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The Tate Conjecture for Cubic Fourfolds over a Finite Field

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Abstract. We prove the Tate conjecture for codimension 2 cycles on an ordinary cubic fourfold over a finite field. The proof involves the construction of canonical coordinates on the formal deformation space via a crystalline period map.

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Key words. Tate conjecture, algebraic cycles, canonical coordinates, period maps

1. Introduction

If $X/\mathbb{F}$ is a smooth projective variety over a finite field $\mathbb{F}$ of characteristic $p > 0$ and $\overline{X} = X \otimes \overline{\mathbb{F}}$, there is a cycle class map $CH^i(X) \to H^{2i}_{et}(\overline{X}, \mathbb{Q}_\ell(i))$ for $\ell \neq p$ from the Chow group of codimension i cycles on X to étale cohomology. The image of this map lies in the subspace of $H^{2i}_{et}(\overline{X}, \mathbb{Q}_\ell(i))$ which is invariant under the natural Galois action. In [T3], Tate conjectures that, in fact, this subspace is actually generated by the image of this cycle class map.

This conjecture has been proven only in a very small number of special cases, e.g. for divisors on Abelian varieties [T2], certain Fermat hypersurfaces [T1], and non-supersingular or elliptic $K3$ surfaces [N, NO, ASD]. Here we will prove Tate's conjecture for *codimension two* cycles of ordinary cubic hypersurfaces in $\mathbb{P}^5$. (A result of Illusie [I] shows that the set of ordinary cubic fourfolds is a dense open set in the moduli space.)

The central idea of the proof is the construction of a lifting of the variety to characteristic zero, where we can use the fact the Hodge conjecture is known for this class of varieties. To be more precise, let $X_0/\mathbb{F}$ be an ordinary cubic hypersurface in $\mathbb{P}^5$. Let X_W be a lifting of X_0 to the Witt vectors W of $\mathbb{F}$. An embedding $W \hookrightarrow \mathbb{C}$ then gives a complex cubic fourfold $X_\mathbb{C}$. We thus have the following diagram:

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_W & \longleftarrow & X_\mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{F} & \longrightarrow & \operatorname{Spec} W & \longleftarrow & \operatorname{Spec}\mathbb{C} \end{array}$$

where both squares are pullback diagrams. This produces natural vector space isomorphisms

$$H^{2i}_{et}(\overline{X}_0, \mathbb{Q}_\ell(i)) \cong H^{2i}_{et}(X_{\mathbb{C}}, \mathbb{Q}_\ell(i)) \cong H^{2i}_{sing}(X_{\mathbb{C}}, \mathbb{Q}(i)) \otimes \mathbb{Q}_\ell$$

between the étale cohomology of X_0 and the singular cohomology of $X_{\mathbb{C}}$ tensored with $\mathbb{Q}_\ell$.

However, this isomorphism is not generally compatible with the extra structure on these cohomology groups. In particular, the Frobenius map on $H^{2i}_{et}(\overline{X}_0, \mathbb{Q}_\ell(i))$ does not usually respect the Hodge structure on $H^{2i}_{sing}(X_{\mathbb{C}}, \mathbb{Q}(i))$ (even after tensoring with $\mathbb{Q}_\ell$).

The proof of the Tate conjecture described here then begins by finding a special lifting of X_0 which has the property that the Frobenius map acts as an endomorphism of the rational Hodge structure of $H^4_{sing}(X_{\mathbb{C}}, \mathbb{Q}(2))$. It has the additional property that the subspace of $H^4_{sing}(X_{\mathbb{C}}, \mathbb{Q}(2))$ that is fixed under the action of Frobenius lies in the $(0, 0)$ part of the Hodge structure. Since the Hodge conjecture is known for codimesion 2 cycles on a cubic fourfold [Z], this implies that the Galois invariant subspace of $H^4_{et}(\overline{X}_0, \mathbb{Q}_\ell(2))$ is generated by algebraic cycles.

Now let X_W be an arbitrary lifting of X_0 to W. The de Rham cohomology of X_W has a natural Hodge filtration and a semilinear Frobenius action induced by the natural isomorphism with the crystalline cohomology of X_0. A lifting satisfying the property described in the previous paragraph must at least have the property that the Frobenius map preserves the Hodge filtration. (A general lifting will not have this property.)

This lifting is then produced by constructing a ‘p-adic period map’ from the universal deformation space of X_0 to a ‘period’ space that, loosely speaking, parameterizes admissible filtrations that can be placed on the crystalline cohomology of X_0. However, the construction of this period map is not as straightforward as it is in Hodge theory. One constructs a p-divisible group $\mathcal{P}$ defined over $\mathbb{F}$ out of the crystalline cohomology of X_0. The period map then arises as a map between the universal deformation space of X_0 and the universal deformation space of $\mathcal{P}$. We can then use the fact that deformations of p-divisible groups are parameterized (roughly) by the Hodge filtrations induced on their associated Dieudonné module [M].

Once the map is constructed, it is not too hard to show that it is isomorphism. Now, p-divisible groups over a finite field can be written as a direct sum of a connected part and an étale part. The desired lifting X_{can} of X_0 then corresponds to the lifting of $\mathcal{P}$ that preserves this direct sum structure.

In fact more can be said. The connected and étale pieces of $\mathcal{P}$ are rigid (i.e they have no non-trivial deformations), so deformations of $\mathcal{P}$ are entirely determined by extension data. This gives a canonical group structure on the universal deformation space of X_0. The required lifting of X_0 then corresponds to the origin of this group structure. This is completely analogous to the situation described by Deligne and Illusie [DI] and Nygaard [N] for ordinary K3 surfaces. This is not surprising, since the middle cohomology of a cubic fourfold looks very similar to that of a K3 surface (its Hodge numbers are $h^{0,4} = h^{4,0} = 0$, $h^{1,3} = h^{3,1} = 1$, and $h^{2,2} = 21$). In fact, the methods of this paper can be used to give another proof of these results as well.

To conclude the proof of the Tate conjecture, one now only has to show that the Frobenius map actually induces an endomorphism of the Hodge structure on the rational cohomology of the complex cubic fourfold $X_{\mathbb{C}}$ associated to X_{can}. This follows by using the fact that complex cubic fourfolds have an associated Kuga–Satake–Deligne Abelian variety and using an absolute Hodge cycles argument as in [N].

2. Preliminaries

Let $\mathbb{F}$ denote a perfect field, $A_0 = \mathbb{F}[[t_1, \ldots, t_n]]$, and $A = W[[t_1, \ldots, t_n]]$, where W denotes the Witt vectors of $\mathbb{F}$. Let $S = \operatorname{Spec} A$ and $S_0 = \operatorname{Spec} A_0$. Let σ denote a lifting of the absolute Frobenius map σ_0 on A_0 to A.

DEFINITION 2.1. An σ-F-crystal on A is a triple (E, ∇, F), where

(1) E is a finitely generated free A-module.
(2) $\nabla: E \to E \otimes \widehat{\Omega}_{A/W}$ is a nilpotent, integrable connection, where $\widehat{\Omega}_{A/W}$ is the module of p-adically complete differentials.
(3) $F: \sigma^*(E, \nabla) \to (E, \nabla)$ is a horizontal morphism which becomes an isomorphism after tensoring with $\mathbb{Q}$.

