Proceedings of the Edinburgh Mathematical Society (2005) **48**, 485–498 ©
DOI:10.1017/S0013091503000993 Printed in the United Kingdom

ON THE BEHAVIOUR OF THE OSCILLATORY SOLUTIONS OF SECOND-ORDER LINEAR UNSTABLE TYPE DELAY DIFFERENTIAL EQUATIONS

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(Received 19 November 2003)

Abstract Second-order linear (non-autonomous as well as autonomous) delay differential equations of unstable type are considered. In the non-autonomous case, sufficient conditions are given in order that all oscillatory solutions are bounded or all oscillatory solutions tend to zero at ∞. In the case where the equations are autonomous, necessary and sufficient conditions are established for all oscillatory solutions to be bounded or all oscillatory solutions to tend to zero at ∞.

Keywords: delay differential equation; oscillatory solutions; behaviour of oscillatory solutions; boundedness; asymptotic decay

2000 *Mathematics subject classification:* Primary 34K11; 34K12; 34K20; 34K25

1. Introduction

This paper deals with the boundedness and the asymptotic decay of the oscillatory solutions of second-order linear (non-autonomous or autonomous) delay differential equations of unstable type. The oscillation theory of delay differential equations has developed very rapidly in the last few decades. For this theory, we choose to refer to the books by Erbe, Kong and Zhang [**3**], Gopalsamy [**4**] and Györi and Ladas [**6**]. For the general theory of delay differential equations, the reader is referred to the books by Diekmann *et al.* [**1**], Driver [**2**] and Hale and Verduyn Lunel [**7**].

First of all, let us consider the first-order linear *autonomous* delay differential equations:

$$x'(t) + px(t-\tau) = 0 \tag{1.1}$$

and

$$x'(t) = px(t-\tau), \tag{1.2}$$

where p *and* τ *are positive constants.*

In spite of the fact that the solutions of the above autonomous delay differential equations are defined for all $t \in \mathbb{R}$, here we will confine our attention only to solutions on $[t_0, \infty)$, for $t_0 \in \mathbb{R}$, of (1.1) or (1.2). For any $t_0 \in \mathbb{R}$, by a *solution on* $[t_0, \infty)$ of (1.1) (respectively, of (1.2)) we mean a continuous real-valued function x defined on the interval $[t_0-\tau,\infty)$, which is continuously differentiable on $[t_0,\infty)$ and satisfies (1.1) (respectively, (1.2)) for every $t \geqslant t_0$.

For the delay differential equation (1.1), the following result is well known (see, for example, [**2**, **4**, **8**, **10**]).

(I) If $p\tau < \frac{1}{2}\pi$, then all solutions of (1.1) tend to zero at ∞.

(II) If $p\tau = \frac{1}{2}\pi$, then all solutions of (1.1) are bounded and, moreover, (1.1) has a bounded oscillatory solution that does not tend to zero at ∞.

(III) If $p\tau > \frac{1}{2}\pi$, then (1.1) has an unbounded oscillatory solution.

It is clear that *every non-oscillatory solution of (1.1) is always bounded*. Furthermore, it is not difficult to show that *all non-oscillatory solutions of (1.1) tend to zero at* ∞. So, we can easily verify that the above result may equivalently be formulated as follows.

(i) All solutions of (1.1) are bounded if and only if $p\tau \leqslant \frac{1}{2}\pi$.

(ii) All solutions of (1.1) tend to zero at ∞ if and only if $p\tau < \frac{1}{2}\pi$.

The result that the condition $p\tau < \frac{1}{2}\pi$ is sufficient for all solutions of (1.1) to tend to zero at ∞ has been extended, for the more general case of first-order linear non-autonomous delay differential equations, by Ladas, Sficas and Stavroulakis [**9**].

Györi [**5**] studied the existence and the growth of oscillatory solutions for first-order linear delay differential equations of unstable type. In particular, for the autonomous delay differential equation (1.2), Györi [**5**, Theorem 4.2] proved the next result.

(I) If $p\tau < \frac{3}{2}\pi$, then all oscillatory solutions of (1.2) tend to zero at ∞.

(II) If $p\tau = \frac{3}{2}\pi$, then all oscillatory solutions of (1.2) are bounded and, moreover, (1.2) has a bounded oscillatory solution that does not tend to zero at ∞.

(III) If $p\tau > \frac{3}{2}\pi$, then (1.2) has an unbounded oscillatory solution.

It is easy to check that this result can equivalently be formulated as follows.

(i) All oscillatory solutions of (1.2) are bounded if and only if $p\tau \leqslant \frac{3}{2}\pi$.

(ii) All oscillatory solutions of (1.2) tend to zero at ∞ if and only if $p\tau < \frac{3}{2}\pi$.

Note that the preceding results for the autonomous delay differential equations (1.1) and (1.2) are obtained by the study of the roots of their characteristic equations.

Our work in this paper is motivated by the results for (1.1) and (1.2) mentioned previously, and especially by the one due to Györi [**5**] for the *first-order* linear autonomous delay differential equation of unstable type (1.2). Here, we consider *second-order* linear (autonomous as well as non-autonomous) delay differential equations of unstable type and we study the boundedness and the asymptotic decay of the oscillatory solutions.

Consider the second-order linear delay differential equation

$$x''(t) = p(t)x(t - \tau), \tag{1.3}$$

where p *is a positive continuous real-valued function on the interval* $[0, \infty)$, *and* τ *is a positive constant.*

Let $t_0 \geqslant 0$. By a *solution on* $[t_0, \infty)$ of the delay differential equation (1.3) we mean a continuous real-valued function x defined on the interval $[t_0 - \tau, \infty)$, which is twice continuously differentiable on $[t_0, \infty)$ and satisfies (1.3) for all $t \geqslant t_0$.

