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OPTIMALITY CONDITIONS FOR SYSTEMS WITH INSUFFICIENT DATA

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In this paper we use the Dubovitski–Milyutin formalism to establish necessary and sufficient conditions for optimality in a nonlinear, distributed parameter control system, with convex cost criterion and initial condition not given *a priori* (that is it is not a known function but instead it belongs to a specified set). Our result extends a recent theorem of Lions. Finally a concrete example is worked out in detail.

1. INTRODUCTION

In this paper, we consider a nonlinear distributed control system, with time varying control constraints and an initial condition which is not determined by an *a priori* given function, but instead is assumed to belong to a certain specified set (Lions [5] calls them "systems with insufficient data"). The cost criterion is a general convex integral functional.

Using the Dubovitski–Milyutin formalism, we are able to obtain a necessary and sufficient condition for the existence of an optimal solution. A very comprehensive presentation of the Dubovitski–Milyutin theory can be found in the monograph of Girsanov [3]. Our result extends Theorem 2.1 of Lions [5], since we allow for nonlinear dynamics and a nonquadratic cost criterion.

2. PRELIMINARIES

The mathematical setting is the following. Let $T = [0, b] \subseteq \mathbf{R}_+$ (a bounded time interval) and H be a separable Hilbert space. Also let $X \subseteq H$ be a subspace of H carrying the structure of a separable reflexive Banach space, which imbeds continuously and densely into H. Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings being continuous and dense. Such a triple (X, H, X^*) of spaces is sometimes called a "Gelfand triple" or "spaces in normal position". By $\|\cdot\|$ (respectively $|\cdot|$, $\|\cdot\|_*$) we will denote the norm of X (respectively of H, X^*). Also by $(\cdot, \cdot)$ we will denote the inner product in H, and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X, X^*).

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The two are compatible in the sense that if $x \in X \subseteq H$ and $h \in H \subseteq X^*$, we have $(x, h) = \langle x, h\rangle$. Also let Y be another separable Banach space modelling the control space. By $P_{fc}(Y)$ we will denote the nonempty, closed, convex subsets of Y.

The optimal control problem under consideration is the following:

$$(*) \qquad \left\{\begin{array}{c} \text{Minimise } J(x,u) = \int_0^b L(t, x(t), u(t))dt \\ \text{such that } \dot{x}(t) + A(t, x(t)) = B(t, x(t))u(t) \text{ a.e.} \\ x(0) \in C,\ u(t) \in U(t) \text{ a.e.} \end{array}\right\}$$

We will need the following hypotheses concerning the data of (*).

$\underline{H(A)}$: $A: T \times X \to X^*$ is an operator such that

(1) $t \to A(t, x)$ is measurable,

(2) $x \to A(t, x)$ is continuously Fréchet differentiable and strongly monotone uniformly in $t \in T$,

(3) $\|A(t, x)\|_* \leqslant a(t) + b\|x\|$ a.e. with $a(\cdot) \in L^2_+$, $b > 0$,

(4) $\langle A(t, x), x\rangle \geqslant c\|x\|^2$, $c > 0$.

$\underline{H(B)}$: $B: T \times H \to \mathcal{L}(Y, X^*)$ is an operator such that

(1) $t \to B(t, x)u$ is measurable for all $(x, u) \in H \times Y$,

(2) $x \to B(t, x)$ is continuous,

(3) $x \to B(t, x)u$ is continuously Fréchet differentiable,

(4) $\|B(t, x)u\|_* \leqslant \beta_1(t) + \beta_2(t)|x| + \beta_3(t)\|u\|$ a.e. with $\beta_1(\cdot)$, $\beta_2(\cdot) \in L^2_+$, $\beta_3(\cdot) \in L^\infty_+$.

$\underline{H(L)}$: $L: T \times H \times Y \to \mathbb{R}$ is an integrand such that

(1) $t \to L(t, x, u)$ is measurable,

(2) $(x, u) \to L(t, x, u)$ is convex and continuously Gateaux differentiable,

(3) for every $(x, u) \in L^\infty(H) \times L^2(Y)$, $J(x, u)$ is finite.