We can construct several useful filtrations on such a crystal. Let $E^{(p)} = \sigma^* E$, $E_0 = E \otimes A_0$, and $E_0^{(p)} = E^{(p)} \otimes A_0$. We then define the filtrations:

$$M^k E^{(p)} = \{x \in E^{(p)}: F(x) \in p^k E\},$$
$$M^k E_0^{(p)} = \operatorname{Im}[M^k E^{(p)} \to E_0^{(p)}],$$
$$N_k E = \{p^{-k} F(x): x \in M^k E^{(p)}\},$$
$$N_k E_0 = \operatorname{Im}[N_k E \to E_0],$$

where $N_k E_0$ is an increasing filtration called the conjugate filtration on E_0 and $M^k E^{(p)}$ is a decreasing filtration. If we regard E_0 as a submodule of $\sigma_{0_*} E_0^{(p)}$ via the adjunction map, we can also construct a Hodge filtration

$$Fil^k_{Hodge} E_0 = E_0 \cap M^k E_0^{(p)}.$$

The names for the Hodge and conjugate filtrations can be justified in the following way. Let X_0 be a smooth proper variety over S_0 and let X be a lifting of X_0 to S, (i.e. X is smooth over S and $X_0 = X \times_S S_0$). Recall that we can construct the crystalline cohomology groups $H^i_{cris}(X_0/S)$ which have the property that $H^i_{cris}(X_0/S)/$

(*torsion*) is a crystal over over A and there is a canonical isomorphism $H^i_{cris}(X_0/S) \cong H^i_{DR}(X/S)$. Note that this implies that the de Rham cohomology of X depends only on its reduction mod p and that it inherits a Frobenius action from $H^i_{cris}(X_0/S)$. We then have the following theorem, due to Mazur and Ogus [O, 2.2]:

THEOREM 2.2 (Mazur, Ogus). *Suppose that X_0 is a smooth, proper variety over S_0, that all the modules $H^i_{cris}(X_0/S)$ are locally free, and that for every point $s \in S_0$, the Hodge to de Rham spectral sequence of $X(s)/k(s)$ degenerates at E_1. Then if we let $E = H^n_{cris}(X_0/S)$ we obtain:*

$$Fil^k_{Hodge}E_0 = Fil^k_{Hodge}H^n_{DR}(X_0/S_0),$$
$$N_kE_0 = Fil^{con}_{n-k}H^n_{DR}(X_0/S_0),$$

where the filtrations on the right-hand side are the usual Hodge and conjugate filtrations.

Let

$$gr^k_M E^{(p)}_0 = M^k E^{(p)}_0 / M^{k+1} E^{(p)}_0 \quad \text{and} \quad gr^k_N E_0 = N_k E_0 / N_{k-1} E_0.$$

Suppose that $gr^k_M E^{(p)}$ is a free A_0 module for all k (i.e., the case $A_0 = \mathbb{F}$). We define the Hodge numbers of E to be $h^i = \mathrm{rk}_{\mathbb{F}} gr^i_M E^{(p)}$. Note that if we are in the situation of Theorem 2.2, we have $\sigma^*(Fil^i_{Hodge}E_0) = M^i E^{(p)}_0$, so

$$h^i = \mathrm{rk}_{A_0} Fil^i_{Hodge}H^n_{DR}(X_0/S_0)/Fil^{i+1}_{Hodge}H^n_{DR}(X_0/S_0)$$
$$= \mathrm{rk}_{A_0} H^{n-i}(X_0, \Omega^i_{X_0/S_0}).$$

The set of the i's which are non-zero are called the Hodge slopes of E. Define the Hodge polygon of E to be the convex polygon is the plane whose left-most point is the origin and which has slope i on the interval $[h^0 + \cdots + h^{i-1}, h^0 + \cdots + h^i]$.

Suppose we are in the case $A_0 = \mathbb{F}$. Let K be the maximal unramified extension of the fraction field of $W(\mathbb{F})$, let E be a crystal over A, and let $E_K = E \otimes K$. We can regard E_K as a module over the noncommutative polynomial ring $K[T]$ with $\sigma(x)T = Tx$ for $x \in K$. It is then a classical theorem of Dieudonné and Manin that E_K can be written uniquely as a direct sum of $K[T]$-modules:

$$E_K = \oplus^n_{i=1} E_{r_i, s_i}, \tag{1}$$

where $r_i > 0$ and $s_i > 1$ are integers, $r_i/s_i \leqslant r_{i+1}/s_{i+1}$, $\sum^n_{i=1} s_i = \mathrm{rk}_K E_K$ and $E_{r,s}$ is the module $K[T]/(T^s - p^r)$. The numbers r_i/s_i are called the Newton slopes of E and are said to occur with multiplicity s_i. The Newton polygon of E is the convex polygon in the plane whose left-most point is the origin and which has slope r_i/s_i over the interval $[s_1 + \cdots + s_{i-1}, s_1 + \cdots + s_i]$.

In the case where E is an ordinary crystal (described below) over W, the Newton slopes of E are actually integers and we can write the decomposition (1) over W

itself rather than its fraction field. In this case we can define a filtration on E, called the slope filtration, by $Fil^i_{slope}E = \oplus_j E_{r_j,s_j}$, where $r_j/s_j \geqslant i$.

The following theorem was proven by Deligne and Illusie [DI, 1.3.2]

THEOREM 2.3 (Deligne, Illusie). *Let E be a σ-F-crystal over A such that $N_{i+1}E_0/N_iE_0$ is a free A_0-module for all i. Then the following conditions are equivalent:*

(1) *The Newton and Hodge polygons of the crystal s^*E induced by E at the closed point $s: A_0 \to \mathbb{F}$ coincide.*
(2) *The Hodge and conjugate filtrations of E_0 are opposed, i.e. $E_0 = N_iE_0 \oplus Fil^{i+1}_{Hodge}E_0$ for every i*
(3) *There exists a unique filtration of E by subcrystals*

$$0 \subset U_1 \subset U_2 \subset \cdots \subset U_i \subset U_{i+1} \cdots$$

such that the Newton and Hodge polygons of U_i and U_i/U_{i+1} are constant for every i and such that $U_i \otimes A_0 = N_iE_0$

DEFINITION 2.4. We say a crystal E is ordinary if it satisfies any of the above conditions. If X_0 is a smooth projective variety over SpecA_0 we say X_0 is ordinary if all of its crystalline cohomology groups are ordinary is the above sense.

THEOREM 2.5 (Newton–Hodge decomposition [K2]). *Suppose E is a crystal on A. Suppose that $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ is a break point of the Newton polygon at every point of S, and that (a, b) lies on the Hodge polygon at every point of S. Then there exists a unique subcrystal P of E, locally free of rank a, and a quotient crystal $Q = E/P$ locally free of rank $r - a$ such that*

– *at every point of S, the Hodge (resp. Newton) slopes of P are the a smallest of the Hodge (resp. Newton) slopes of E.*
– *at every point of S, the Hodge (resp. Newton) slopes of Q are the a largest of the Hodge (resp. Newton) slopes of E.*

Furthermore, when $A = \mathbb{F}$, the exact sequence of crystals

$$0 \to P \to E \to Q \to 0$$

admits a unique splitting.

THEOREM 2.6. *Suppose that E and U_i are as in Theorem 2.3 with $gr_{N_k}E_0$ free A_0-modules for all k. Let $Q = E/U_i$. Then the Hodge filtration on Q_0 is determined* *by the Hodge filtration on E_0; i.e.*

$$Fil^k_{Hodge}E_0 \twoheadrightarrow Fil^k_{Hodge}Q_0.$$

Proof. It is obvious from the definition of M^k that $M^kE \twoheadrightarrow M^kQ$. which implies that $M^kE_0 \twoheadrightarrow M^kQ_0$. This implies that the image of $Fil^k_{Hodge}E_0$ in Q_0 is contained in $Fil^k_{Hodge}Q_0$.

On the other hand, the fact that $gr_{N_k}E_0$ is free implies that $gr^k_M Q_0^{(p)}$ is free by [O, 1.6.2] which implies that $M^kE_0^{(p)}$ is free for all k and is in fact a direct summand of $E_0^{(p)}$ (by induction, since M^k is a finite filtration). But $\sigma^* Fil^k_{Hodge}E_0 \cong M^kE_0^{(p)}$ and since σ is faithfully flat, we must have that $Fil^k_{Hodge}E_0$ is a direct summand of E_0 (c.f. [O, 1.12.1]).

Condition 2 of Theorem 2.3 and the fact that $gr_{N_k}E_0$ is free implies that $gr_{N_k}Q_0$ is free, which implies that $Fil^k_{Hodge}Q_0$ is a direct summand of Q_0, as above. Condition 2 then implies that $Fil^{i+1}_{Hodge}E_0$ maps isomorphically onto Q_0, which implies that the image of each $Fil^k_{Hodge}E_0$ in Q_0 is a direct summand. The theorem then follows from comparing the Hodge numbers of E_0 and Q_0 and using the Newton–Hodge decomposition described in Theorem 2.5.