Consider also the second-order linear *autonomous* delay differential equation

$$x''(t) = px(t - \tau), \tag{1.4}$$

where p *and* τ *are positive constants.*

It is well known that every solution of the autonomous delay differential equation (1.4) is defined on the whole real line. But, in what follows, we will restrict ourselves to considering only solutions on $[t_0, \infty)$, for $t_0 \in \mathbb{R}$, of (1.4). If t_0 is any real number, by a *solution on* $[t_0, \infty)$ of (1.4) we mean a continuous real-valued function x defined on $[t_0 - \tau, \infty)$, which is twice continuously differentiable on $[t_0, \infty)$ and satisfies (1.4) for every $t \geqslant t_0$.

With the differential equation (1.4) we associate its *characteristic equation*

$$\lambda^2 = p\mathrm{e}^{-\lambda\tau}. \tag{1.5}$$

As usual, a continuous real-valued function defined on an interval $[T, \infty)$ is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is said to be *non-oscillatory*.

For the delay differential equation (1.3) we will establish a sufficient condition for all oscillatory solutions to be bounded together with their first-order derivatives as well as a condition under which all oscillatory solutions tend to zero at ∞ together with their first-order derivatives. Moreover, in the case of the autonomous delay differential equation (1.4), we will prove that $\sqrt{p}\tau \leqslant \pi$ is a necessary and sufficient condition for all oscillatory solutions to be bounded and that $\sqrt{p}\tau < \pi$ is a necessary and sufficient condition in order that all oscillatory solutions tend to zero at ∞. Our results and their proofs in the case of the autonomous delay differential equation (1.4) are motivated by the analogous results and their proofs given by Györi [**5**] for the first-order linear autonomous delay differential equation (1.2). The main results of the paper and some related comments are presented in § 2. The proofs of the main results are given in § 3.

2. Statement of the main results and comments

The main results of this paper are Propositions 2.1 and 2.2, Theorems 2.3 and 2.4, Proposition 2.5 and Theorem 2.6 below. Proposition 2.1 establishes that the boundedness of an oscillatory solution of (1.3) is implied by the boundedness of its derivative and that an oscillatory solution of (1.3) tends to zero at ∞ if its derivative tends to zero at ∞. Proposition 2.2 gives a condition under which, for any oscillatory solution of (1.3), the boundedness of the solution implies the boundedness of its derivative, and the convergence to zero at ∞ of the solution ensures the same for its derivative. Theorem 2.3 provides a sufficient condition for all oscillatory solutions of (1.3) to be bounded together with their derivatives of first order. Theorem 2.4 establishes a condition, which is sufficient for all oscillatory solutions of (1.3) to tend to zero at ∞ together with their first-order derivatives. Proposition 2.5 and Theorem 2.6 concern the autonomous case, i.e. the case of the delay differential equation (1.4). Proposition 2.5 gives necessary and sufficient conditions, via the roots of the characteristic equation (1.5) of (1.4), for all oscillatory solutions of (1.4) to be bounded or for all oscillatory solutions of (1.4) to tend to zero at ∞. Theorem 2.6 provides sufficient conditions, on the coefficient p and the delay τ, in order that all oscillatory solutions of (1.4) tend to zero at ∞ or in order that all oscillatory solutions of (1.4) are bounded and (1.4) has a bounded oscillatory solution which does not tend to zero at ∞ or in order that (1.4) has an unbounded oscillatory solution. Note that Theorem 2.6 establishes a necessary and sufficient condition (on the coefficient p and the delay τ) for all oscillatory solutions of (1.4) to be bounded and also a necessary and sufficient condition (on p and τ) for all oscillatory solutions of (1.4) to tend to zero at ∞.

Proposition 2.1. *For any oscillatory solution x of the delay differential equation (1.3), the following statements hold.*

(i) *If x' is bounded, then x is also bounded.*

(ii) *If $\lim_{t\to\infty} x'(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$.*

Proposition 2.2. *Suppose that*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,\mathrm{d}s < \infty. \tag{2.1}$$

Then, for any oscillatory solution x of the delay differential equation (1.3), the following statements hold.

(i) *If x is bounded, then x' is also bounded.*

(ii) *If $\lim_{t\to\infty} x(t) = 0$, then $\lim_{t\to\infty} x'(t) = 0$.*

In the special case of the autonomous delay differential equation (1.4), condition (2.1) holds by itself. So, for the autonomous case, Propositions 2.1 and 2.2 can be unified in the following result.

For any oscillatory solution x of the autonomous delay differential equation (1.4), the following statements hold.

(i) *x is bounded if and only if x' is bounded.*

(ii) *$\lim_{t\to\infty} x(t) = 0$ if and only if $\lim_{t\to\infty} x'(t) = 0$.*

Theorem 2.3. *Suppose that*

$$\int_{t-\tau}^{t} (t-s)p(s)\,\mathrm{d}s \leqslant 1 \quad \textit{for all large } t. \tag{2.2}$$

Then all oscillatory solutions of the delay differential equation (1.3) are bounded together with their first-order derivatives.

Theorem 2.4. *Suppose that*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} (t-s)p(s)\,\mathrm{d}s < 1. \tag{2.3}$$

Then all oscillatory solutions of the delay differential equation (1.3) tend to zero at ∞ together with their first-order derivatives.

For the special case of the autonomous delay differential equation (1.4), Theorem 2.3 establishes that the condition $\sqrt{p}\tau \leqslant \sqrt{2}$ is sufficient for all oscillatory solutions to be bounded, and Theorem 2.4 guarantees that $\sqrt{p}\tau < \sqrt{2}$ is a sufficient condition in order that all oscillatory solutions tend to zero at ∞. These conditions are not sharp. Indeed, as is established by Theorem 2.6 below, a necessary and sufficient condition for all oscillatory solutions of (1.4) to be bounded is that $\sqrt{p}\tau \leqslant \pi$ and also a necessary and sufficient condition for all oscillatory solutions of (1.4) to tend to zero at ∞ is $\sqrt{p}\tau < \pi$.