$\underline{H(U)}$: $U: T \to P_{fc}(Y)$ is a multifunction such that

(1) $GrU = \{(t, u) \in T \times Y : u \in U(t)\} \in B(T) \times B(Y)$ (where $B(T)$ is the Borel σ-field of T and $B(Y)$ the Borel σ-field of Y),

(2) $t \to |U(t)| = \sup\{\|u\| : u \in U(t)\}$ belongs to L^2_+, and if $S^2_U = \{u(\cdot) \in L^2(Y): u(t) \in U(t) \text{ a.e.}\}$, then $\operatorname{int} S^2_U \neq \emptyset$.

$\underline{H(C)}$: $C \subseteq H$ is a closed, convex set with a nonempty interior.

Following Lions [4], we define $W(T) = \{x(\cdot) \in L^2(X): \dot{x} \in L^2(X^*)\}$. This is a Banach space with norm $\|x\|_{W(T)} = [\int_0^b \|x(t)\|^2 dt + \int_0^b \|\dot{x}(t)\|_*^2 dt]^{1/2}$. It is well known that $W(T) \hookrightarrow C(T, H)$, that is, the elements of $W(T)$ are continuous maps with values in H, possibly after changing each function on a set of measure zero.

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Since our necessary and sufficient conditions, will involve the adjoint state, we need the following existence result.

Proposition 2.1. *If hypotheses $H(A)$, $H(B)$, $H(L)$ hold, $B_x(t, x(t), u(t))\,|_X\,(\cdot)$ is dissipative, and $t \to L_x(t, x(t), u(t))$ belongs to $L^2(H)$, then there exists $p(\cdot) \in W(T)$ such that*

$$-\dot{p}(t) + A_x^*(t, x(t))p(t) = B_x^*(t, x(t), u(t))p(t) - L_x(t, x(t), u(t)) \text{ a.e.}$$
$$p(b) \in H$$

Proof: From the strong monotonicity of $A(t, \cdot)$, uniformly in $t \in T$, we have:

$$\langle A(t, x') - A(t, x(t)), x' - x(t)\rangle \geqslant \theta \left\|x' - x(t)\right\|^2 \theta > 0$$

and so

$$\langle A_x(t, x(t))(x' - x(t)) + o(\|x' - x(t)\|), x' - x(t)\rangle \geqslant \theta \left\|x' - x(t)\right\|^2.$$

Putting $x' - x(t) = \varepsilon p$, we see that

$$\langle A_x(t, x(t))\varepsilon p + o(\varepsilon \|p\|), \varepsilon p\rangle \geqslant \theta\varepsilon^2 \|p\|^2.$$

Divide by ε^2 and let $\varepsilon \to 0^+$. We get:

$$\langle A_x(t, x(t))p, p\rangle = \langle A_x^*(t, x(t))p, p\rangle \geqslant b'' \|p\|^2.$$

Also, by hypothesis, $\langle -B_x^*(t, x(t), u(t))p, p\rangle \geqslant 0$. Since $t \to L_x(t, x(t), u(t))$ belongs in $L^2(H)$, we can invoke Theorem 4.2, p.167 of Barbu [1], and get that indeed there exists $p(\cdot) \in W(T)$ solving our problem. $\square$

3. Necessary and sufficient conditions

The next result gives us necessary and sufficient conditions for a triple $(x_0, x, u) \in H \times W(T) \times L^2(Y)$ to be a solution of (*).