DEFINITION 2.7. Let E be a crystal over A and let F^kE_n be a filtration of $E_n = E/p^{n+1}E$. We say that F^kE_n is an admissible filtration if each F^kE_n is a direct summand of E_n and if $F^kE_n \otimes A_0 = Fil^k_{Hodge}E_0$.

DEFINITION 2.8. A Dieudonné module over A is a crystal E which possesses a horizontal A-linear map $V: E \to \sigma^*E$ which satisfies

$$FV = VF = \textit{multiplication by } p$$

and whose Hodge filtration $Fil^1E_0 \subset E_0$ is a direct summand. A filtered Dieudonné module is a Dieudonné module endowed with an admissible filtration $H \subset E$.

Remark 2.9. This is equivalent to the definition given by De Jong(see [dJ, 2.5.2]).

The importance of Dieudonné modules for us is their connection to p-divisible groups via the Dieudonné functor $\mathbb{D}$ from the category of p-divisible groups to the category of Dieudonné modules. More specifically, we have the following theorem:

THEOREM 2.10 (Bloch–Kato, de Jong). *Assume char* $\mathbb{F} \neq 2$ *and let* $S = \mathrm{Spec}A$ *and* $S_0 = \mathrm{Spec}A_0$. *Then the (covariant) Dieudonné functor* $\mathbb{D}$ *induces an equivalence of categories between the category of p-divisible groups over* S_0 *and the category of Dieudonné modules over* S *and between the category of filtered Dieudonné modules and p-divisible groups over* S.

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Proof. The equivalence for Dieudonné modules is proven in [dJ, 4.1.1, 2.4.4, and 2.4.8.1]. The proof for filtered Dieudonné modules then follows from Messing's description of deformations of p-divisible groups in terms of the induced Hodge filtration on their Dieudonné modules [M, V.1.6]. The restriction on the characteristic of $\mathbb{F}$ arises because if char$\mathbb{F} = 2$, the ideal generated by p^n in W does not have nilpotent divided powers.

Remark 2.11. We will denote the inverse functor by $\mathbb{M}$.

Remark 2.12. De Jong uses the contravariant Dieudonné functor as opposed to the covariant functor in his paper, but this poses no difficulties (see [BBM, 5.3.3]).

Remark 2.13. For later use, we note that if G is a divisible group, then $\mathbb{D}(G)_0$ fits into the exact sequence:

$$0 \to \omega_{G^*} \to \mathbb{D}(G)_0 \to Lie(G) \to 0 \tag{2}$$

where $Lie(G)$ is the Lie algebra of G and ω_{G^*} is the group of invariant differentials of the dual of G.

3. Deformation Theory

Let $X_{\mathbb{F}}$ be a cubic fourfold over a perfect field $\mathbb{F}$. In this section, we will describe the formal moduli space parameterizing deformations of $X_{\mathbb{F}}$ to Artin local W-algebras. In order to do this, we first need a few general facts about the cohomology of cubic fourfolds.

THEOREM 3.1. *Let X be a smooth cubic fourfold defined over an affine scheme S. Then the de Rham cohomology $H^4_{DR}(X/S)$ is torsion free with Hodge numbers:*

$$h^{0,4} = h^{4,0} = 0, \qquad h^{1,3} = h^{3,1} = 1, \qquad h^{2,2} = 21.$$

Furthermore, the Hodge to de Rham spectral sequence degenerates at E_1.

Proof. This follows from [SGA7, Exposé II] and [R, section 2]. □

Remark 3.2. If X is a cubic fourfold over a perfect field, the above description of the Hodge numbers implies that the only non-zero Hodge slopes of $H^4_{cris}(X/W)$ are 1, 2 and 3.

Let T_X denote the tangent space to X.

LEMMA 3.3. $H^i(X, \Omega^3_{\mathbb{P}^5}|_X(3)) \cong H^{i+1}(\mathbb{P}^5, \Omega^3_{\mathbb{P}^5})$ if $0 < i < 4$.

Consider the exact sequence of sheaves on $\mathbb{P}^5$:

$$0 \to \mathcal{I} \cong \mathcal{O}_{\mathbb{P}^5}(-3) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_X \to 0, \tag{3}$$

where $\mathcal{I}$ is the ideal in $\mathbb{P}^5$ which defines X. We can tensor this sequence with $\Omega^i_{\mathbb{P}^5}(3)$ to get the long exact sequence:

$$\to H^i(\mathbb{P}^5, \Omega^3_{\mathbb{P}^5}(3)) \to H^i(X, \Omega^3_{\mathbb{P}^5}|_X(3)) \to H^{i+1}(\mathbb{P}^5, \Omega^3_{\mathbb{P}^5}) \to H^{i+1}(\mathbb{P}^5, \Omega^3_{\mathbb{P}^5}(3)) \tag{4}$$

But now $H^i(\mathbb{P}^5, \Omega^j_{\mathbb{P}^5}(n)) = 0$ if $i \neq 0, 5$ and $n \neq 0$ (Bott Vanishing) so both ends of the above exact sequence vanish. This proves the lemma. □

Let $H^i(X, \Omega^j_X)_{prim}$ denote the primitive part of $H^i(X, \Omega^j_X)$.

THEOREM 3.4. *The cup product map induces an isomorphism*

$$H^1(X, T_X) \cong \mathrm{Hom}(H^1(X, \Omega^3_X), H^2(X, \Omega^2_X)_{prim}).$$

Proof. It is sufficient to show that the natural pairing

$$H^1(X, T_X) \otimes H^1(X, \Omega^3_X) \to H^2(X, \Omega^2_X)_{prim}$$

is an isomorphism. Since X is a cubic hypersurface in $\mathbb{P}^5$, we know that the canonical bundle ω_X is isomorphic to $\mathcal{O}_X(-3)$, and the normal bundle $\mathcal{N}$ is isomorphic to $\mathcal{O}_X(3)$. We can the write the normal bundle sequence as:

$$0 \to T_X \to T_{\mathbb{P}^5}|_X \to \mathcal{N} \to 0 \tag{5}$$

and its dual

$$0 \to \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^5}|_X \to \Omega_X{}^1 \to 0, \tag{6}$$

where T_X and $T_{\mathbb{P}^5/\mathbb{F}}$ denote the tangent spaces of X and $\mathbb{P}^5$ respectively. Taking exterior powers of (6) and tensoring with $\mathcal{O}_X(3)$ then gives:

$$0 \to \Omega_X{}^2 \to \Omega^3_{\mathbb{P}^5}|_X(3) \to \Omega_X{}^3(3) \to 0. \tag{7}$$

On the other hand we have the natural isomorphism $T_X \otimes \omega_X \cong \Omega^3_X$ which allows us to write (7) as

$$0 \to \Omega_X{}^2 \to \Omega^3_{\mathbb{P}^5}|_X(3) \to T_X \to 0. \tag{8}$$

This, in turn, induces the long exact sequence

$$\to H^1(X, \Omega^3_{\mathbb{P}^5}|_X(3)) \to H^1(X, T_X) \xrightarrow{\delta} \delta H^2(X, \Omega^2_X) \to H^2(X, \Omega^3_{\mathbb{P}^5}|_X(3)) \to, \tag{9}$$

where δ is induced by cup product with the extension class of (6) via its identification with the extension class of (7).

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But now the previous lemma implies that

$$H^1(X, \Omega^3_{\mathbb{P}^5}|_X(3)) \cong H^2(\mathbb{P}^5, \Omega^3_{\mathbb{P}^5}) = 0, \tag{10}$$

$$H^2(X, \Omega^3_{\mathbb{P}^5}|_X(3)) \cong H^3(\mathbb{P}^5, \Omega^3_{\mathbb{P}^5}) \cong \mathbb{F}, \tag{11}$$

so δ induces an isomorphism $H^1(X, T_X) \cong H^2(X, \Omega^2_X)_{prim}$. Since

$$\begin{aligned} \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X(-3)) &\cong \mathrm{Ext}^1(\mathcal{O}_X, T_X(-3)) \\ &\cong \mathrm{Ext}^1(\mathcal{O}_X, \Omega^3_X) \cong H^1(X, \Omega^3_X), \end{aligned}$$

the theorem is proved.