Proposition 2.5 below is needed for the proof of Theorem 2.6. But, this proposition is also interesting by itself.

Proposition 2.5.

(i) *All oscillatory solutions of (1.4) are bounded if and only if for any root $\lambda = \mu + \mathrm{i}\nu$ of (1.5) we have $\mu \leqslant 0$ whenever $\nu > 0$.*

(ii) *All oscillatory solutions of (1.4) tend to zero at ∞ if and only if for any root $\lambda = \mu + \mathrm{i}\nu$ of (1.5) we have $\mu < 0$ whenever $\nu > 0$.*

From parts (i) and (ii) of Proposition 2.5 we can easily obtain the following result.

All oscillatory solutions of (1.4) are bounded and (1.4) has a bounded oscillatory solution which does not tend to zero at ∞ if and only if for any root $\lambda = \mu + \mathrm{i}\nu$ of (1.5) we have $\mu \leqslant 0$ whenever $\nu > 0$ and (1.5) has a root $\lambda = \mathrm{i}\nu$ with $\nu > 0$.

Theorem 2.6.

(I) *If $\sqrt{p}\tau < \pi$, then all oscillatory solutions of (1.4) tend to zero at ∞.*

(II) *If $\sqrt{p}\tau = \pi$, then all oscillatory solutions of (1.4) are bounded and, moreover, (1.4) has a bounded oscillatory solution that does not tend to zero at ∞.*

(III) *If $\sqrt{p}\tau > \pi$, then (1.4) has an unbounded oscillatory solution.*

It is not difficult to see that Theorem 2.6 can equivalently be formulated as follows.

(i) *All oscillatory solutions of (1.4) are bounded if and only if* $\sqrt{p}\tau \leqslant \pi$.

(ii) *All oscillatory solutions of (1.4) tend to zero at* ∞ *if and only if* $\sqrt{p}\tau < \pi$.

Before closing this section, let us introduce the conditions

$$\int_{t-\tau}^{t} (t-s)p(s)\,\mathrm{d}s \leqslant \tfrac{1}{2}\pi^2 \quad \text{for all large } t \tag{2.4}$$

and

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} (t-s)p(s)\,\mathrm{d}s < \tfrac{1}{2}\pi^2, \tag{2.5}$$

which are weaker than (2.2) and (2.3), respectively. In the case of the autonomous delay differential equation (1.4), conditions (2.4) and (2.5) take the forms $\sqrt{p}\tau \leqslant \pi$ and $\sqrt{p}\tau < \pi$, respectively. So, it is an interesting problem to examine if we can establish the conclusions of Theorems 2.3 and 2.4 with the conditions (2.4) and (2.5) in place of (2.2) and (2.3), respectively.

3. Proofs of the main results

Proof of Proposition 2.1. Let x be an oscillatory solution on an interval $[t_0, \infty)$, $t_0 \geqslant 0$, of the delay differential equation (1.3). If the solution x is eventually identically zero, then x is bounded and $\lim_{t\to\infty} x(t) = 0$. So, we may (and do) assume that x is not eventually identically zero. Now, let us consider a sequence $(t_\nu)_{\nu\geqslant 1}$ of zeros of the solution x with $t_0 + \tau \leqslant t_1 < t_2 < \cdots$ and $\lim_{\nu\to\infty} t_\nu = \infty$, and such that x is not identically zero on $[t_\nu, t_{\nu+1}]$ for any $\nu \in \{1, 2, \dots\}$. Clearly, the proof of statements (i) and (ii) can be accomplished by showing that

$$\max_{t_\nu \leqslant t \leqslant t_{\nu+1}} |x(t)| \leqslant \tau \max_{t_\nu - \tau \leqslant t \leqslant t_{\nu+1}} |x'(t)| \quad (\nu = 1, 2, \dots).$$

To prove this fact, let us consider an arbitrary $\nu \in \{1, 2, \dots\}$. Let $T_\nu \in (t_\nu, t_{\nu+1})$ be such that

$$|x(T_\nu)| = \max_{t_\nu \leqslant t \leqslant t_{\nu+1}} |x(t)|.$$

Obviously, $x(T_\nu) \neq 0$. Furthermore, as the negative of a solution of (1.3) is also a solution of the same equation, we can restrict ourselves only to the case where $x(T_\nu) > 0$. Since x has a maximum at T_ν, we must have $x'(T_\nu) = 0$ and $x''(T_\nu) \leqslant 0$. By the last inequality, from (1.3) it follows that $x(T_\nu - \tau) \leqslant 0$. This inequality together with $x(T_\nu) > 0$ imply the existence of a point $T_\nu^* \in [T_\nu - \tau, T_\nu)$ such that $x(T_\nu^*) = 0$. Hence, we obtain

$$\begin{aligned} x(T_\nu) = \int_{T_\nu^*}^{T_\nu} x'(s)\,\mathrm{d}s &\leqslant (T_\nu - T_\nu^*) \max_{T_\nu^* \leqslant t \leqslant T_\nu} |x'(t)| \\ &\leqslant \tau \max_{T_\nu - \tau \leqslant t \leqslant T_\nu} |x'(t)| \leqslant \tau \max_{t_\nu - \tau \leqslant t \leqslant t_{\nu+1}} |x'(t)|, \end{aligned}$$

and so the proof is complete. □

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Proof of Proposition 2.2. By hypothesis (2.1), we have

$$\int_{t-\tau}^{t} p(s)\,\mathrm{d}s \leqslant K \quad \text{for all } t \geqslant \tau,$$

where K is a positive constant.