Theorem 3.1. *If hypotheses $H(A)$, $H(B)$, $H(L)$, $H(U)$, $H(C)$ hold, for the pair $(x, u) \in W(T) \times L^2(Y)$ we have*

(1) $\|A_x(t, x(t))\|_{\mathcal{L}(X, X^*)} \leqslant \eta_1$,
(2) $\|B_x(t, x(t), u(t))\|_{\mathcal{L}(H, X^*)} \leqslant \eta_2$, $B_x(t, x(t))$,
(3) $(u(t))\,|_X\,(\cdot)$ *is dissipative,*

then the triple $(x(0) = x_0, x, u) \in H \times W(T) \times L^2(Y)$ is a solution of (*) if and only if

(a) $$\left\{\begin{array}{l} \dot{x}(t) + A(t, x(t)) = B(t, x(t))u(t) \text{ a.e.} \\ x(0) = x_0 \in C,\ u(t) \in U(t) \end{array}\right\},$$

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(b) *there exists* $p(\cdot) \in W(T)$ *satisfying the "adjoint equation"*

$$\left\{\begin{array}{l} -\dot{p}(t) + A_x^*(t, x(t))p(t) = B_x^*(t, x(t), u(t))p(t) - L_x(t, x(t), u(t)) \text{ a.e.} \\ p(b) = 0 \end{array}\right\}$$

(c) *and the following "minimum principles" hold*

(3) $\quad (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v - u(t))_{Y, Y^*} \leqslant 0$ *for all* $v \in U(t)$ a.e.

and $\quad (-p(0), c - x_0) \geqslant 0$ *for all* $c \in C$.

PROOF: As we already mentioned in the introduction, our approach is based on the Dubovitski–Milyutin formalism. So we need to analyse the cost criterion, the equality constraint (that is the evolution equation) and the initial data–control constraints (regarded here as an inequality constraint, by determining the cone of directions of decrease, the tangent cone, and the cone of feasible directions, respectively.

We will start with cost criterion $J(\cdot, \cdot)$. Recalling that $J(\cdot, \cdot)$ is convex and using the monotone convergence theorem we get that

$$\nabla J(x, u)(h, v) = \int_0^b \nabla L(t, x(t), u(t))(h(t), v(t))dt.$$

But since hy hypothesis $H(L)(2)$, $L(t, \cdot, \cdot)$ is continuously Gateaux differentiable, from the total differential rule we have:

$$\nabla L(t, x(t), u(t))(h(t), v(t)) = L'_x(t, x(t), u(t))h(t) + L'_u(t, x(t), u(t))v(t).$$

Invoking Theorem 7.4 of Girsanov [**3**], we get that the cone of directions of decrease of the cost criterion $J(\cdot, \cdot)$ at (x, u) is given by

$$K_d\{(h, v) \in W(T) \times L^2(Y) \colon J'(x, u)(h, v) < 0\}.$$

Assume $K_d \neq \emptyset$. Then we have:

$$K_d^* = \{-\lambda J(x, u) \colon \lambda \in \mathrm{R}_+\}.$$

Now we pass to the analysis of the equality constraint. This is determined by the dynamical equation of the system. Consider the map $P \colon H \times W(T) \times L^2(Y) \to L^2(X^*) \times H$ defined by

$$P(x'_0, x', u') = (\dot{x}'(t) + A(t, x'(t)) - B(t, x'(t))u'(t), x'(0) - x'_0).$$

Observe that because of our hypotheses both $\hat{A}\colon W(T) \to X^*$ defined by $\left(\hat{A}x'\right)(t) = A(t, x'(t))$ and $\hat{B}\colon W(T) \times L^2(Y) \to X^*$ defined by $\hat{B}(x', u')(t) = B(t, x'(t))u'(t)$ are continuously Fréchet differentiable at (x_0, x, u). So $P(\cdot, \cdot, \cdot)$ is continuously Fréchet differentiable at (x_0, x, u) and furthermore

$$\begin{aligned} P'(x_0, x, u)(h_0, h, v)(t) = \Big(&\dot{h}(t) + A_x(t, x(t))h(t) \\ &- B_x(t, x(t), u(t))h(t) - B(t, x(t))v(t),\ h(0) - h_0\Big). \end{aligned}$$

We will show that $P'(x_0, x, u)$ is surjective. Let $(g, v, h_0, h_1) \in L^2(X^*) \times L^2(Y) \times H \times H$ be given and consider the following Cauchy problem.