THEOREM 3.5. *Let X be a cubic fourfold over a field k. Then*

(1) $H^0(X, T_X) = 0$
(2) $\mathrm{rk}_k H^1(X, T_X) = 20$
(3) $H^i(X, T_X) = 0, i \geqslant 2$

Proof. The Kodaira–Akizuk–Nakano vanishing theorem (which is valid for hypersurfaces over an arbitrary field [SGA7]) implies that $H^i(X, {\Omega_X}^k(-s)) = 0$ for $s > 0$ and and $i + k < 4$. By Serre duality we have

$$H^i(X, T_X) = H^{4-i}(X, {\Omega_X}^1 \otimes \mathcal{O}_X(-3))$$

so $H^i(X, T_X) = 0$ if $4 - i + 1 < 4$, i.e. $i > 1$. This proves (3). (2) follows immediately from Theorem 3.4, so it remains to show (1).

Consider the following classical exact sequences:

$$0 \to T_X \to T_{\mathbb{P}^5}|_X \to \mathcal{O}_X(3) \to 0, \tag{12}$$

$$0 \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_{\mathbb{P}^5}(1)^6 \to T_{\mathbb{P}^5} \to 0. \tag{13}$$

These sequences give relationships between the Euler characteristics χ of the sheaves involved as follows:

$$\chi(X, T_X) = \chi(X, T_{\mathbb{P}^5}|_X) - \chi(X, \mathcal{O}_X(3)), \tag{14}$$

$$\begin{aligned} \chi(\mathbb{P}^5, T_{\mathbb{P}^5}) &= \chi(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)^6) - \chi(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}) \\ &= 6\chi(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)) - \chi(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}). \end{aligned} \tag{15}$$

Twisting 13 also gives

$$\chi(\mathbb{P}^5, T_{\mathbb{P}^5}(-3)) = 6\chi(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(-2)) - \chi(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(-3)). \tag{16}$$

We want to use the sequence (3) to relate these two equations. If we tensor the sequence (3) with $\mathcal{O}_{\mathbb{P}^5}(3)$ we get:

$$\chi(X, \mathcal{O}_X(3)) = \chi(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)) - \chi(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}). \tag{17}$$

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If we tensor (3) with $T_{\mathbb{P}^5}$ we get

$$\chi(X, T_{\mathbb{P}^5}|_X) = \chi(\mathbb{P}^5, T_{\mathbb{P}^5}) - \chi(\mathbb{P}^5, T_{\mathbb{P}^5}(-3)). \tag{18}$$

But now we can calculate these numbers by by using the fact that

$$H^i(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(k)) = 0 \text{ if } 0 < i < r, \tag{19}$$

$$\chi(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(k)) = \binom{5+k}{5}. \tag{20}$$

Putting all this together implies that

$$\chi(X, T_X) = -20. \tag{21}$$

But we know that $\mathrm{rk}_k H^i(X, T_X) = 20$ if $i = 1$ and is equal to 0 if $i > 1$. This implies that $H^0(X, T_X) = 0$. □

From now on we will assume that $X_{\mathbb{F}}$ is a cubic fourfold over a perfect field $\mathbb{F}$. We want to consider the formal moduli space of deformations of $X_{\mathbb{F}}$ to Artin rings over $W = W(\mathbb{F})$, the Witt vectors of F. Since $H^0(X_{\mathbb{F}}, T_{X_{\mathbb{F}}}) = H^2(X_{\mathbb{F}}, T_{X_{\mathbb{F}}}) = 0$, classical deformation theory implies that there exists a smooth formal scheme $\mathcal{S}$ and a formal scheme $\mathcal{X} \to \mathcal{S}$ such that any deformation X_A of $X_{\mathbb{F}}$ to an Artin W-algebra A is given by pulling back $\mathcal{X} \to \mathcal{S}$ along a uniquely determined map $\mathrm{Spec} A \to \mathcal{S}$. Furthermore, the tangent space $T_{\mathcal{S}}$ of $\mathcal{S}$ is isomorphic to $H^1(X_{\mathbb{F}}, T_{X_{\mathbb{F}}})$, so we can conclude:

COROLLARY 3.6. $\mathcal{S} \cong \mathrm{Spf}\, W[[t_1 \ldots t_{20}]]$.

COROLLARY 3.7. *The Kodaira–Spencer map induces isomorphisms:*

$$T_{\mathcal{S}} \cong H^1(X_{\mathbb{F}}, T_{X_{\mathbb{F}}}) \cong \mathrm{Hom}(H^1(X_{\mathbb{F}}, \Omega^3_{X_{\mathbb{F}}}), H^2(X_{\mathbb{F}}, \Omega^2_{X_{\mathbb{F}}})_{prim}).$$

It is a bit inconvenient to work with formal schemes, and it turns out in this case we don't have to. Recall that the obstructions to lifting an ample line bundle from $X_{\mathbb{F}}$ to $\mathcal{X}$ lives in $H^2(X_{\mathbb{F}}, \mathcal{O}_{X_{\mathbb{F}}})$. Since $X_{\mathbb{F}}$ is a hypersurface in $\mathbb{P}^5$, this cohomology group vanishes, so any ample line bundle can be lifted. Grothendieck's existence theorem then tells us that $\mathcal{X}$ is algebraizable, i.e., that there is a formally smooth cubic fourfold $X \to S = \mathrm{Spec}\, W[[t_1 \ldots t_{20}]]$ with $\mathcal{X} \cong \mathcal{S} \times_S X$.

4. Construction of the Period Map

We will keep the notation of the previous section, but we will now assume that $X_{\mathbb{F}}$ is ordinary. By definition, this means that $H^4_{cris}(X_{\mathbb{F}}/W)$ is an ordinary crystal. We will actually want to consider the primitive cohomology $P^4_{cris}(X_{\mathbb{F}}/W)$, which is a subcrystal of $H^4_{cris}(X_{\mathbb{F}}/W)$, and hence also ordinary. The Hodge polygon and the Newton polygon of $P^4_{cris}(X_{\mathbb{F}}/W)$ then coincide, so by Theorem 2.5 we can find a subcrystal $P \subset P^4_{cris}(X_{\mathbb{F}}/W)$ such that the Newton and Hodge slopes of P are equal

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to 1 and a quotient crystal Q with all slopes > 1 which fit in the exact sequence:

$$0 \to P \to P^4_{cris}(X_{\mathbb{F}}/W) \to Q \to 0. \tag{14}$$

Furthermore, since $\mathbb{F}$ is perfect, this sequence splits canonically, so we can regard Q as a subcrystal of $P^4_{cris}(X_{\mathbb{F}}/W)$ in a natural way. Examination of the Hodge polygon shows that $\mathrm{rk}_W P = 1$ and $\mathrm{rk}_W Q = 21$ with all slopes equal to 2 or 3. Thus the crystal $Q(2)$ has all its slopes equal to 0 or 1, so by [O, 1.6.4] it must be a Dieudonné module. So by Theorem 2.10, there is a unique p-divisible group $\mathcal{P}_{\mathbb{F}}$ over $\mathbb{F}$ with $\mathbb{D}(\mathcal{P}_{\mathbb{F}}) = Q(2)$.

Let $\mathcal{T}$ be the fine formal moduli space parameterizing deformations of $\mathcal{P}_{\mathbb{F}}$ to Artin rings over W. This is isomorphic to $\mathrm{Spf} A$, where $A = W[[t_1 ... t_{20}]]$ (where $20 = \mathrm{rk}_{\mathbb{F}} \mathrm{Hom}(\omega_{\mathcal{P}^*_{\mathbb{F}}}, Lie(\mathcal{P}_0)))$. We would like to show that the deformation theories of $X_{\mathbb{F}}$ and $\mathcal{P}_{\mathbb{F}}$ are the same. To do this we need to construct a 'period map' $\mathcal{S} \to \mathcal{T}$ and show it is an isomorphism. The categories of p-divisible groups over $\mathrm{Spec} A$ and $\mathrm{Spf} A$ are equivalent [dJ, 2.4.4], so we can work in the category of schemes, rather than formal schemes.

