \le j \le \ell - 1$,

(5.8b)
$$(\partial_1 - \ell - 1)b_{j-1}(a) - S_2 b_j(a) = 0.$$

By pairing the equations (5.7a) and (5.8a), or the ones (5.7b) and (5.8b), we obtain the following equations: for $0 \le j \le \ell - 1$,

$$\left(\partial_2 - 2\sqrt{-1}\chi(c)a_2^2 - j - 1\right) b_j(a) = 0 \quad \text{and} \quad (\partial_1 - \ell + j)b_j(a) = 0.$$

If we put $b_j(a) = a_1^{\ell-j} a_2^{j+1} e^{\sqrt{-1}\chi(c)a_2^2} c_j(a)$, $0 \le j \le \ell - 1$, then these equations tell us that all $c_j(a) = c_j$ are constants. To determine these constants we use again (5.6a), (5.6b), (5.6c), and (5.6d). Since $h_1 \neq 0$, we get only the trivial solution: $c_j = 0$ for $0 \le j \le \ell - 1$.

6 Observation on Some Relations With Nilpotent Orbits

6.1

Let us consider a derived functor module. We will observe that the existence of a non-zero (N, ψ)-Whittaker embedding of it could be related to a nilpotent G-orbit in $\mathfrak{g}_{\mathbb{R}}^*$ which is attached to this module.

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Let G be a real semisimple Lie group and $\mathfrak{g}_{\mathbb{R}}$ its Lie algebra. Matumoto [Ma] studied the $C^{-\infty}$-Whittaker vectors of a Harish-Chandra G-module $(\pi, \mathcal{H}_\pi)$ for a non-degenerate unitary character ψ of a maximal nilpotent Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_{\mathbb{R}}$. He obtained a necessary and sufficient condition for these vectors not to vanish: such a non-zero $C^{-\infty}$-Whittaker vector of π exists if and only if the asymptotic support $\mathrm{Asym}(\pi)$ of π [B-V] contains the principal real nilpotent G-orbit in $\mathfrak{g}_{\mathbb{R}}^*$ to which ψ belongs; in particular, π should be (quasi) large. More precise statements can be found in [Ma] Theorems 5.5.1 and 5.5.2. For unitary highest weight modules we can refer [Y2] for a related subject. Kawanaka [K] conjectured some relation between Whittaker vectors and the wave front set for a smooth representation of the group of rational points of a reductive group over a finite or a p-adic field. Works of Rodier, originally, Howe, and Mœglin-Waldspurger are also concerned with this philosophy. It might be partially stated as a conjecture in our case:

Conjecture 6.2 *Let π be an irreducible admissible Hilbert space representation of a real semisimple Lie group G, and $\mathrm{Asym}(\pi)$ be the asymptotic support of π. Consider the (N, ψ)-Whittaker embedding $\mathrm{Wh}_{\chi\boxtimes\psi}(\pi)$ of π, where ψ is a unitary character of a unipotent subgroup N and χ a unitary representation of $M(\psi)$. Moreover we consider a subspace $\mathrm{Wh}^{\mathrm{mod}}_{\chi\boxtimes\psi}(\pi)$ of functionals in $\mathrm{Wh}_{\chi\boxtimes\psi}(\pi)$ which have images consisting of analytic and at most moderate growth functions on G. Then it might hold that*

(i) $\mathrm{Wh}^{\mathrm{mod}}_{\chi\boxtimes\psi}(\pi) \neq \{0\}$ *for some χ if and only if the real nilpotent coadjoint G-orbit $\mathcal{O}^{\mathbb{R}}_{\psi}$ of ψ in $\sqrt{-1}\mathfrak{g}_{\mathbb{R}}^*$ is contained in* $\mathrm{Asym}(\pi)$. *Here we use the maps $\mathfrak{n}_{\mathbb{R}}^* \simeq \mathfrak{n}_{\mathbb{R}}^{\mathrm{opp}} \hookrightarrow \mathfrak{g}_{\mathbb{R}} \simeq \mathfrak{g}_{\mathbb{R}}^*$ to obtain the orbit $\mathcal{O}^{\mathbb{R}}_{\psi}$, where $\mathfrak{n}_{\mathbb{R}}^{\mathrm{opp}}$ is the nilpotent subalgebra in the opposite position to $\mathfrak{n}_{\mathbb{R}}$.*

(ii) *If $\dim_{\mathbb{C}} \mathrm{Ass}(\pi) \leq \dim_{\mathbb{C}} \mathfrak{n}$ and the condition in (i) is satisfied, then there are only finitely many irreducible χ up to equivalence such that $\mathrm{Wh}^{\mathrm{mod}}_{\chi\boxtimes\psi}(\pi) \neq \{0\}$. And for such a χ, the dimension of $\mathrm{Wh}^{\mathrm{mod}}_{\chi\boxtimes\psi}(\pi)$ is finite.*