Let us consider an oscillatory solution x on an interval $[t_0, \infty)$, $t_0 \geqslant 0$, of the differential equation (1.3). Obviously, x' is also oscillatory. In the case where x' is eventually identically zero, x' is bounded and $\lim_{t\to\infty} x'(t) = 0$. Thus, we can suppose that x' is not eventually identically zero. Next, we choose a sequence $(t_\nu)_{\nu=1,2,\dots}$ of zeros of x' with $t_0 + \tau \leqslant t_1 < t_2 < \cdots$ and $\lim_{\nu\to\infty} t_\nu = \infty$, such that x' is not identically zero on any interval $[t_\nu, t_{\nu+1}]$ ($\nu = 1, 2, \dots$). In order to establish statements (i) and (ii), it suffices to show that

$$\max_{t_\nu \leqslant t \leqslant t_{\nu+1}} |x'(t)| \leqslant K \max_{t_\nu - 2\tau \leqslant t \leqslant t_{\nu+1} - \tau} |x(t)| \quad (\nu = 1, 2, \dots).$$

To this end, let ν be an arbitrary positive integer. Consider a point $T_\nu \in (t_\nu, t_{\nu+1})$ such that

$$|x'(T_\nu)| = \max_{t_\nu \leqslant t \leqslant t_{\nu+1}} |x'(t)|.$$

It is obvious that $x'(T_\nu) \neq 0$. Since $-x$ is also a solution of (1.3), we may (and do) assume that $x'(T_\nu) > 0$. As x' has a maximum at T_ν, we have $x''(T_\nu) = 0$ and, moreover, there exists a (sufficiently small) $\epsilon > 0$ such that $x''(t) \geqslant 0$ for $t \in (T_\nu - \epsilon, T_\nu)$ and $x''(t) \leqslant 0$ for $t \in (T_\nu, T_\nu + \epsilon)$. Thus, from (1.3) it follows immediately that $x(T_\nu - \tau) = 0$. We now claim that there exists a point $T_\nu^* \in [T_\nu - \tau, T_\nu)$ with $x'(T_\nu^*) = 0$. Otherwise, as $x'(T_\nu) > 0$, the function x' is necessarily positive on the whole interval $[T_\nu - \tau, T_\nu)$, which ensures that x is strictly increasing on $[T_\nu - \tau, T_\nu)$. Thus, in view of the fact that $x(T_\nu - \tau) = 0$, we can conclude that $x(t) > 0$ for $t \in (T_\nu - \tau, T_\nu)$. By this inequality, from (1.3) we can arrive at the contradiction $x''(t) > 0$ for $t \in (T_\nu, T_\nu + \tau)$, which establishes our claim. Finally, we obtain

$$\begin{aligned}
x'(T_\nu) &= \int_{T_\nu^*}^{T_\nu} x''(s)\,\mathrm{d}s = \int_{T_\nu^*}^{T_\nu} p(s)x(s-\tau)\,\mathrm{d}s \\
&\leqslant \left[\int_{T_\nu^*}^{T_\nu} p(s)\,\mathrm{d}s\right] \max_{T_\nu^* \leqslant t \leqslant T_\nu} |x(t-\tau)| \\
&\leqslant \left[\int_{T_\nu-\tau}^{T_\nu} p(s)\,\mathrm{d}s\right] \max_{T_\nu - \tau \leqslant t \leqslant T_\nu} |x(t-\tau)| \\
&\leqslant K \max_{t_\nu - \tau \leqslant t \leqslant t_{\nu+1}} |x(t-\tau)| = K \max_{t_\nu - 2\tau \leqslant t \leqslant t_{\nu+1} - \tau} |x(t)|,
\end{aligned}$$

which completes the proof. □

Proof of Theorem 2.3. First of all, by taking into account condition (2.2), we consider a $T_0 \geqslant \tau$ such that

$$\int_{t-\tau}^{t} (t-s)p(s)\,\mathrm{d}s \leqslant 1 \quad \text{for all } t \geqslant T_0.$$

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Next, we observe that, by Proposition 2.1, it suffices to show that all oscillatory solutions of the delay differential equation (1.3) have bounded first-order derivatives.

Assume, for the sake of contradiction, that the differential equation (1.3) admits an oscillatory solution x on an interval $[t_0,\infty)$, $t_0 \geqslant 0$, such that x' is unbounded. As the negative of a solution of (1.3) is also a solution of the same equation, we can suppose that x' is unbounded from above. It is clear that x' is also oscillatory. Now, it is not difficult to conclude that there exists a sufficiently large point $T > \max\{t_0+2\tau, T_0\}$ such that

$$x'(T) = \max_{t_0\leqslant t\leqslant T} |x'(t)|,$$
$$|x'(t)| < x'(T) \quad \text{for every } t \in [t_0, T)$$

and, for some (sufficiently small) $\epsilon > 0$,

$$x''(t) \leqslant 0 \quad \text{for } t \in (T, T+\epsilon).$$

The fact that x' has a maximum at T implies that $x''(T) = 0$ and so from (1.3) it follows immediately that

$$x(T-\tau) = 0.$$

Furthermore, we claim that

$$x'(T^*) = 0$$

for some $T^* \in [T-\tau, T)$. Otherwise, since $x'(T) > 0$, the function x' is always positive on $[T-\tau, T)$ and hence x is strictly increasing on the interval $[T-\tau, T)$. Thus, as $x(T-\tau) = 0$, we must have $x(t) > 0$ for $t \in (T-\tau, T)$. So, (1.3) gives $x''(t) > 0$ for $t \in (T, T+\tau)$, which contradicts the fact that $x''(t) \leqslant 0$ for $t \in (T, T+\epsilon)$. This contradiction proves our claim. Next, we derive

$$\begin{aligned}
0 < x'(T) &= \int_{T^*}^{T} x''(s)\,\mathrm{d}s = \int_{T^*}^{T} p(s)x(s-\tau)\,\mathrm{d}s \\
&= \int_{T^*}^{T} p(s)\biggl[\int_{T-\tau}^{s-\tau} x'(r)\,\mathrm{d}r\biggr]\mathrm{d}s \leqslant \int_{T^*}^{T} p(s)\biggl[\int_{s-\tau}^{T-\tau} |x'(r)|\,\mathrm{d}r\biggr]\mathrm{d}s \\
&< \biggl\{\int_{T^*}^{T} p(s)\biggl[\int_{s-\tau}^{T-\tau} \mathrm{d}r\biggr]\mathrm{d}s\biggr\}x'(T) = \biggl[\int_{T^*}^{T} (T-s)p(s)\,\mathrm{d}s\biggr]x'(T) \\
&\leqslant \biggl[\int_{T-\tau}^{T} (T-s)p(s)\,\mathrm{d}s\biggr]x'(T) \leqslant x'(T).
\end{aligned}$$