$$\left\{\begin{array}{l} \dot{h}(t) + A_x(t, x(t))h(t) = B_x(t, x(t), u(t))h(t) + B(t, x(t))v(t) + g(t) \text{ a.e.} \\ h(0) = h_0 + h_1 \end{array}\right\}.$$

As in the proof of Proposition 2.1, we can check that all the hypotheses of Theorem 4.2 of Barbu [**1**] are satisfied. Hence the above Cauchy problem has a solution $h(\cdot) \in W(T)$. So for any $(g, h_1) \in L^2(X^*) \times H$, we can find $(h_0, h, v) \in H \times H \times L^2(Y)$ such that $P'(x_0, x, u)(h_0, h, v) = (g, h_1)$ that is $P(x_0, x, u)$ is surjective. Hence we can apply Lyusternik's theorem (see Girsanov [**3**, Theorem 9.1]) and deduce that if

$$Q_1 = \{(x_0', x', u') \in H \times W(T) \times L^2(Y)\colon P(x_0', x', u') = 0\} \text{ (equality constraint set)},$$

then the tangent space to Q_1 at (x_0, x, u) is given by

$$\begin{aligned} T(Q_1) &= \{(h_0, h, v) \in H \times L^2(Y)\colon P'(x_0, x, u)(h_0, h, v) = 0\} \\ &= \ker P'(x_0, x, u) \end{aligned}$$

and so

$$(T(Q_1))^* = \{x^* \in H \times W(T)^* \times W(T) \times L^2(Y^*)\colon (\forall (h_0, h, v) \in T(Q_1))w^*(h_0, h, v) = 0\}$$

Finally we will analyse the initial data–control constraints. Set

$$Q_2 = C \times S_U^2 \subseteq H \times L^2(Y).$$

By hypothesis, int $C \neq \emptyset$ and int $S_U^2 \neq \emptyset$, and $C \times S_U^2$ is convex. So Theorem 10.5 of Girsanov [**3**], tells us that the dual to the cone of feasible directions of Q_2 at (x_0, u) is given by

$$K(Q_2)_f^* = \left(C \times S_U^2\right)^* = C^* \times S_U^{2*}.$$

Hence $(c^*, u^*) \in K(Q_2)_f^*$ if and only if c^* supports C at x_0 and u^* support S_U^2 at u.

Now that we have in our disposal all the appropriate cones, we can apply the Dubovitski–Milyutin theorem [**2**] (see also Girsanov [**3**, Theorem 6.1]) and get $y^* \in K_d^*$, $w^* \in T(Q_1)^*$ and $(c^*, u^*) \in K(Q_2)_f^*$, not all simultaneously zero such that

$$(0, y^*) + w^* + (c^*, 0, u^*) = 0.$$

Hence $y^*(h, v) + w^*(h_0, h, v) + (c^*, h_0) + u^*(v) = 0$ for all $(h_0, h, v) \in H \times W(T) \times L^2(Y)$.

Recall from the analysis of the equality constraint that if $(h_0, h, v) \in T(Q_1)$ (that is if $P'(x_0, x, u)(h_0, h, v) = 0$), then $w^*(h_0, h, v) = 0$. This means then that if for any $(h_0, v) \in H \times L^2(Y)$, we choose $h \in W(T)$, so that (h_0, h, v) solves the Cauchy problem

$$\left\{\begin{array}{l} \dot{h}(t) + A_x(t, x(t))h(t) = B_x(t, x, (t), u(t))h(t) + B(t, x(t))v(t) \text{ a.e.} \\ h(0) = h_0 \end{array}\right\}.$$

(we already saw that such an $h \in W(T)$ always exists), then $w^*(h_0, h, v) = 0$ and in this case the Euler–Lagrange equation becomes

$$h^*(h, v) + (c^*, h_0) + u^*(v) = 0$$

and so

$$-\lambda J'(x, u)(h, v) + (c^*, h_0) + u^*(v) = 0.$$

Since $(h_0, v) \in H \times L^2(Y)$ is arbitrary, if $\lambda = 0$, then $c^* = 0$, $u^* = 0$ and so $w^* = 0$, a contradiction to the Dubovitski–Milyutin theorem. So $\lambda > 0$ and, without any loss of generality, we can take $\lambda = 1$.