We show that the results in Sections 4 and 5 are consistent with the conjecture and Matumoto's results. We use the notations in Section 2, 3. By a result of Schmid and Vilonen [S-V], given an irreducible admissible representation π of G, we can compute its asymptotic support $\mathrm{Asym}(\pi)$ [B-V] from the associated variety $\mathrm{Ass}(\pi)$ of π by Kostant-Sekiguchi correspondence. Then we can obtain $\mathrm{Asym}\big(A_{\mathfrak{q}_7}(\lambda)\big)$ as the union of the real nilpotent coadjoint G-orbits in $\sqrt{-1}\mathfrak{g}_{\mathbb{R}}^*$ which corresponds to $\mathrm{Ass}\big(A_{\mathfrak{q}_7}(\lambda)\big) = \mathrm{Cl}\big(\mathcal{O}_{\tiny\begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline\end{array}}\big)$ by the correspondence. Let us put $\Psi_h := \begin{pmatrix} 0_2 & 0_2 \\ \sqrt{-1}h & 0_2 \end{pmatrix} \in \sqrt{-1}\mathfrak{g}_{\mathbb{R}}$ with $h = {}^t h \in M_2(\mathbb{R})$. It determines a linear form $\psi_h \in \sqrt{-1}\mathfrak{g}_{\mathbb{R}}^*$ by $\psi_h(X) = \mathrm{tr}(\Psi_h X)$, $X \in \mathfrak{g}_{\mathbb{R}}$. We can check that the real nilpotent G-orbit $\mathcal{O}^{\mathbb{R}}_{\psi_h}$ of ψ_h coincides with the image of the orbit $\mathcal{O}_{\tiny\begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline\end{array}}$ by the correspondence if and only if $\det(h) < 0$. Therefore it occurs in $\mathrm{Asym}\big(A_{\mathfrak{q}_7}(\lambda)\big)$ if and only if ψ_h defines an "indefinite" unitary character of the unipotent radical N_1 of the Siegel parabolic subgroup. On the other hand, ψ_h with $\det(h) > 0$, which defines a "definite" unitary characters of N_1, belongs to another real nilpotent orbit; if h is a positive, or negative, definite, then $\mathcal{O}^{\mathbb{R}}_{\psi_h}$ is the real orbit which corresponds to the $K_{\mathbb{C}}$-orbit $\mathcal{O}_{\tiny\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline\end{array}}$, or $\mathcal{O}_{\tiny\begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline\end{array}}$, respectively. Both of these definite real orbits are not contained in $\operatorname{Asym}\big(A_{\mathfrak{q}_7}(\lambda)\big)$. On the other hand, the orbit $\mathcal{O}^{\mathbb{R}}_{\psi_h}$ for a positive definite h meets with $\operatorname{Asym}\big(A_{\mathfrak{q}_6}(\lambda)\big)$, because $\operatorname{Ass}\big(A_{\mathfrak{q}_6}(\lambda)\big) = \operatorname{Cl}\Big(\mathcal{O}_{\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline\end{array}}\Big)$. For a linear form ψ given in 3.4, we can check that the orbit $\mathcal{O}^{\mathbb{R}}_{\psi}$ agrees with the image of the $K_{\mathbb{C}}$-orbit $\mathcal{O}_{\begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline\end{array}}$ by the correspondence.

Putting together our observations on the nilpotent orbits and the statements in Theorem 4.6, 4.10, 5.2, and 5.4, we can state the following:

Proposition 6.3 *Fix a non-degenerate unitary character of N_0, N_1, or N_2. Assume that $\pi = A_{\mathfrak{q}_6}(\lambda)$ or $A_{\mathfrak{q}_7}(\lambda)$ with good and integral λ has a non-trivial realization into C^∞-$\operatorname{Ind}^G_{M_i(\psi)\ltimes N_i}(\chi\cdot\psi)$ for some character χ of $M_i(\psi)$, $i \in \{0,1,2\}$ ($M_0(\psi)$: trivial), and also that its image consists of holomorphic and moderate growth functions on G. Then the real nilpotent coadjoint G-orbit $\mathcal{O}^{\mathbb{R}}_{\psi}$ of ψ should be contained in $\operatorname{Asym}(\pi)$.*

Proof The statement for the (N_0,ψ)-Whittaker embedding of π is a well known theorem of Kostant [Ko] and Vogan [V2]. It says each of $A_{\mathfrak{q}_6}(\lambda)$ and $A_{\mathfrak{q}_7}(\lambda)$ has no (N_0,ψ)-Whittaker embedding. But we know, in this case, $\mathcal{O}^{\mathbb{R}}_{\psi}$ is of maximal dimension, which is not contained in $\operatorname{Asym}(\pi)$.

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