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A Note on the Antipode for Algebraic Quantum Groups

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Abstract. Recently, Beattie, Bulacu ,and Torrecillas proved Radford's formula for the fourth power of the antipode for a co-Frobenius Hopf algebra.

In this note, we show that this formula can be proved for any regular multiplier Hopf algebra with integrals (algebraic quantum groups). This, of course, not only includes the case of a finite-dimensional Hopf algebra, but also that of any Hopf algebra with integrals (co-Frobenius Hopf algebras). Moreover, it turns out that the proof in this more general situation, in fact, follows in a few lines from well-known formulas obtained earlier in the theory of regular multiplier Hopf algebras with integrals.

We discuss these formulas and their importance in this theory. We also mention their generalizations, in particular to the (in a certain sense) more general theory of locally compact quantum groups. Doing so, and also because the proof of the main result itself is very short, the present note becomes largely of an expository nature.

Introduction

Let H be a finite-dimensional Hopf algebra over a field k. Radford's formula for the fourth power of the antipode says that $S^4(h) = g(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1})g^{-1}$ for any element $h \in H$ where g and α are the so-called distinguished group-like elements in H and its dual H', respectively.

The formula was first proved by Larson in a special case [13] and later extended by Radford to general finite-dimensional Hopf algebras [15]. Since Radford proved the formula in 1976, it has been subject to various generalizations. Recently, in an article by Beattie, Bulacu, and Torrecillas [1], the formula was proved for any co-Frobenius Hopf algebra. See the introduction to that paper for more about the history and recent work on Radford's formula.

Consider the finite-dimensional case and observe the following. If φ is a left integral on H, then the element g is characterized by the property that $\varphi(S(h)) = \varphi(hg^{-1})$ for all $h \in H$. Similarly, when φ' is a left integral on the dual Hopf algebra H', the element α is characterized by a similar formula, namely, $\varphi'(S'(h')) = \varphi'(h'\alpha)$ for all $h' \in H'$, where S' is the antipode of the dual. We use a slightly dif-

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ferent terminology than the one common in the theory of Hopf algebras. However, doing so, we immediately see that Radford's formula can be stated for any regular multiplier Hopf algebra with integrals (A, Δ). Indeed, also in this case, we have a unique group-like element δ with the property that $\varphi(S(a)) = \varphi(a\delta)$ for all $a \in A$. Now $\delta \in M(A)$, the multiplier algebra of A (see the preliminaries). Because for such a multiplier Hopf algebra, one can construct the dual $(\widehat{A}, \widehat{\Delta})$ which is again a regular multiplier Hopf algebra with integrals, we also have the group-like element $\widehat{\delta} \in M(\widehat{A})$, characterized by $\widehat{\varphi}(S(b)) = \widehat{\varphi}(b\widehat{\delta})$ for all $b \in \widehat{A}$, where now $\widehat{\varphi}$ is a left integral on the dual and where S is also used to denote the antipode on the dual. We refer to the literature and also to the preliminaries for more details about δ and $\widehat{\delta}$.

Usually, when a formula is true for finite-dimensional Hopf algebras and when it makes sense for regular multiplier Hopf algebras with integrals, it is also true in this more general setting. And, indeed, this turns out also to be the case for Radford's formula. This is the main result discussed in the present note.

However, there is more. Not only do we obtain a substantial generalization of the formula, we also show how it follows very quickly from other formulas, some of which are unpublished, but well known (for some time) to researchers in the field of multiplier Hopf algebras with integrals and locally compact quantum groups. The remark that the theory of multiplier Hopf algebras was developed first as a by-product during the process of the search for a good notion of what is now called a locally compact quantum group. This explains the fact that the terminology is not completely the same as the one used in Hopf algebra theory. On the other hand, this disadvantage is compensated by the availability of new techniques.

Also, in this note, we will use both the slightly different terminology on the one hand, but also other techniques than the ones common in Hopf algebra theory. For the convenience of the reader, we will clearly explain the difference in terminology. And we hope the reader will be convinced that these other techniques are valuable.

We have, apart from this introduction, two sections in this paper. In Section 1 we will briefly describe the basic notions and results in the theory. In Section 2 we will prove the result. We will also comment on the various formulas that are used and in the meantime, we will refer to various places in the literature where such formulas can be found, either in special cases, or in more general cases.

Because the proof of the result itself is quite short and because of the focus on these other aspects, this note is mainly expository. We refer to Section 1 for basic references, notations, and terminology.