Consider the following adjoint Cauchy problem.

$$\left\{\begin{array}{l} -\dot{p}(t) + A_x^*(t, x(t))p(t) = B_x^*(t, x(t), u(t))p(t) - L_x(t, x(t), u(t)) \text{ a.e.} \\ p(b) = 0 \end{array}\right\}$$

From Proposition 2.1 we know that the Cauchy problem above has a solution $p(\cdot) \in W(T)$. Using this adjoint state $p(\cdot)$, we get:

$$\begin{aligned} &\int_0^b (L_x(t, x(t), u(t)), h(t))dt \\ &\quad = \int_0^b \langle \dot{p}(t) - A_x^*(t, x(t))p(t) + B_x^*(t, x(t))p(t), h(t)\rangle dt \\ &\quad = \int_0^b \langle \dot{p}(t), h(t)\rangle dt - \int_0^b \langle A_x^*(t, x(t))p(t), h(t)\rangle dt \\ &\qquad + \int_0^b \langle B_x^*(t, x(t))p(t), h(t)\rangle dt. \end{aligned}$$

From Lemma 5.5.1 of Tanabe [6], we know that:

$$\int_0^b \langle \dot{p}(t),\, h(t)\rangle dt = (p(t),\, h(t))\,|_0^b - \int_0^b \langle p(t),\, \dot{h}(t)\rangle dt$$
$$= -(p(0),\, h_0) - \int_0^b \langle p(t),\, \dot{h}(t)\rangle dt.$$

Also we have

$$\int_0^b \langle A_x^*(t, x(t))p(t),\, h(t)\rangle dt = \int_0^b \langle p(t),\, A_x(t,\, x(t))h(t)\rangle dt$$

and

$$\int_0^b \langle B_x^*(t, x(t),\, u(t))p(t),\, h(t)\rangle dt = \int_0^b \langle p(t),\, B_x(t, x(t),\, u(t))h(t)\rangle dt.$$

Using these facts, we get:

$$\int_0^b (L_x(t, x(t),\, u(t)),\, h(t))dt$$
$$= \int_0^b \langle p(t),\, -\dot{h}(t) - A_x(t, x(t))h(t) + B_x(t, x(t),\, u(t))h(t)\rangle dt - (p(0),\, h_0).$$

Recalling the choice of $h(\cdot) \in W(T)$, we get:

$$\int_0^b (L_x(t, x(t),\, u(t)), h(t))dt = \int_0^b \langle p(t),\, -B(t, x(t))v(t)\rangle dt - (p(0),\, h_0).$$

Substitute this back into the Euler–Lagrange equation, to get:

$$u^*(v) + (c^*,\, h_0) = \int_0^b \langle p(t),\, -B(t, x(t))v(t)\rangle dt$$
$$+ \int_0^b (L_u(t, x(t),\, u(t)),\, v(t))_{Y,Y^*} dt - (p(0),\, h_0)$$

for every $v \in L^2(Y)$ and every $h_0 \in H$. Hence clearly

$$u^*(v) = \int_0^b (L_u(t, x(t),\, u(t)) - B^*(t, x(t))p(t),\, v(t))_{Y,Y^*} dt$$

and

$$c^*(h_0) = -(p(0),\, h_0).$$

Recall that u^* supports S_U^2 at u and c^* supports C at x_0. So we have:

$$\int_0^b (L_u(t, x(t),\, u(t)) - B^*(t, x(t))p(t),\, v(t) - u(t))dt \geqslant 0 \text{ for all } v \in S_U^2$$

and

$$(-p(0),\, c - x_0) \geqslant 0 \text{ for all } c \in C.$$

Suppose that for some $E \subseteq T$ with $\lambda(E) > 0$, we have:

$$\inf_{v \in U(t)} (L_u(t, x(t)), u(t)) - B^*(t, x(t))p(t), v - u(t)_{Y,Y^*} < 0, t \in E.$$