To construct a map, it is then sufficient to construct a p-divisible group $\mathcal{P}$ on S which reduces to $\mathcal{P}_{\mathbb{F}}$ at the closed point. This would give a map $S \to T = \mathrm{Spec} A$ (and hence a map $\mathcal{S} \to \mathcal{T}$). Let $S_0 = S \otimes \mathbb{F}$. We will first construct a p-divisible group $\mathcal{P}_0$ over S_0 with the required properties and then show that it lifts canonically to S.

Let $X_0 = X \times_S S_0$. we want to repeat the above argument using $P^4_{cris}(X_0/S)$. By the base change theorems for crystalline cohomology, $P^4_{cris}(X_0/S)$ specializes to $P^4_{cris}(X_{\mathbb{F}}/W)$ at the closed point of S_0, so $P^4_{cris}(X_0/S)$ is ordinary by Theorem 2.3. Again by Theorem 2.3, we can write an exact sequence of crystals

$$0 \to P' \to P^4_{cris}(X_0/S) \to Q' \to 0, \tag{23}$$

with P' and Q' satisfying the same slope conditions as P and Q. By uniqueness, these must pull back via the specialization map to P and Q. Just as before, $Q(2)'$ is a Dieudonné module and there is a unique p-divisible group $\mathcal{P}_0$ over S_0 with $\mathbb{D}(\mathcal{P}_0) = Q(2)'$. (The fact that the Hodge filtration on $Q(2)'_0$ is a direct summand follows from Theorem 2.6; see Theorem 4.4 below for more details.) Since the Dieudonné functor commutes with base change, $\mathcal{P}_0$ will specialize to $\mathcal{P}_{\mathbb{F}}$ at the closed point of S_0.

In order to construct a p-divisible group over S, it suffices to put a Hodge filtration on $Q(2)'$. To simplify notation, we will identify $P^4_{cris}(X_0/S)$ with $P^4_{DR}(X/S)$ in (23), i.e., we will regard P' and Q' as a submodule and a quotient module respectively of $P^4_{DR}(X/S)$. We begin with the following result.

LEMMA 4.1. *$P^4_{DR}(X/S) = P' \oplus Fil^2_{Hodge} P^4_{DR}(X/S)$.*

Proof. Since all of the cohomology groups of X are free, and since the Hodge to de Rham spectral sequence degenerates at E_1, we know that

$$P^4_{DR}(X/S)\otimes S_0 \cong P^4_{DR}(X_0/S_0), \tag{24}$$

$$Fil^k_{Hodge}P^4_{DR}(X/S)\otimes S_0 \cong Fil^k_{Hodge}P^4_{DR}(X_0/S_0). \tag{25}$$

Since $P^4_{DR}(X/S)$ is ordinary, we know from Theorem 2.3 and Theorem 2.2 that the theorem holds true mod p. Nakayama's lemma then implies that

$$P' + Fil^2_{Hodge}P^4_{DR}(X_0/S) = P^4_{DR}(X/S).$$

But now both P' and $Fil^2_{Hodge}P^4_{DR}(X/S)$ are direct summands of $P^4_{DR}(X/S)$ with

$$\mathrm{rk}_A P' + \mathrm{rk}_A Fil^2_{Hodge}P^4_{DR}(X/S) = \mathrm{rk}_A P^4_{DR}(X/S),$$

so we must have

$$P' \cap Fil^2_{Hodge}P^4_{DR}(X/S) = \emptyset,$$

since it is true after tensoring with the fraction field of A. This proves the lemma. □

COROLLARY 4.2. *The exact sequence (23) induces an isomorphism*

$$Fil^2_{Hodge}P^4_{DR}(X/S) \cong Q'.$$

COROLLARY 4.3. *The image of $Fil^3_{Hodge}P^4_{DR}(X/S)$ in Q' is a direct summand.*

Let H be the image of $Fil^3_{Hodge}P^4_{DR}(X/S)$ in Q' and let H_0 denote its reduction mod p. Since $Fil^i_{Hodge}P^4_{DR}(X_0/S_0) = 0$ for all $i > 3$, Theorem 2.6 implies that the entire Hodge filtration on Q'_0 is given by $Q'_0 \supset H_0 \supset 0$ with $Fil^3_{Hodge}Q'_0 = H_0$. Since twisting by 2 lowers the Hodge slopes by 2, we then get that $Fil^1_{Hodge}Q(2)' = H_0$. This proves:

THEOREM 4.4. *$H \subset Q(2)'$ is an admissible filtration.*

By Theorem 2.10, there exists a p-divisible group $\mathcal{P}$ over S with the property that $\mathbb{D}(\mathcal{P}) = Q(2)'$. Furthermore, $\mathcal{P}$ has the property that it specializes to $\mathcal{P}_{\mathbb{F}}$ over the closed point of S. We have finally proven:

THEOREM 4.5. *There exists a map $f: \mathcal{S} \to \mathcal{T}$.*

THEOREM 4.6. *The map $\mathcal{S} \to \mathcal{T}$ is smooth.*

Proof. It is sufficient to show the induced map on tangent spaces is surjective. Corollary 3.7 implies that the Kodaira–Spencer map induces an isomorphism

$$T_{\mathcal{S}} \cong \mathrm{Hom}(H^1(X_{\mathbb{F}}, \Omega^3_{X_{\mathbb{F}}}), H^2(X_{\mathbb{F}}, \Omega^2_{X_{\mathbb{F}}})_{prim}).$$

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On the other hand, we have already mentioned that the tangent space $T_{\mathcal{T}}$ to $\mathcal{T}$ is isomorphic to $\mathrm{Hom}(\omega_{\mathcal{P}^*_{\mathbb{F}}}, Lie\mathcal{P}_{\mathbb{F}})$ But now the Dieudonné module $Q(2)'$ was constructed in such a way that

$$\omega_{\mathcal{P}^*_{\mathbb{F}}} \cong Fil^1_{Hodge}Q(2)_0 \cong Fil^3_{Hodge}P^4_DR(X_{\mathbb{F}}/\mathbb{F}) \cong H^1(X_{\mathbb{F}}, \Omega^3_{X_{\mathbb{F}}}), \tag{26}$$

$$Lie\mathcal{P}_{\mathbb{F}} \cong gr^0_{Hodge}Q(2)_0 \cong gr^2_{Hodge}P^4_DR(X_{\mathbb{F}}/\mathbb{F}) \cong H^2(X_{\mathbb{F}}, \Omega^2_{X_{\mathbb{F}}})_{prim}), \tag{27}$$

and so that the map $S \to \mathcal{T}$ induces the now obvious map on tangent spaces. This proves the theorem.

Remark 4.7. It is probably worth elaborating on what is really going on here. Let X_ε be a deformation of $X_{\mathbb{F}}$ to the dual numbers $\mathrm{Spec}\mathbb{F}[\varepsilon]$ Since $\mathrm{Spec}\mathbb{F} \hookrightarrow \mathrm{Spec}\mathbb{F}[\varepsilon]$ is a divided power extension, there is a canonical isomorphism between $P^4_{DR}(X_\varepsilon/\mathbb{F}[\varepsilon])$ and $P^4_{cris}(X_{\mathbb{F}}/\mathbb{F}[\varepsilon]) \cong P^4_{cris}(X_{\mathbb{F}}/W) \otimes \mathbb{F}[\varepsilon]$, which implies that there is a canonical isomorphism

$$Fil^2_{Hodge}P^4_{DR}(X_\varepsilon/\mathbb{F}[\varepsilon]) \cong Q(2)_\varepsilon = Q(2) \otimes \mathbb{F}[\varepsilon].$$

Lemma 3.4 then implies that giving deformation X_ε as above is equivalent to giving a one step filtration on $Q(2)_\varepsilon$ which reduces to the usual filtration under the reduction map.