1 Preliminaries

In this preliminary section, we will recall some of the basic notions and results in the theory of regular multiplier Hopf algebras with integrals (sometimes called *algebraic quantum groups*). We will be rather brief and refer to the literature for more details. However, we will be somewhat more explicit about the differences in terminology between the usual Hopf algebra theory and the theory of multiplier Hopf algebras and algebraic quantum groups.

1.1 Multiplier Hopf Algebras

Let A be an algebra over $\mathbb{C}$, with or without identity, but with a non-degenerate product, *i.e.*, if $a \in A$ and either $ab = 0$ for all $b \in A$ or $ba = 0$ for all $b \in A$, then $a = 0$. If the algebra has an identity, of course, the product is automatically non-degenerate. The multiplier algebra $M(A)$ of A is characterized as the largest algebra with identity, containing A as a two-sided ideal with the property that if $x \in M(A)$ and $xa = 0$ for all $a \in A$ or $ax = 0$ for all $a \in A$, then $x = 0$. Again, if A has an identity, then $M(A) = A$.

Consider the tensor product $A \otimes A$ of A with itself. Then $A \otimes A$ is an algebra with a non-degenerate product and we can consider the multiplier algebra $M(A \otimes A)$. A coproduct Δ on A is a homomorphism from A to $M(A \otimes A)$ satisfying some extra conditions (such as coassociativity).

A *multiplier Hopf algebra* is a pair (A, Δ) consisting of an algebra A over $\mathbb{C}$ with a non-degenerate product and a coproduct Δ such that the two linear maps T_1, T_2 defined from $A \otimes A$ to $M(A \otimes A)$ by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

are injective, map into $A \otimes A$, and have range all of $A \otimes A$. A multiplier Hopf algebra is called *regular* if (A, Δ') is still a multiplier Hopf algebra with Δ' obtained from Δ by composing it with the flip.

For any multiplier Hopf algebra (A, Δ) there is a counit ε and an antipode S satisfying (and characterized by) properties very similar to the case of Hopf algebras. Regularity is equivalent with the antipode being a bijective map from A to itself.

Any Hopf algebra is a multiplier Hopf algebra. Conversely, if (A, Δ) is a multiplier Hopf algebra and if A has an identity, then it is a Hopf algebra.

The motivating example comes from a group G. Let A be the algebra $K(G)$ of complex functions on G with finite support and pointwise product. Identify $A \otimes A$ with $K(G \times G)$ and $M(A \otimes A)$ with the algebra $C(G \times G)$ of all complex functions on $G \times G$. Then A becomes a (regular) multiplier Hopf algebra if the coproduct Δ is defined by $\Delta(f)(p, q) = f(pq)$ for $p, q \in G$ and $f \in K(G)$.

The theory of multiplier Hopf algebras has been developed for algebras over the field $\mathbb{C}$ (because of an analytical background) but this is not essential and any other field would do as well.

We refer to [16] for details about the theory of multiplier Hopf algebras. For more examples, [17, 19] and for some constructions, see [4–6].

1.2 Multiplier Hopf Algebras with Integrals

Assume in what follows that (A, Δ) is a regular multiplier Hopf algebra. A linear functional φ on A is called *left invariant* if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ in $M(A)$ for all $a \in A$. We use ι to denote the identity map and the formula is given a meaning by multiplying from left or right with any element in A. A non-zero left invariant functional φ is called a *left integral on A*. A right integral is defined similarly. The motivating example, explained before, justifies the terminology. Indeed, if $A = K(G)$

for a group G and φ is defined by $\varphi(f) = \sum_{p\in G} f(p)$, then φ is a left (and a right) integral.

In general, one can show that left and right integrals are unique (up to a scalar) if they exist. If a left integral φ exists, a right integral also exists, namely $\varphi \circ S$. Left and right integrals may be different, but they are related. For a left integral φ, there is a unique group-like element δ in $M(A)$ such that $\varphi(S(a)) = \varphi(a\delta)$ for all $a \in A$. It is (the inverse of) the so-called distinguished group-like element, known in Hopf algebra theory. In the theory of multiplier Hopf algebras, it is called the *modular element* because it relates the left and the right integral just as, in the theory of locally compact groups, the modular function relates the left and the right Haar integrals.