Consider the multifunction $V: E \to 2^Y \setminus \{\emptyset\}$, defined by:

$$V(t) = \{v \in U(t): (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v - u(t))^*_{Y,Y} < 0\}$$

From our hypotheses $H(B)$ and $H(L)$, it is easy to see that $(t, v) \to r(t, v) = (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v - u(t))_{Y,Y^*}$, is measurable in t, continuous in v, hence jointly measurable. Thus

$$GrV = \{(t, v) \in E \times Y: r(t, v) < 0\} \cap GrU \in B(E) \times B(Y).$$

Apply Aumann's selection theorem (see Wagner [7]), to get $v_1: E \to Y$ measurable such that $v_1(t) \in V(t)$, $t \in E$. Let $v: T \to Y$ be defined by setting $v(t) = v_1(t)$ for $t \in E$ and $v(t) = u(t)$ for $t \in T \setminus E$. Clearly $v \in S^2_U$ and furthermore

$$\int_0^b (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v(t) - u(t))_{Y,Y^*} dt < 0,$$

a contradiction. So we have:

$$\inf_{v \in U(t)} (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v - u(t)) \geqslant 0 \text{ a.e.}$$

while

$$\inf_{c \in C} (-p(0), c) = (-p(0), x_0).$$

Finally we remove the hypothesis $K_d \neq \emptyset$. If $K_d = \emptyset$, then

$$\int_0^b (L_x(t, x(t), u(t)), h(t))dt + \int (L_u(t, x(t), u(t)), v(t))_{Y,T^*} dt = 0$$

for all $(h, v) \in W(T) \times L^2(Y)$. Let $\lambda = 1$ and working as before we get

$$\int_0^b (L_x(t, x(t), u(t)), h(t))dt = \int_0^b \langle p(t), B(t, x(t))v(t)\rangle dt$$

with $p(\cdot) \in W(T)$ being the adjoint state. So we have:

$$\int_0^b (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v(t))_{Y,Y^*} dt = 0$$

and so the first minimum principle follows.

This completes the necessity part of the proof.

For the sufficiency part, we apply Theorem 15.2 of Girsanov [3]. Note that $J(\cdot, \cdot)$ is a convex function, which is finite everywhere. Also, through a simple application of Fatou's lemma, we can check that $J(\cdot, \cdot)$ is l.s.c.. A convex, l.s.c. function which is finite everywhere is continuous. So $J(\cdot, \cdot)$ is continuous and convex. The Slater type requirement of Theorem 15.2 of Girsanov [3], is automatically satisfied, since by hypothesis int $S^2_U \neq \phi$ and int $C \neq \phi$. Thus an application of Theorem 15.2 of Girsanov [3], gives us the sufficiency part. $\square$

REMARK. If $U(\cdot)$ is not L^2-bounded (that is $t \to |U(t)|$ is not in L^2_+), then the minimum principle has integral form, that is $\int_0^b (L_u(t, x(t)), u(t)) - B^*(t, x(t))p(t), v(t) - u(t))_{Y,Y^*} \geqslant 0$ for all $v \in S^2_U$.

4. AN EXAMPLE

In this section we work out a concrete example of a parabolic distributed parameter control system, to which our result applies.

Let $T = [0, b]$ and let V be a bounded domain in $\mathbb{R}^n$, with a smooth boundary $\partial V = \Gamma$. We consider the following distributed parameter optimal control problem defined on $\Gamma \times V$

$$(**) \qquad \left\{ \begin{array}{l} \text{Minimise } \hat{J}(x, u) = \displaystyle\int_0^b \int_V \hat{L}(t, z, x(t, z), u(t, z)) dz dt \\ \text{such that } \dfrac{\partial x(t, z)}{\partial t} + \hat{A}(t) x(t, z) = u(t, z) \text{ on } (0, b) \times V \\ x(t, z) = 0 \text{ on } T \times \Gamma \\ x(0, \cdot) \in C \subseteq L^2(V), \left(\displaystyle\int_V \|u(t, v)\|^2 \, dv \right)^{1/2} \leqslant r(t) \end{array} \right\}$$