On the other hand, a theorem of Messing [M, V.1.6] implies that giving a deformation of $\mathcal{P}_{\mathbb{F}}$ to $\mathrm{Spec}\mathbb{F}[\varepsilon]$ is equivalent to putting an admissible filtration on

$$\mathbb{D}(\mathcal{P}_{\mathbb{F}}) \otimes \mathbb{F}[\varepsilon] = Q(2)_\varepsilon.$$

But now X_ε is given by a map $f: \mathrm{Spec}\mathbb{F}[\varepsilon] \to S$. If we pull back $Q(2)'$ via this map we get an admissible filtration on $Q(2)_\varepsilon$ induced by the de Rham cohomology of X_ε, which gives us a deformation of $\mathcal{P}_{\mathbb{F}}$ to $\mathrm{Spec}\mathbb{F}[\varepsilon]$. But it is clear that this is exactly the p-divisible group that arises by pulling back $\mathcal{P}$ along f. Since all possible admissible filtrations occur in this manner, it is easy to see that the period map induces a surjective map on tangent spaces, and hence must be a isomorphism.

THEOREM 4.8. *There is a unique lift X_{can} of $X_{\mathbb{F}}$ to W with the property that that the slope filtration on the crystalline cohomology group $H^4_{cris}(X_{\mathbb{F}}/W)$ agrees with the Hodge filtration on de Rham cohomology group $H^4_{DR}(X_{can}/W)$ via the canonical isomorphism between the two cohomology theories.*

Proof. Consider the p-divisible group $\mathcal{P}_{\mathbb{F}}$ and its Dieudonné module $\mathbb{D}(\mathcal{P}_{\mathbb{F}})$ that we constructed before. $Q(2)$ is ordinary and only has Hodge slopes of 0 and 1, so by Theorem 2.5 there are subcrystals M and N of $Q(2)$ such that $Q(2) = M \oplus N$, where M has all Hodge slopes equal to 0 and N has all Hodge slopes equal to 1. This decomposition corresponds to the splitting of $\mathcal{P}_{\mathbb{F}}$ into a connected part $\mathcal{P}_{\mathbb{F},conn} = \mathbb{M}(M)$ and an étale part $\mathcal{P}_{\mathbb{F},et} = \mathbb{M}(N)$.

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We want to consider liftings of $\mathcal{P}_{\mathbb{F},conn}$ and $\mathcal{P}_{\mathbb{F},et}$ to W. By Theorem 2.10, this is equivalent to putting an admissible filtration on their Dieudonné modules, M and N. But $Fil^1_{Hodge}M_0 = 0$ so the only possible admissible filtration on M is $Fil^1_{Hodge}M = 0$. Similarly, $Fil^1_{Hodge}N_0 = N_0$ so the only admissible filtration on N is $Fil^1_{Hodge}N = N$. Thus $\mathcal{P}_{\mathbb{F},conn}$ and $\mathcal{P}_{\mathbb{F},et}$ have unique liftings to W, which we denote by $\mathcal{P}_{conn}$ and $\mathcal{P}_{et}$ respectively.

Let $\mathcal{P}_{can} = \mathcal{P}_{conn} \oplus \mathcal{P}_{et}$. This is clearly a lifting of $\mathcal{P}_{\mathbb{F}}$ with $\mathbb{D}(\mathcal{P}_{\mathbb{F}}) = Q(2)$ and with $Fil^1_{Hodge}Q(2) = N$. On the other hand, by construction we have $Fil^1_{slope}Q(2) = N$, so the Hodge and slope filtrations coincide on $Q(2)$. Since the Dieudonné functor is an equivalence of categories, this is the only lifting of $\mathcal{P}_{\mathbb{F}}$ with this property.

Since the period map of Theorem 4.5 is an isomorphism, $\mathcal{P}_{can}$ corresponds to a unique cubic fourfold over W, which we denote by X_{can}. By construction, it has the property that $Fil^3_{Hodge}H^4_{DR}(X_{can}/W) = Fil^3_{slope}H^4_{DR}(X_{can}/W)$. But now $Fil^2_{Hodge}H^4_{DR}(X_{can}/W)$ is the orthogonal complement of $Fil^3_{Hodge}H^4_{DR}(X_{can}/W)$ in $H^4_{DR}(X_{can}/W)$ under the cup product pairing, and $Fil^2_{slope}H^4_{DR}(X_{can}/W)$ is the orthogonal complement to $Fil^3_{slope}H^4_{DR}(X_{can}/W)$ in $H^4_{DR}(X_{can}/W)$ under the cup product pairing, so we have

$$Fil^2_{slope}H^4_{DR}(X_{can}/W) = Fil^2_{Hodge}H^4_{DR}(X_{can}/W).$$

Since both filtrations are three step filtrations with

$$Fil^1_{slope}H^4_{DR}(X_{can}/W) = Fil^1_{Hodge}H^4_{DR}(X_{can}/W) = H^4_{DR}(X_{can}/W)$$

we are done. □

Remark 4.9. We can get a little more information out of our construction. One can show that $\mathcal{P}_{\mathbb{F},conn} \cong \widehat{\mathbb{G}}^{20}_{m,\mathbb{F}}$ and $\mathcal{P}_{\mathbb{F},et} = \mathbb{Q}_p/\mathbb{Z}_p$. Both of these groups are rigid (i.e. they have no nontrivial deformations), so the deformations of $\mathcal{P}_{can}$ are entirely determined by the possible extensions between them. This gives a canonical group structure on $\mathcal{T}$, and hence on S. The canonical lift X_{can} then corresponds to the unit of this group structure, i.e the split extension.

Remark 4.10. There is analogous notion of a canonical liftings for Abelian varieties. Suppose A is an ordinary Abelian variety over Spec$\mathbb{Z}/p\mathbb{Z}$. There is then a unique lifting A_{can} of A to Spec$W(\mathbb{Z}/p\mathbb{Z})$ which has the property that the Frobenius map on $H^1_{DR}(A_{can}/W)$ preserves the Hodge filtration (c.f. Katz's appendix to [DI]). By a result of Messing [M, Appendix, cor. 1.2], this is equivalent to having a lifting of the Frobenius endomorphism to A_{can}. It is interesting to note that this can never happen for cubic fourfolds.

THEOREM 4.11. *Let $X_{\mathbb{F}}$ be a cubic fourfold over a perfect field $\mathbb{F}$ and let X be a lifting of $X_{\mathbb{F}}$ to W. Then the Frobenius map on $X_{\mathbb{F}}$ admits no lifting to a map on X.*

Proof. By (9) and the isomorphism $T_{X_{\mathbb{F}}} \cong \Omega_{X_{\mathbb{F}}}{}^3(3)$ we see that $H^1(X, \Omega_{X_{\mathbb{F}}}{}^3(3)) \neq 0$. So X does not satisfy Bott Vanishing. Thus by [BT, theorem 3] Frobenius does not lift.

Remark 4.12. It would follow from the Hodge conjecture for self products of cubic fourfolds and Theorem 6.3, that the Frobenius map does lift to a correspondence (up to homolgical equivalence) on $X_{can} \times X_{can}$. Since any lifting of $X_{\mathbb{F}}$ with this property would also have the property that the Hodge and slope filtrations agree, this is enough to characterize the cannonical lifting. In any case, one can show (using [A, 1.5.3]) that the Frobenius map lifts to a 'motivated', hence absolute Hodge, cycle on $X_{can} \times X_{can}$.

Remark 4.13. Period maps analogous to the one constructed here can be constructed under quite general circumstances. They exist for any ordinary variety over a perfect field whose deformation space is sufficiently nice (smooth, for example). In particular, one can perform this construction on ordinary $K3$ (where the period map is again an isomorphism) surfaces and construct the canonical lift of [DI, N] using these techniques.

5. The Kuga–Satake–Deligne Abelian Variety

Let A be an integral domain and let V be a free A-module with a non-degenerate quadratic form q. Let $C(V)$ denote the Clifford algebra of V constructed with respect to the quadratic form q and let $C^+(V)$ denote the even part of the Clifford algebra. Let $CSpin(V)$ denote the Clifford group of V. This is the subgroup of invertible elements g in $C^+(V)$ with the property that $gVg^{-1} = V$, where V is regarded as a subspace of $C(V)$ via the natural inclusion.