Let φ be a left integral and ψ a right integral. There are automorphisms σ and σ' of A satisfying

$$\varphi(ab) = \varphi(b\sigma(a)) \qquad \psi(ab) = \psi(b\sigma'(a))$$

for all $a, b \in A$. It is known that the integrals are faithful, *i.e.,* if $a \in A$ and $\varphi(ab) = 0$ for all $b \in A$ or $\varphi(ba) = 0$ for all $b \in A$, then $a = 0$ and similarly for ψ. So these automorphisms are unique and characterized by these formulas. This property is called *the weak K.M.S. property* and the automorphisms are called the *modular automorphisms*. This terminology comes from the theory of operator algebras and finds its origin in physics. In Hopf algebra theory, these automorphisms are known as the Nakayama automorphisms.

Finally, there is a scalar $\tau \in \mathbb{C}$ satisfying $\varphi(S^2(a)) = \tau\varphi(a)$ for all $a \in A$. It exists because $\varphi\circ S^2$ is also a left integral and therefore a multiple of φ. For obvious reasons, it is called the *scaling constant*.

There are many formulas relating these objects. We will state them when we use them in the next section. For details about regular multiplier Hopf algebras with integrals we refer to [17], see also [19]. The example $K(G)$ is too trivial to illustrate these various objects. See [17, 19] for examples that do illustrate various features.

1.3 Duality for Multiplier Hopf Algebras with Integrals

In what follows, (A, Δ) is a regular multiplier Hopf algebra with integrals and φ and ψ are a left and a right integral, respectively.

Define $\widehat{A}$ as the space of linear functionals on A of the form $\varphi(\,\cdot\, a)$, where $a \in A$. Then $\widehat{A}$ can be made into an algebra with a product dual to the coproduct Δ on A. This product is non-degenerate. The product on A can also be dualized and yields a coproduct $\widehat{\Delta}$ on $\widehat{A}$. It is shown that the pair $(\widehat{A}, \widehat{\Delta})$ becomes a regular multiplier Hopf algebra. It is called the *dual of* (A, Δ).

There exist integrals on $(\widehat{A}, \widehat{\Delta})$. A right integral $\widehat{\psi}$ on $\widehat{A}$ is defined by $\widehat{\psi}(\omega) = \varepsilon(a)$ if $\omega = \varphi(\,\cdot\, a)$ and $a \in A$. The various objects associated with $(\widehat{A}, \widehat{\Delta})$ are denoted as for (A, Δ) but with a *hat*. So we have $\widehat{\varphi}$, $\widehat{\psi}$, $\widehat{\delta}$, $\widehat{\sigma}$, $\widehat{\sigma}'$. However, we use ε and S also for the counit and antipode on the dual.

Repeating the procedure, *i.e.*, if we take the dual of $(\widehat{A}, \widehat{\Delta})$, we find the original pair (A, Δ). This result is not very deep. It essentially follows from the formula

$$\widehat{\psi}(\omega'\omega) = \omega'(S^{-1}(a))$$

for all $a \in A$ and $\omega' \in \widehat{A}$, where $\omega = \varphi(\cdot\, a)$. This formula in turn follows quite easily from the definitions (see [17, Theorem 4.12]). It is refered to as *biduality*.

Duality for regular multiplier Hopf algebra with integrals generalizes duality for finite-dimensional Hopf algebras. The results are quite similar, but, of course, the class of objects is much bigger. It can be applied to Hopf algebras with integrals (the co-Frobenius type) but also to the duals. In the theory of multiplier Hopf algebras, we call the first multiplier Hopf algebras of compact type and the second multiplier Hopf algebras of discrete type. The terminology has its origin in analysis. Think of Pontryagin duality for abelian locally compact groups. The dual of a compact group is a discrete group. And to any compact group G is associated the Hopf algebra of polynomial functions on G. For a discrete group, we have the multiplier Hopf algebra $K(G)$ as introduced before. It is indeed of discrete type.

This is perhaps a good occasion to say something more about terminology related to integrals. For us, an integral is always a linear functional *on* the algebra. This is because the algebra is considered as an "algebra of functions" on a "quantum space". And in classical analysis, an integral is a linear functional on a space of functions. What is often called an integral *in* a Hopf algebra, we will call a *co-integral* because it should be considered as an integral *on* the dual Hopf algebra. We use this terminology consistently in the theory of algebraic quantum groups. For an algebraic quantum group (A, Δ), an integral is a linear functional on A. If the integral on the dual $\widehat{A}$ turns out to be an element in A, then we call it a co-integral in A. This is precisely the case when A is of discrete type. For more details about duality, we again refer to [17].