Here $\hat{A}(t)$ is the formal second order elliptic partial differential operator in divergence form, defined by $\hat{A}(t) y = -\sum_{i,j=1}^n \partial/\partial z_i (a_{ij}(t, z)(\partial y(t, z))/(\partial z_j))$. We assume that $a_{ij}(\cdot, \cdot) \in L^\infty(T \times V)$ and that they satisfy the following strong ellipticity condition:

$$\sum_{i,j=1}^n a_{ij}(t, z) \eta_i \eta_j \geqslant \theta \sum_{i=1}^n \eta_i^2$$

for all $(t, z) \in T \times V$, $\eta = (\eta_i)_{i=1}^n \in \mathbb{R}^n$ and with $\theta > 0$.

For this example $X = H_0^1(V)$, $H = L^2(V)$ and $X^* = H^{-1}(V)$. Clearly (X, H, X^*) is a Gelfand triple. On $X \times X$ we consider the following bilinear Dirichlet form:

$$a(t, x, y) = \int_V \sum_{i,j=1}^n a_{ij}(t, z) \frac{\partial x(t, z)}{\partial z_i} \frac{\partial x(t, z)}{\partial z_j} dz.$$

Since $a_{ij}(\cdot, \cdot) \in L^\infty(T \times V)$ and using Poincaré's inequality, we have:

$$|a(t, x, y)| \leqslant c \|x\|_{H_0^1(V)} \|y\|_{H_0^1(V)}.$$

Let $A(t) \colon H_0^1(V) = X \to H^{-1}(V) = X^*$ be the continuous linear operator defined by

$$a(t, x, y) = \langle A(t) x, y \rangle x, y \in H_0^1(V).$$

Making use of the strong ellipticity condition, we can show that:

$$\langle A(t)x,\, x\rangle \geqslant c\,\|x\|^2_{H^1_0(V)}\,.$$

We set $Y = L^2(V)$ (the control space) and set $U(t) = \{u \in L^2(V)\colon \|u\|_2 \leqslant r(t)\}$. Assume $r(\cdot) \in L^2_+$ and $0 < \delta \leqslant r(t)$. Let $\overset{\circ}{B}(\delta/(\max(b,1))) = \overset{\circ}{B} = \{u \in L^2(Y)\colon \|u\|_{L^2(Y)} < \delta/(\max(b,1))\}$. Then $\overset{\circ}{B} \subseteq S^2_U$ and so int $S^2_U \neq \phi$. Also we assume that $C \subseteq L^2(V)$ is nonempty, closed, convex, solid (that is int $C \neq \phi$).

Finally let $\hat{L}\colon T \times V \times \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ be an integrand such that

(i) $(t,\, z) \to \hat{L}(t,\, z,\, x,\, u)$ is measurable,

(ii) $(x,\, u) \to \hat{L}(t,\, z,\, x,\, u)$ is convex and continuously differentiable,

(iii) for every $x \in L^\infty(T, L^2(V))$ and every $u \in L^2(T, L^2(V)) = L^2(T \times V)$, $\hat{J}(x,\, u)$ is finite.

Define $L\colon T \times L^2(V) \times L^2(V) \to \mathrm{R}$ by $L(t,\, x,\, u) = \int_V \hat{L}(t,\, z,\, x(z),\, u(z))dz$. Using the above hypotheses (i) – (iii) about $\hat{L}$, we have that $L(t,\, x,\, u)$ satisfies $H(L)$. Furthermore $L_x(t,\, x,\, u)(h) = \int_V \hat{L}_x(t,\, z,\, x(z),\, u(z))h(z)dz$ and $L_u(t,\, x,\, u)(v) = \int_V \hat{L}_u(t,\, z,\, x(z),\, u(z))v(z)dz$