We have two representations of $CSpin(V)$ on $C(V)$, one given by multiplication on the left, which we will denote by $C(V)_s$, and one given by conjugation, which we will denote by $C(V)_{ad}$. Right multiplication is compatible with the action of $CSpin(V)$ on $C(V)_s$ so we get the following isomorphism of representations:

THEOREM 5.1. *Let* $\mathcal{A} = C(V)$. *Then* $C(V)_{ad} \cong \mathrm{End}_{\mathcal{A}}(C(V)_s)$ *where* $C(V)_s$ *is regarded as a right* $\mathcal{A}$*-module.*

Proof. Send $x \in C(V)_{ad}$ to the element $L_x \in \mathrm{End}_{C(V)}(C(V)_s)$ corresponding to left multiplication by x. □

If A is a field, then the action of $CSpin(V)$ by conjugation on V induces a short exact sequence

$$1 \to \mathbb{G}_m \xrightarrow{w} CSpin(V) \xrightarrow{m} SO(V, q) \to 1 \tag{28}$$

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and the spinor norm $N\colon CSpin(V) \to \mathbb{G}_m$ induces a map

$$1 \to Spin(V) \to CSpin(V) \xrightarrow{t} \mathbb{G}_m \to 1, \tag{29}$$

where $t(x) = \frac{1}{N(x)}$.

Now assume V is a free $\mathbb{Z}$-module of rank $n+2$ underlying a polarized $\mathbb{Z}$-Hodge structure of weight 0 such that:

(1) The Hodge structure on V is of type $\{(-1,1),(0,0),(1,-1)\}$.
(2) The index of the bilinear form underlying the polarization is $(n+, 2-)$

Let $\mathbb{S}$ be the group $\mathbb{C}^\times$ regarded as a real algebraic group. We have maps $w\colon \mathbb{G}_m \to \mathbb{S}$ given by the natural inclusion and $t\colon \mathbb{S} \to \mathbb{G}_m$ given by the inverse to the norm. The Hodge structure on V then gives a map $h\colon \mathbb{S} \to SO(V, q)$.
We have the following theorem of Deligne [D, 4.2–4.3]:

THEOREM 5.2 (Kuga–Satake–Deligne). *For V satisfying the above conditions, there is a unique map $\rho\colon \mathbb{S} \to CSpin(V_\mathbb{R})$ such that $m \circ \rho = h$ and such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathbb{G}_m & \xrightarrow{w} & \mathbb{S} & \xrightarrow{w} & \mathbb{G}_m \\ \| & & \downarrow{\rho} & & \| \\ \mathbb{G}_m & \xrightarrow{w} & CSpin(V_\mathbb{R}) & \xrightarrow{t} & \mathbb{G}_m \end{array} \tag{30}$$

(*Note: m is the map defined in* (28).)

The map ρ then induces a polarizable Hodge structure of type $\{(0,1),(1,0)\}$ on $C(V)_s$, which in turn defines a complex Abelian variety A of dimension 2^{n+1} (called the Kuga–Satake Abelian variety) with the property that $H^1(A,\mathbb{Z}) = C(V)_s$ [D, 2.8]. Substituting in (5.1) then gives the following isomorphism of Hodge structures:

$$C(V)_{ad} \cong \bigoplus_{i=0}^{2n+2} \bigwedge^i V \cong \mathrm{End}_{\mathcal{A}}(H^1(A,\mathbb{Z})). \tag{31}$$

(We again take $\mathcal{A} = C(V)$.) Furthermore, right multiplication of $C(V)_s$ by $\mathcal{A}$ preserves the Hodge structure, which gives $\mathcal{A} \subset \mathrm{End}(A)$.

Remark 5.3. The Abelian variety constructed here is actually isogenous to the product of two copies of the Abelian variety constructed by Deligne (see [A, 4.1.3])

6. The Tate Conjecture

Let X_0 be an ordinary cubic fourfold over the finite field $\mathbb{F}_q$, and X_{can} is its canonical lift to W. Let K_0 be the fraction field of W and fix an embedding $K_0 \hookrightarrow \mathbb{C}$ with the

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property that $\overline{K_0} = \mathbb{C}$. This embedding lets us associate a complex cubic fourfold $X_{\mathbb{C}}$ to X_{can}. We can then apply the results of the previous section to get an Abelian variety A_{can} with the property that $H^1(A_{can}, \mathbb{Q}) \cong CP^4(X_{\mathbb{C}}, \mathbb{Q}(2))$ as $\mathbb{Q}$-vector spaces. Furthermore, our choice of embedding enables us to assume that A_{can} is defined over some finite extension K of K_0.

Let $\mathcal{A} = CP^4(X_{\mathbb{C}}, \mathbb{Q}(2))$ and $\mathcal{A}_\ell = CP^4(X_{\mathbb{C}}, \mathbb{Q}(2)) \otimes \mathbb{Q}_\ell$. Applying the results of the previous section gives us isomorphisms

$$\oplus \bigwedge^i P^4(X_{\mathbb{C}}, \mathbb{Q}(2)) \cong \mathrm{End}_{\mathcal{A}}(H^1(A_{can}, \mathbb{Q}))) \subset H^2(A_{can} \times A_{can}, \mathbb{Q}) \tag{32}$$

as Hodge structures. This isomorphism is actually induced by an absolute Hodge cycle (see [A, 1.7.1 and 4.1.3]), which gives the isomorphism of $Gal(\overline{K}/K)$-modules:

$$\oplus \bigwedge^i P^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_\ell(2)) \cong \mathrm{End}_{\mathcal{A}_\ell}(H^1_{et}(A_{can}, \mathbb{Q}_\ell)) \subset H^2_{et}(A_{can} \times A_{can}, \mathbb{Q}_\ell) \tag{33}$$

after possibly replacing K by a finite extension.

THEOREM 6.1 (Deligne). *A_{can} has potentially good reduction.*

Proof. Let $I \subset Gal(\overline{K}/K)$ be the inertia subgroup and choose ℓ to be prime to the characteristic of the residue field of K. Since X_0 is smooth, we know (by Grothendieck's monodromy theorem) that I acts trivially on $P^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_\ell(2))$. The isomorphism (33) then implies that I acts trivially on $\mathrm{End}_{\mathcal{A}_\ell}(H^1_{et}(A, \mathbb{Q}_\ell))$.

Since $\mathcal{A}_\ell \subset \mathrm{End}_K(A_{can})$ (see [A, 6.51]), the action of $\mathcal{A}_\ell$ on $H^1_{et}(A_{can}, \mathbb{Q}_\ell)$ commutes with the Galois action. So each element $g \in I$ is an element in $\mathrm{End}_{\mathcal{A}_\ell}(H^1_{et}(A, \mathbb{Q}_\ell))$ and hence can be regarded as acting on $H^1_{et}(A_{can}, \mathbb{Q}_\ell)$ by left multiplication by an element of $CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_\ell(2))$. Since I acts trivially on $\mathrm{End}_{\mathcal{A}_\ell}(H^1_{et}(A, \mathbb{Q}_\ell))$, we can even assume that g acts by an element in the center of $CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_\ell(2))$.

But Grothendieck's monodromy theorem says that there is a subgroup of finite index $I_1 \subset I$ such that the action of I_1 on $H^1_{et}(A_{can}, \mathbb{Q}_\ell)$ is unipotent. Since the center of $CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_\ell(2))$ is a field and thus contains no non-trivial unipotent elements, it follows that I_1 must act trivially on $H^1_{et}(A_{can}, \mathbb{Q}_\ell)$. Thus we can replace K by a finite extension which will insure that the action of the inertia group on $H^1_{et}(A_{can}, \mathbb{Q}_\ell)$ is trivial, which is enough to insure that A_{can} has good reduction. □

Assume K is large enough to insure that A_{can} has good reduction and let A_0 denote the reduction.

Remark 6.2. Using [A, 9.3.1], one can show that A_{can} has good reduction, not just potentially good reduction.