We have thus arrived at the contradiction $x'(T) < x'(T)$, which completes the proof. □

Proof of Theorem 2.4. By condition (2.3), we can consider a positive number γ such that

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} (t-s)p(s)\,\mathrm{d}s < \gamma < 1.$$

Then there exists a point $T_0 \geqslant \tau$ such that

$$\int_{t-\tau}^{t} (t-s)p(s)\,\mathrm{d}s \leqslant \gamma \quad \text{for all } t \geqslant T_0.$$

Furthermore, in view of Proposition 2.1, it is enough to establish that $\lim_{t\to\infty} x'(t) = 0$ for all oscillatory solutions x of the delay differential equation (1.3).

Let x be an oscillatory solution on an interval $[t_0, \infty)$, $t_0 \geqslant 0$, of the differential equation (1.3). If x' is eventually identically zero, then $\lim_{t\to\infty} x'(t) = 0$. So, we may (and do) suppose that x' is not eventually identically zero. Obviously, x' is also oscillatory. Hence, we can consider a sequence $(t_\nu)_{\nu\geqslant 1}$ of zeros of x' with $\max\{t_0 + 2\tau, T_0\} \leqslant t_1 < t_2 < \cdots$ and $\lim_{\nu\to\infty} t_\nu = \infty$, and such that $t_{\nu+1} - t_\nu \geqslant 2\tau$ $(\nu = 1, 2, \dots)$ and x' is not identically zero on any interval $[t_\nu, t_{\nu+1}]$ $(\nu = 1, 2, \dots)$. Set

$$M_\nu = \max_{t_\nu \leqslant t \leqslant t_{\nu+1}} |x'(t)| \quad (\nu = 1, 2, \dots).$$

Clearly, $M_\nu > 0$ $(\nu = 1, 2, \dots)$. The proof will be accomplished by showing that

$$\lim_{\nu\to\infty} M_\nu = 0.$$

To this end, it suffices to establish that

$$M_\nu \leqslant \gamma M_{\nu-1} \qquad (\nu = 2, 3, \dots).$$

Indeed, in this case, we can obtain

$$M_\nu \leqslant \gamma^{\nu-1} M_1 \quad (\nu = 1, 2, \dots),$$

which implies that $\lim_{\nu\to\infty} M_\nu = 0$, since $\lim_{\nu\to\infty} \gamma^{\nu-1} = 0$.

Now, let us consider an arbitrary $\nu \in \{2, 3, \dots\}$. For this fixed ν, we will show that $M_\nu \leqslant \gamma M_{\nu-1}$. For this purpose, let $T_\nu \in (t_\nu, t_{\nu+1})$ be such that

$$|x'(T_\nu)| = M_\nu \equiv \max_{t_\nu \leqslant t \leqslant t_{\nu+1}} |x'(t)|.$$

It is obvious that $x'(T_\nu) \neq 0$. Furthermore, as $-x$ is also a solution of (1.3), we can confine our discussion only to the case where $x'(T_\nu) > 0$. Since x' has a maximum at T_ν, we have $x''(T_\nu) = 0$ and, moreover, there exists a (sufficiently small) $\epsilon > 0$ such that $x''(t) \geqslant 0$ for $t \in (T_\nu - \epsilon, T_\nu)$ and $x''(t) \leqslant 0$ for $t \in (T_\nu, T_\nu + \epsilon)$. Thus, (1.3) gives

$$x(T_\nu - \tau) = 0.$$

Assume that $x'(t) \neq 0$ for $t \in [T_\nu - \tau, T_\nu)$. Then, as $x'(T_\nu) > 0$, the function x' is always positive on $[T_\nu - \tau, T_\nu)$, which guarantees that x is strictly increasing on $[T_\nu - \tau, T_\nu)$. So, since $x(T_\nu - \tau) = 0$, we have $x(t) > 0$ for $t \in (T_\nu - \tau, T_\nu)$ and hence from (1.3) we can arrive at the contradiction $x''(t) > 0$ for $t \in (T_\nu, T_\nu + \tau)$. This contradiction shows that there exists a point $T_\nu^* \in [T_\nu - \tau, T_\nu)$ such that