Now rewrite the optimal control problem (**) in the following abstract form:

$$(**)\qquad \left\{ \begin{array}{l} \text{Minimise } \displaystyle\int_0^b L(t,\, x(t),\, u(t))dt \\ \text{such that } \dot{x}(t) + A(t)x(t) = u(t) \text{ a.e.} \\ \qquad\qquad x(0) \in C,\, u(t) \in U(t) \text{ a.e.} \end{array} \right\}$$

This is a particular case of the more general problem studied in Section 3. So we can apply Theorem 3.1 and get the following necessary and sufficient conditionf for a triple $(x_0,\, x,\, u) \in L^2(V) \times W(T) \times L^2(T \times V)$, to be a solution of (**). Recall that $W(T) = \{x \in L^2(T,\, H^1_0(V)) : \dot{x} \in L^2(T,\, H^{-1}(V))\}$. Also $A^*(t)$ is the formal adjoint of the operator $A(t)$.

Theorem 4.1. *If the above hypotheses hold, then $(x_0,\, x,\, u) \in L^2(V) \times W(T) \times L^2(T \times V)$ solves (**) if and only if*

(i) $\partial x(t,z)/\partial t + A(t)x(t,\, z) = u(t,z)$ *on* $T \times V$

$$x\,|_{T\times\Gamma}\,(t,\, z) = 0,\, x(0,\, \cdot) = x_0(\cdot) \in C,\, \left(\int_V |u(t,\, z)|^2\, dz\right)^{1/2} \leqslant r(t);$$

(ii) *there exists $p(\cdot) \in W(T)$ satisfying the "adjoint equation"*

$$-\frac{\partial p(t,z)}{\partial t} + A^*(t)p(t,z) = \hat{L}_x(t,z,x(t,z),\, u(t,z)) \text{ on } T \times V$$

$$p(t,z) = 0 \text{ on } T \times \Gamma,\, p(b,z) = 0,\, z \in V;$$

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(iii) *the following "minimum principles" hold*

$$\int_V \Big(-p(t,z) + \hat{L}_u(t,z,x(t,z),\, u(t,z))\Big)(v(z) - u(t,z))dz \geqslant 0 \text{ a.e.}$$

$$\textit{for all } v \in L^2(V) \textit{ such that } \|v\|_2 \leqslant r(t)$$

and

$$\int_V -p(0,z)(c(z) - x(0,z))dz \geqslant 0 \textit{ for all } c(\cdot) \in C.$$

REMARK. If $r(\cdot)$ is not in L^2_+, but simply measurable, then the minimum principle has an integral form $\int_0^b \int_V \Big(-p(t,z) + \hat{L}_u(t,z,x(t,z),\, u(t,z))\Big)(v(t,z) - u(t,z)) \geqslant 0$ for all $v \in L^2(T \times V)$ such that $\|v(t,\,\cdot)\|_{L^2(V)} \leqslant r(t)$ a.e..

Finally we will conclude with some special cases of the problem studied in this paper

(1) $C = H$, $S^2_U = L^2(Y)$:
Then from the maximum principles we get

$$B^*(t,x(t))p(t) = L_u(t,x(t),u(t)) \text{ a.e.}$$

and $p(0) = 0$.

(2) $C \subseteq L^2(Y)$ with int $C \neq \emptyset$, $S^2_U = L^2(Y)$:
The maximum principles give us

$$B^*(t,x(t))p(t) = L_u(t,x(t),\, u(t)) \text{ a.e.}$$

and $(-p(0),\, c - x(0)) \geqslant 0$ for all $x \in C$.

Finally if $C = \{0\}$, then although int $C = \emptyset$, it can be easily seen looking at the proof of Theorem 3.1 that the second minimum principle disappears and we have:

(3) $C = \{0\}$, $S^2_U = L^2(U)$:

The first minimum principle tells us that

$$B^*(t,x(t))p(t) = L_u(t,x(t),\, u(t)) \text{ a.e.}$$

In the particular case of our example we have in all cases that the adjoint state is $p(t,z) = \hat{L}_u(t,z,x(t,z),\, u(t,z))$ a.e.

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