THEOREM 6.3. $\mathrm{End}(A_0)\otimes\mathbb{Q} \subset \mathrm{End}(A_{can})\otimes\mathbb{Q}$

Proof. Let $CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2)) = \mathcal{A}_p$, where $p = \mathrm{char}\,\mathbb{F}_q$. $P^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2))$ is even dimensional, so $CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2))$ is a simple ring. Thus, there is a unique simple left

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$CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2))$-module M up to isomorphism. The structure theorem for simple rings then implies that there exists a division algebra $D \subset CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2))$ such that

$$CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2)) \cong \mathrm{End}_D(M)$$

as $\mathbb{Q}_p$-algebras and

$$CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2)) \cong \oplus M$$

as left $CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2))$-modules. Substituting in (5.1) then gives the isomorphism

$$\mathrm{End}_D(M) \cong \mathrm{End}_{\mathcal{A}_p}(\oplus M), \tag{34}$$

where the map is given by $g \to \oplus g$. If we then use the isomorphism $H^1_{et}(A_{can}, \mathbb{Q}_p) \cong CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2))$ the above equation becomes

$$\mathrm{End}_D(M) \cong \mathrm{End}_{\mathcal{A}_p}(H^1_{et}(A_{can}, \mathbb{Q}_p)). \tag{35}$$

Now suppose $g \in Gal(\overline{K}/K)$. $\mathcal{A}_p \subset \mathrm{End}_K(A_{can})$ so the action of g on $H^1_{et}(A_{can}, \mathbb{Q}_p)$ commutes with the action of $\mathcal{A}_p$ and thus gives rise to an element $g_M \in \mathrm{End}_D(M)$. This gives a Galois action (via conjugation) on $\mathrm{End}_D(M)$ which makes (35) an isomorphism of Galois modules. The diagonal map

$$\Delta\colon \mathrm{End}_D(M) \hookrightarrow \mathrm{End}(\oplus M) \tag{36}$$

given by $g_M \to \oplus g_M$ then induces a Galois module structure on $\mathrm{End}(\oplus M)$ which makes the square:

$$\begin{array}{ccc} \mathrm{End}_D(M) & \xrightarrow{\sim} & \mathrm{End}_{\mathcal{A}_p}(H^1_{et}(A_{can}, \mathbb{Q}_p)) \\ \Delta\downarrow & & \downarrow \\ \mathrm{End}(\oplus M) & \xrightarrow{\sim} & \mathrm{End}(H^1_{et}(A_{can}, \mathbb{Q}_p)) \end{array} \tag{37}$$

a commutative diagram of Galois modules.

This tells us that

$$\mathrm{End}(H^1_{et}(A_{can}, \mathbb{Q}_p)) \cong \mathrm{End}(\oplus M) \cong \oplus \oplus \mathrm{End}(M) \cong \oplus CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2))$$

as Galois modules. But we also know that

$$\mathrm{End}(H^1_{et}(A_{can}, \mathbb{Q}_p)) \cong H^1_{et}(A_{can}, \mathbb{Q}_p) \otimes H^1_{et}(A_{can}, \mathbb{Q}_p)(1)$$

and

$$CP^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2)) \cong \bigoplus_i \wedge^i P^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2))$$

as Galois modules so we get the isomorphism

$$H^1_{et}(A_{can}, \mathbb{Q}_p) \otimes H^1_{et}(A_{can}, \mathbb{Q}_p)(1) \cong \oplus \bigoplus_i \wedge^i P^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_p(2)). \tag{38}$$

Recall that Fontaine has constructed a functor from a subcategory of the category of

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p-adic $Gal(\overline{K}/K)$-representations to a subcategory of the category of filtered K-vector spaces endowed with a Frobenius action. When this functor is applied to the étale cohomology of an algebraic variety Y over K with good reduction, its output is the de Rham cohomology of Y with the induced Frobenius action. This functor is compatible with direct sums and all tensor constructions, so when it is applied to (38) we get

$$H^1_{DR}(A_{can}/K) \otimes H^1_{DR}(A_{can}/K)(1) \cong \oplus \bigoplus_i \wedge^i H^4_{DR}(X_{can} \otimes K/K)(2). \tag{39}$$

This isomorphism preserves the natural filtrations on both sides and the natural Frobenius action induced by crystalline cohomology.

Since we know that $H^4_{DR}(X_{can} \otimes K/K)(2)$ has the property that the Hodge filtration coincides with the slope filtration, we must have that the same property holds for $\oplus \bigoplus_i \wedge^i H^4_{DR}(X_{can} \otimes K/K)(2)$ and hence for

$$H^1_{DR}(A_{can}/K) \otimes H^1_{DR}(A_{can}/K)(1) \subset H^2_{DR}(A_{can} \times A_{can}/K)(1).$$

But now every endomorphism of A_0 gives rise to an element in

$$Fil^1_{slope} H^2_{cris}(A_{can} \times A_{can}/W)$$

(Frobenius acts by multiplication by p), so it must live in

$$F^1_{Hodge} H^2_{DR}(A_{can} \times A_{can}/K).$$

Since every cycle on $A_0 \times A_0$ whose image in $H^2_{DR}(A_{can} \times A_{can}/K)$ is in

$$F^1_{Hodge} H^2_{dR}(A_{can} \times A_{can}/K)$$

has some multiple that lifts to $A_{can} \times A_{can}$ by [BO2, 3.8], we have proven the theorem. □

COROLLARY 6.4. *A_{can} is of CM-type.*

THEOREM 6.5. *The Tate conjecture holds for ordinary cubic fourfolds over $\mathbb{F}_q$*

Proof. As a consequence of the Tate and Hodge conjectures for Abelian varieties we have

$$\mathrm{End}(A_{can}) \otimes \mathbb{Q} \cong \mathrm{End}(H^1(A_{can}, \mathbb{Q}))^{(0,0)}, \tag{40}$$

$$\begin{aligned} \mathrm{End}(A_0) \otimes \mathbb{Q}_l &\cong \mathrm{End}(H^1_{et}(A_0, \mathbb{Q}_l))^{Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)} \\ &\cong \mathrm{End}(H^1_{et}(A_{can}, \mathbb{Q}_\ell))^{Gal(\overline{K}/K)} \end{aligned} \tag{41}$$

where the endomorphisms on the right-hand side are those preserving the Hodge structure and the Galois-module structure respectively.

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So the previous theorem tells us that

$$\mathrm{End}(H^1_{et}(A_{can}, \mathbb{Q}_\ell))^{Gal(\overline{K}/K)} \subset \mathrm{End}(H^1(A_{can}, \mathbb{Q}))^{(0,0)} \otimes \mathbb{Q}_\ell.$$

So every Galois invariant class in $\mathrm{End}(H^1_{et}(A_{can}, \mathbb{Q}_\ell))$ comes from a Hodge class in $\mathrm{End}(H^1(A_{can}, \mathbb{Q}))$. But we have

$$P^4(X_{\mathbb{C}}, \mathbb{Q}(2)) \hookrightarrow \mathrm{End}(H^1(A_{can}, \mathbb{Q})) \tag{42}$$

$$P^4_{et}(X_{\mathbb{C}}, \mathbb{Q}_\ell(2)) \hookrightarrow \mathrm{End}(H^1_e t(A_{can}, \mathbb{Q}_\ell)) \tag{43}$$

which preserve the appropriate structures. Thus since all of the Galois invariant classes of $\mathrm{End}(H^1_{et}(A_{can}, \mathbb{Q}_\ell))$ come from Hodge classes, the same must be true for $P^4(X_{\mathbb{C}}, \mathbb{Q}(2))$. But we know that the Hodge conjecture is true for cubic fourfolds [Z], so these classes must correspond to algebraic cycles. □

Remark 6.6. The proof of the Tate conjecture in this section only requires the existence of a canonical lifting of the variety from $\mathbb{F}_q$ to $W(\mathbb{F}_q)$, the existence of a suitable Kuga–Satake variety (see [A] for other examples of varieties with this property), and the truth of the Hodge conjecture for the lifted variety. In particular, the same proof works for ordinary K3 surfaces. Theorem 6.5 also implies the Tate conjecture for divisors on the Fano variety of a ordinary cubic fourfold (see, for instance, [A]).

Remark 6.7. The Tate conjecture for cubic fourfolds over a finitely generated field of characteristic zero was proved in [A].

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