$$x'(T_\nu^*) = 0.$$

We now obtain

$$\begin{aligned}
M_\nu = x'(T_\nu) &= \int_{T_\nu^*}^{T_\nu} x''(s)\,\mathrm{d}s = \int_{T_\nu^*}^{T_\nu} p(s)x(s-\tau)\,\mathrm{d}s \\
&= \int_{T_\nu^*}^{T_\nu} p(s)\biggl[\int_{T_\nu-\tau}^{s-\tau} x'(r)\,\mathrm{d}r\biggr]\mathrm{d}s \\
&\leqslant \int_{T_\nu^*}^{T_\nu} p(s)\biggl[\int_{s-\tau}^{T_\nu-\tau} |x'(r)|\,\mathrm{d}r\biggr]\mathrm{d}s \\
&\leqslant \biggl\{\int_{T_\nu^*}^{T_\nu} p(s)\biggl[\int_{s-\tau}^{T_\nu-\tau} \mathrm{d}r\biggr]\mathrm{d}s\biggr\} \max_{T_\nu^*-\tau\leqslant t\leqslant T_\nu-\tau} |x'(t)| \\
&= \biggl[\int_{T_\nu^*}^{T_\nu} (T_\nu-s)p(s)\,\mathrm{d}s\biggr] \max_{T_\nu^*-\tau\leqslant t\leqslant T_\nu-\tau} |x'(t)| \\
&\leqslant \biggl[\int_{T_\nu-\tau}^{T_\nu} (T_\nu-s)p(s)\,\mathrm{d}s\biggr] \max_{T_\nu-2\tau\leqslant t\leqslant T_\nu-\tau} |x'(t)| \\
&\leqslant \gamma \max_{T_\nu-2\tau\leqslant t\leqslant T_\nu-\tau} |x'(t)| \\
&\leqslant \gamma \max_{t_\nu-2\tau\leqslant t\leqslant t_{\nu+1}-\tau} |x'(t)| \\
&\leqslant \gamma \max_{t_\nu-2\tau\leqslant t\leqslant t_{\nu+1}} |x'(t)| \\
&\leqslant \gamma \max_{t_{\nu-1}\leqslant t\leqslant t_{\nu+1}} |x'(t)| \\
&= \gamma \max\Bigl\{\max_{t_{\nu-1}\leqslant t\leqslant t_\nu} |x'(t)|, \max_{t_\nu\leqslant t\leqslant t_{\nu+1}} |x'(t)|\Bigr\}
\end{aligned}$$

and, consequently,

$$M_\nu \leqslant \gamma \max\{M_{\nu-1}, M_\nu\}.$$

Since $\gamma < 1$, we have $M_\nu < \max\{M_{\nu-1}, M_\nu\}$ and hence $\max\{M_{\nu-1}, M_\nu\}$ is always equal to $M_{\nu-1}$. So, we have proved that $M_\nu \leqslant \gamma M_{\nu-1}$ and therefore our proof is complete. □

Proof of Proposition 2.5. Let $\lambda = \mu + \mathrm{i}\nu$ with $\nu > 0$ be a root of (1.5). Then the function x defined by

$$x(t) = \mathrm{e}^{\mu t}\cos\nu t, \quad t \geqslant -\tau,$$

is an oscillatory solution on $[0,\infty)$ of (1.4). Clearly, x is unbounded if $\mu > 0$, and x does not tend to zero as $t \to \infty$ if $\mu \geqslant 0$. Thus, if there exists a root $\lambda = \mu + \mathrm{i}\nu$ of (1.5) with $\mu > 0$ and $\nu > 0$, then (1.4) has an unbounded oscillatory solution, which proves the 'only if' part of (i). Moreover, the existence of a root $\lambda = \mu + \mathrm{i}\nu$ of (1.5) with $\mu \geqslant 0$ and $\nu > 0$ guarantees the existence of an oscillatory solution of (1.4) not tending to zero at ∞, which establishes the 'only if' part of (ii).

Now we proceed to the proof of the 'if' parts of (i) and (ii).

First of all, we notice that, for any real number γ, (1.5) has at most finitely many roots λ with $\mathrm{Re}\,\lambda > \gamma$ (see, for example, [**1**, Chapter I, Theorem 4.4]). Our technique is based on the use of the following well-known asymptotic result (see, for example, [**1**, Chapter I, Theorem 5.4]). Let x be a solution of (1.4). For any real number γ such that (1.5) has no roots on the line $\operatorname{Re}\lambda = \gamma$, we have the asymptotic expansion

$$x(t) = \sum_{j=1}^{\ell} P_j(t)\mathrm{e}^{\lambda_j t} + o(\mathrm{e}^{\gamma t}) \quad \text{for } t \to \infty,$$

where $\lambda_1, \dots, \lambda_\ell$ are the finitely many roots of (1.5) with real part exceeding γ and where, for any $j \in \{1, \dots, \ell\}$, $P_j(t)$ is a (complex-valued) polynomial in t of degree less than or equal to $m_j - 1$ with m_j the multiplicity of λ_j as a root of (1.5).

Next, we show that in the interval $[0, \infty)$ there exists exactly one root of (1.5). In addition, this root is positive and simple, and will be denoted by λ_0. To this end, we set $F(\lambda) = \lambda^2 - p\mathrm{e}^{-\lambda\tau}$ for $\lambda \geqslant 0$ and we observe that $F'(\lambda) = 2\lambda + p\tau\mathrm{e}^{-\lambda\tau} > 0$ for $\lambda \geqslant 0$. So, F is strictly increasing on $[0, \infty)$. Moreover, we have $F(0) = -p < 0$ and $F(\infty) = \infty$. Thus, $\lambda = 0$ is not a root of (1.5), and (1.5) has exactly one positive root λ_0. The root λ_0 of (1.5) is simple, since $F'(\lambda_0) > 0$.

Furthermore, we claim that, if (1.5) admits purely imaginary roots, then (1.5) has exactly two purely imaginary roots, the following ones $\mathrm{i}\sqrt{p}$ and $-\mathrm{i}\sqrt{p}$, which are simple roots of (1.5). To prove this claim, let us consider a real number ν with $\nu \neq 0$. Then $\mathrm{i}\nu$ is a root of (1.5) if and only if

$$(\mathrm{i}\nu)^2 = p\mathrm{e}^{-\mathrm{i}\nu\tau}, \quad \text{i.e. } -\nu^2 = p(\cos\nu\tau - \mathrm{i}\sin\nu\tau),$$

or, equivalently,

$$-\nu^2 = p\cos\nu\tau \quad \text{and} \quad \sin\nu\tau = 0.$$

The last equations hold if and only if

$$\nu^2 = p \quad \text{and} \quad \cos\nu\tau = -1,$$

namely if and only if

$$\nu = \pm\sqrt{p} \quad \text{and} \quad \cos\sqrt{p}\tau = -1.$$

In addition, by setting $F(\lambda) = \lambda^2 - p\mathrm{e}^{-\lambda\tau}$ for complex λ, we have

$$\begin{aligned} F'(\pm\mathrm{i}\sqrt{p}) &= \pm 2\mathrm{i}\sqrt{p} + p\tau\mathrm{e}^{\mp\mathrm{i}\sqrt{p}\tau} \\ &= \pm 2\mathrm{i}\sqrt{p} + p\tau(\cos\sqrt{p}\tau \mp \mathrm{i}\sin\sqrt{p}\tau) \\ &= -p\tau \pm 2\mathrm{i}\sqrt{p} \\ &\neq 0, \end{aligned}$$

which completes the proof of our claim.