Also here, there are many formulas relating the various objects for (A, Δ) with those of $(\widehat{A}, \widehat{\Delta})$. Only a few of them are given in [17]. More formulas are found in [8], but it should be mentioned that these are only proved in the case of a multiplier Hopf *-algebra with positive integrals. However, in general, when a formula makes sense in this special case, it turns out to be true also in the general case. We will recall (some of) these formulas (and prove them if necessary) where we use them in the next section.

1.4 Pairing and Actions

We will use the expression $\langle a, b\rangle$ to denote the value of a functional $b \in \widehat{A}$ in the point $a \in A$. Doing so, we get a non-degenerate pairing of multiplier Hopf algebras in the sense of [6]. This pairing gives rise to the natural four actions. We have a left and a right action of A on $\widehat{A}$, given by the formulas

$$\langle a'a, b\rangle = \langle a', a \rightharpoonup b\rangle \qquad \langle aa', b\rangle = \langle a', b \leftharpoonup a\rangle$$

for any $a, a' \in A$ and $b \in \widehat{A}$. Similarly the left and right actions of $\widehat{A}$ on A are given by

$$\langle a, b'b\rangle = \langle b \rightharpoonup a, b'\rangle \qquad \langle a, bb'\rangle = \langle a \leftharpoonup b, b'\rangle$$

for any $a \in A$ and $b, b' \in \widehat{A}$. It is not completely obvious that this can be done, but we refer to [6] for a precise treatment. See also [7] for more information about actions of multiplier Hopf algebras in general.

2 The Main Result and Comments

In this section, the aim is twofold. On the one hand, we will present a proof of the main result, Radford's S^4 formula for regular multiplier Hopf algebras with integrals (algebraic quantum groups). On the other hand, we will also comment on the various formulas, related with and necessary for the proof of Radford's formula. We will make links with other results in the field and compare with results in the (analytic) theory of locally compact quantum groups.

We consider a regular multiplier Hopf algebra (A, Δ) with integrals and its dual $(\widehat{A}, \widehat{\Delta})$ as explained in the previous section.

We have the different objects associated with (A, Δ) and $(\widehat{A}, \widehat{\Delta})$ and many relations among them. In particular, we have a left and a right integral φ and ψ on A and we have a left and a right integral $\widehat{\varphi}$ and $\widehat{\psi}$ on $\widehat{A}$. We fix φ and we normalize the three others in the following way. We let $\psi = \varphi \circ S$ and $\widehat{\psi} = \widehat{\varphi} \circ S$. We define $\widehat{\psi}$ by $\widehat{\psi}(\omega) = \varepsilon(a)$ when $\omega = \varphi(\,\cdot\, a)$ and $a \in A$ as before. One can show that, with these assumptions, we have $\widehat{\varphi}(\omega) = \varepsilon(a)$ when $\omega = \psi(a\,\cdot\,)$ and $a \in A$ (see [17, Proposition 4.8]). It should be mentioned however that for our purposes, this normalization is not so important.

For the modular automorphism σ and σ' we have the two basic formulas

$$\Delta(\sigma(a)) = (S^2 \otimes \sigma)\Delta(a) \qquad \Delta(\sigma'(a)) = (\sigma' \otimes S^{-2})\Delta(a)$$

for all $a \in A$. They are found already in[17, Proposition 3.14]. However, there is also a third formula of this type. It says that not only $\Delta(S^2(a)) = (S^2 \otimes S^2)\Delta(a)$, but also

$$\Delta(S^2(a)) = (\sigma \otimes \sigma'^{-1})\Delta(a)$$

for all $a \in A$. This formula was first proved in [12, Lemma 3.10] in the case of a multiplier Hopf *-algebra with positive integrals, but later simpler and more direct proofs have been given, valid in the general case (see the appendix in [14] and also [3, Proposition 2.7]). The automorphism σ, σ' and S^2 all mutually commute and there are two basic relations between σ and σ'. We have $S(\sigma'(a)) = \sigma^{-1}(S(a))$ and also $\delta\sigma(a) = \sigma'(a)\delta$ for all $a \in A$. These properties are already found in the original paper [17].