Now, it is clear that the proof of the 'if' parts of (i) and (ii) can be accomplished by showing that all oscillatory solutions of (1.4) tend to zero at ∞ (and so all oscillatory solutions of (1.4) are bounded) if

$$\text{for any root } \lambda = \mu + \mathrm{i}\nu \text{ of (1.5) we have } \mu < 0 \text{ whenever } \nu > 0, \tag{3.1}$$

and by showing that all oscillatory solutions of (1.4) are bounded if

$$\text{for any root } \lambda = \mu + \mathrm{i}\nu \text{ of (1.5) we have } \mu \leqslant 0 \text{ whenever } \nu > 0, \\ \text{and (1.5) has at least one root of the form } \mathrm{i}\nu \text{ with } \nu > 0. \tag{3.2}$$

Consider first the case where (3.1) is true. If $\lambda = \mu + \mathrm{i}\nu$ with $\nu \neq 0$ is a root of (1.5), then $\bar{\lambda} = \mu - \mathrm{i}\nu$ is also a root of (1.5). So, for any root $\lambda = \mu + \mathrm{i}\nu$ with $\nu \neq 0$ of (1.5) we have $\mu < 0$. Furthermore, we conclude that $\operatorname{Re}\lambda < 0$ for all roots λ of (1.5) with $\lambda \neq \lambda_0$. Note that (1.5) has finitely many roots λ with $\operatorname{Re}\lambda > -1$. So, we can choose a negative real number γ such that $\operatorname{Re}\lambda < \gamma$ for all roots λ of (1.5) with $\lambda \neq \lambda_0$. For this number γ, (1.5) has no roots λ with $\operatorname{Re}\lambda = \gamma$. Moreover, λ_0 is the unique root of (1.5) with real part exceeding γ and this root is simple. Let x be an arbitrary oscillatory solution of (1.4). Then we have the asymptotic expansion

$$x(t) = P_0 \mathrm{e}^{\lambda_0 t} + o(\mathrm{e}^{\gamma t}) \quad \text{for } t \to \infty,$$

where P_0 is a constant. If $P_0 \neq 0$, then, since $\lambda_0 > 0$ and $\gamma < 0$, we immediately obtain $\lim_{t\to\infty} x(t) = \pm\infty$, which contradicts the oscillatory character of x. So, we must have $P_0 = 0$ and, consequently,

$$x(t) = o(\mathrm{e}^{\gamma t}) \quad \text{for } t \to \infty.$$

As $\gamma < 0$, we conclude that $\lim_{t\to\infty} x(t) = 0$.

Next, assume that (3.2) is true. Then for any root $\lambda = \mu + \mathrm{i}\nu$ with $\nu \neq 0$ of (1.5) we have $\mu \leqslant 0$. Furthermore, it follows that $\operatorname{Re}\lambda < 0$ for all roots λ of (1.5) with $\lambda \neq \lambda_0$ and $\lambda \neq \pm\mathrm{i}\sqrt{p}$. Since (1.5) has finitely many roots λ with $\operatorname{Re}\lambda > -1$, we can consider a real number $\gamma < 0$ such that $\operatorname{Re}\lambda < \gamma$ for any root λ of (1.5) with $\lambda \neq \lambda_0$ and $\lambda \neq \pm\mathrm{i}\sqrt{p}$. Note that λ_0 and $\pm\mathrm{i}\sqrt{p}$ are the only roots of (1.5) with real part exceeding γ and that these roots are simple ones. Consider an arbitrary oscillatory solution x of (1.4). Then

$$x(t) = P_0 \mathrm{e}^{\lambda_0 t} + P_1 \mathrm{e}^{\mathrm{i}\sqrt{p}t} + P_2 \mathrm{e}^{-\mathrm{i}\sqrt{p}t} + o(\mathrm{e}^{\gamma t}) \quad \text{for } t \to \infty,$$

where P_0, P_1 and P_2 are constants. We must have $P_0 = 0$. Otherwise, as $\gamma < 0$ and $\lambda_0 > 0$, we obtain $\lim_{t\to\infty} x(t) = \pm\infty$, a contradiction. Thus, for the solution x we have the asymptotic expansion

$$x(t) = P_1 \mathrm{e}^{\mathrm{i}\sqrt{p}t} + P_2 \mathrm{e}^{-\mathrm{i}\sqrt{p}t} + o(\mathrm{e}^{\gamma t}) \quad \text{for } t \to \infty,$$

from which the boundedness of x follows immediately. □

Proof of Theorem 2.6. (I) Assume that $\sqrt{p}\tau < \pi$. By part (ii) of Proposition 2.5, it suffices to show that for any root $\lambda = \mu + \mathrm{i}\nu$ of (1.5) we have $\mu < 0$ whenever $\nu > 0$. Suppose, for the sake of contradiction, that (1.5) has a root $\lambda = \mu + \mathrm{i}\nu$ with $\mu \geqslant 0$ and $\nu > 0$. Clearly, λ is either a root of the equation

$$\lambda = \sqrt{p}\mathrm{e}^{-\lambda(\tau/2)} \tag{3.3}$$

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or a root of the equation

$$\lambda = -\sqrt{p}\mathrm{e}^{-\lambda(\tau/2)}. \tag{3.4}$$

We first consider the case where λ is a root of (3.3). Then

$$\mu + \mathrm{i}\nu = \sqrt{p}\mathrm{e}^{-\mu\tau/2}[\cos(\tfrac{1}{2}\nu\tau) - \mathrm{i}\sin(\tfrac{1}{2}\nu\tau)]$$

and, consequently,

$$\mu = \sqrt{p}\mathrm{e}^{-\mu\tau/2}\cos(\tfrac{1}{2}\nu\tau) \quad \text{and} \quad \nu = -\sqrt{p}\mathrm{e}^{-\mu\tau/2}\sin(\tfrac{1}{2}\nu\tau).$$

Thus, since $\mu \geqslant 0$ and $\nu > 0$, we have

$$\cos(\tfrac{1}{2}\nu\tau) \geqslant 0 \quad \text{and} \quad \sin(\tfrac{1}{2}\nu\tau) < 0$$

and so there exists an integer k such that

$$2k\pi - \tfrac{1}{2}\pi \leqslant \tfrac{1}{2}\nu\tau < 2k\pi.$$

As $\nu > 0$, we always have $k > 0$. Therefore,

$$\tfrac{1}{2}\nu\tau \geqslant 2\pi - \tfrac{1}{2}\pi = \tfrac{3}{2}\pi, \quad \text{i.e. } \nu \geqslant 3\pi/\tau.$$

On the other hand, because of $\mu \geqslant 0$ and $\sin(\tfrac{1}{2}\nu\tau) < 0$, it holds that

$$\nu = \sqrt{p}\mathrm{e}^{-\mu\tau/2}[-\sin(\tfrac{1}{2}\nu\tau)] \leqslant \sqrt{p}.$$

Hence,

$$3\pi/\tau \leqslant \sqrt{p}, \quad \text{i.e. } \sqrt{p}\tau \geqslant 3\pi,$$

which contradicts our assumption.

Next, let us suppose that λ is a root of (3.4). This means that

$$\mu = -\sqrt{p}\mathrm{e}^{-\mu\tau/2}\cos(\tfrac{1}{2}\nu\tau) \quad \text{and} \quad \nu = \sqrt{p}\mathrm{e}^{-\mu\tau/2}\sin(\tfrac{1}{2}\nu\tau)$$

and hence, as $\mu \geqslant 0$ and $\nu > 0$, we have

$$\cos(\tfrac{1}{2}\nu\tau) \leqslant 0 \quad \text{and} \quad \sin(\tfrac{1}{2}\nu\tau) > 0,$$

which implies that

$$2k\pi + \tfrac{1}{2}\pi \leqslant \tfrac{1}{2}\nu\tau < 2k\pi + \pi$$

for some non-negative integer k. This gives

$$\tfrac{1}{2}\nu\tau \geqslant \tfrac{1}{2}\pi, \quad \text{i.e. } \nu \geqslant \pi/\tau.$$

But, since $\mu \geqslant 0$ and $\sin(\tfrac{1}{2}\nu\tau) > 0$, we have

$$\nu = \sqrt{p}\mathrm{e}^{-\mu\tau/2}\sin(\tfrac{1}{2}\nu\tau) \leqslant \sqrt{p}.$$

So,

$$\pi/\tau \leqslant \sqrt{p}, \quad \text{i.e. } \sqrt{p}\tau \geqslant \pi,$$

a contradiction to our assumption.

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(II) Let us assume that $\sqrt{p}\tau = \pi$. In view of parts (i) and (ii) of Proposition 2.5 (see also the result following Proposition 2.5), it is enough to prove that for any root $\lambda = \mu + \mathrm{i}\nu$ of (1.5) we have $\mu \leqslant 0$ whenever $\nu > 0$ and (1.5) has a root $\lambda = \mathrm{i}\nu$ with $\nu > 0$. We immediately verify that $\lambda = \mathrm{i}\sqrt{p}$ is a root of (1.5), since $\sqrt{p}\tau = \pi$. So, it remains to show that for any root $\lambda = \mu + \mathrm{i}\nu$ of (1.5) we have $\mu \leqslant 0$ whenever $\nu > 0$. For the sake of contradiction, let us suppose that there exists a root $\lambda = \mu + \mathrm{i}\nu$ of (1.5) with $\mu > 0$ and $\nu > 0$. Then λ is either a root of (3.3) or a root of (3.4). If λ is a root of (3.3), then we follow the same procedure as in the proof of (I) to arrive at the contradiction $\sqrt{p}\tau > 3\pi$. Moreover, when λ is a root (3.4), we can follow the same steps as in the proof of (I) to find $\sqrt{p}\tau > \pi$, which is a contradiction.

(III) Suppose that $\sqrt{p}\tau > \pi$. By part (i) of Proposition 2.5, it is sufficient to establish that (1.5) has a root $\lambda = \mu + \mathrm{i}\nu$ with $\mu > 0$ and $\nu > 0$. Furthermore, it is enough to show that (3.4) has a root $\lambda = \mu + \mathrm{i}\nu$ with $\mu > 0$ and $\nu > 0$. So, the proof can be accomplished by using the following well-known lemma.

Lemma 3.1. *Let q and σ be positive constants. If $q\sigma > \frac{1}{2}\pi$, then the equation*

$$\lambda + q\mathrm{e}^{-\lambda\sigma} = 0$$

has a root $\lambda = \mu + \mathrm{i}\nu$ with $\mu > 0$ and $\nu > 0$.

The proof of Theorem 2.6 is now complete. □

Acknowledgements. The authors are grateful to the referee for critical comments which significantly improved the original manuscript.

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