Now, we also have the data for the dual and, of course, all the above results can also be formulated for the dual objects. In the original paper however, very few relations between the objects for (A, Δ) and the ones for the dual $(\widehat{A}, \widehat{\Delta})$ are found. Many such relations (and much more) are found in [8]. Unfortunately, in that paper, it is assumed that (A, Δ) is a multiplier Hopf *-algebra, *i.e.*, A is a *-algebra and Δ a *-homomorphism, and the left integral is assumed to be positive, *i.e.*, $\varphi(a^*a) \geq 0$ for all a. Remember that in that case, the right integral ψ is automatically positive. Therefore, as we mentioned already, the results of [8] cannot be used in the general case we consider here. On the other hand, these results are a good source of inspiration and often they are also true in the more general case.

The first of such formulas we like to consider and discuss are the following.

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Proposition 2.1 *For all $a \in A$ we have*

$$\langle a, \widehat{\delta} \rangle = \varepsilon(\sigma^{-1}(a)) = \varepsilon({\sigma'}^{-1}(a))$$

$$\langle a, \widehat{\delta}^{-1} \rangle = \varepsilon(\sigma(a)) = \varepsilon(\sigma'(a)).$$

Before we give a proof, we need to make a comment on the left-hand side of these formulas. We need to observe that the original pairing of A with $\widehat{A}$ can be extended in a natural way to a pairing between A and $M(\widehat{A})$. If, for instance, m is multiplier in $M(\widehat{A})$, then formally we have

$$\langle a, mb \rangle = \langle b \rightharpoonup a, m \rangle$$

for any $a \in A$ and $b \in \widehat{A}$. This formula can be used to extend the pairing because the action of $\widehat{A}$ on A is unital, *i.e.*, for all $a \in A$ there is a $b \in \widehat{A}$ satisfying $b \rightharpoonup a = a$ (see [6]). For a precise treatment of this extended pairing, see [2].

Now we give the proof of Proposition 2.1.

Proof Take $c \in A$ and let $b = \varphi(\,\cdot\, c)$. We will use the Sweedler notation as justified in [6,7]. We write for any $a, a' \in A$,

$$\varphi(a'c\sigma(a)) = \varphi(aa'c) = \langle aa', b \rangle = \langle a, b_{(1)} \rangle \langle a', b_{(2)} \rangle,$$

so that

$$\varphi(\,\cdot\, c\sigma(a)) = \langle a, b_{(1)} \rangle b_{(2)} = b \leftharpoonup a.$$

From the definition of $\widehat{\psi}$ and the formula $(\iota \otimes \widehat{\psi})\Delta(b) = \widehat{\psi}(b)\widehat{\delta}^{-1}$, we get

$$\varepsilon(c\sigma(a)) = \widehat{\psi}(b_{(2)})\langle a, b_{(1)} \rangle = \langle a, \widehat{\delta}^{-1} \rangle \widehat{\psi}(b) = \langle a, \widehat{\delta}^{-1} \rangle \varepsilon(c),$$

so that $\varepsilon(\sigma(a)) = \langle a, \widehat{\delta}^{-1} \rangle$.

This proves one of the formulas. Using the two relations between σ and σ' formulated earlier and using that $\varepsilon \circ S = \varepsilon$, $\varepsilon(\delta) = 1$ and $S(\widehat{\delta}) = \widehat{\delta}^{-1}$, we easily obtain the other three. ∎

From these formulas, we see that $\widehat{\delta}$ has to be group-like because ε and σ are algebra maps.

In the original paper [17], it was remarked already that $\varepsilon(\sigma(a)) = \varepsilon(\sigma'(a))$ for all a, but it was also stated that *there seems to be no obvious relation between σ and ε* (see the end of [17, §3]). However, these formulas are essentially obtained in [8], but as stated, only for multiplier Hopf *-algebras with positive integrals. Indeed, [8, Proposition 7.10] says that $\widehat{\delta}^{iz} = \varepsilon\sigma_{-z}$ and this will give the results in our Proposition 2.1 when we let $z = i$ and $z = -i$. It was known, however, for some time that these formulas also were true in the general case. Recently, they have been included in [3] for the more general case of algebraic quantum hypergroups.

But, as we saw, the formulas in Proposition 2.1 are easy to obtain. Nevertheless, it is precisely these formulas which will give us very quickly, Radford's formula.

First we need to prove some other formulas. These are equivalent with the formulas in the previous proposition in the sense that it is easy to obtain one set from the other.

Proposition 2.2 *For all $a \in A$ and $b \in \widehat{A}$, we have*

$$\langle \sigma(a), b \rangle = \langle a, S^2(b)\widehat{\delta}^{-1} \rangle,$$
$$\langle \sigma^{-1}(a), b \rangle = \langle a, S^{-2}(b)\widehat{\delta} \rangle,$$
$$\langle \sigma'(a), b \rangle = \langle a, \widehat{\delta}^{-1} S^{-2}(b) \rangle,$$
$$\langle \sigma'^{-1}(a), b) \rangle = \langle a, \widehat{\delta} S^2(b) \rangle.$$

Proof Start with the formula $\Delta(\sigma(a)) = (S^2 \otimes \sigma)\Delta(a)$. If we apply $\iota \otimes \varepsilon$ and use the previous proposition, we get

$$\sigma(a) = S^2(a_{(1)})\varepsilon(\sigma(a_{(2)})) = \langle\, a_{(2)}, \widehat{\delta}^{-1} \,\rangle S^2(a_{(1)}).$$

Pairing with b gives

$$\langle \sigma(a), b \rangle = \langle a, S^2(b)\widehat{\delta}^{-1} \rangle.$$

This gives the first formula. The three others are obtained in a similar fashion. ∎

If in the formulas above we let $b = 1$, we see that we recover the formulas in the previous proposition. One has to be a little bit careful, but this is done by using the "covering technique", needed to work properly with Sweedler's notation in the context of multiplier Hopf algebras [6, 7].

The formulas in Propositions 2.1 and 2.2 have obvious dual forms [3, 8]. The forms we have given are the ones that are needed to prove Radford's formula.

These formulas are very useful. We see that if φ is a trace, that is, when σ is trivial, then it follows from Proposition 2.1 that $\widehat{\delta} = 1$ (meaning that left and right integrals on the dual are the same, *i.e.,* the dual is *unimodular*). In this case the first formula in Proposition 2.2 will give us that $S^2 = \iota$. In particular, when A is abelian we recover the result that the square of the antipode is trivial. By duality, the same is true when A is coabelian. Another conclusion from these formulas is that, if left and right integrals on A coincide (i.e. that A is unimodular and $\delta = 1$), then $\widehat{\sigma}$ and S^2 coincide on $\widehat{A}$.

Again, these formulas can be found already in [8]. Consider Proposition 7.3 in that paper where it is stated that

$$\langle a, \widehat{\sigma}_z(b) \rangle = \langle \tau_z(a)\delta^{-iz}, b \rangle$$

whenever $a \in A$ and $b \in \widehat{A}$. If we take $z = -i$, we get

$$\langle a, \widehat{\sigma}(b) \rangle = \langle S^2(a)\delta^{-1}, b \rangle$$

because $\widehat{\sigma}_{-i}$ and τ_{-i} in [8] are here $\widehat{\sigma}$ and S^2, respectively. So this gives the first formula, in dual form, of Proposition 2.2 above. These formulas were also discovered before and finally proved in the more general setting of algebraic quantum hypergroups in [3].

In the analytic setting of locally compact quantum groups, we have equivalent formulas. Take the first formula in [18, Theorem 4.17]. It comes in the form

$$\nabla^{it} = (\widehat{J}\,\widehat{\delta}^{it}\,\widehat{J})P^{it}$$

for all $t \in \mathbb{R}$. This is an equality of unitary operators on a Hilbert space. With $t = i$ and $t = -i$, they are essentially the same as the first two formulas in Proposition 2.2 here. It is not so easy to see this, and one should notice that the operator ∇ is related to the automorphism σ whereas the operator P is related withto the square of the antipode. These formulas are already found in the original papers by Kustermans and Vaes on locally compact quantum groups [11]. We do not have counterparts for the formulas of Proposition 2.1 because the counit is, from an analytical point of view, a difficult and not very useful object in the theory of locally compact quantum groups.

We now arrive at the main result of this note: Radford's formula for the fourth power of the antipode.

Theorem 2.3 *Let (A, Δ) be a regular multiplier Hopf algebra with integrals (an algebraic quantum group). If δ and $\widehat{\delta}$ denote the modular elements in $M(A)$ and $M(\widehat{A})$, respectively, then*

$$S^4(a) = \delta^{-1}(\widehat{\delta} \rightharpoonup a \leftharpoonup \widehat{\delta}^{-1})\delta$$

for all $a \in A$.

Proof From the previous proposition, we see that

$$\sigma(x) = \widehat{\delta}^{-1} \rightharpoonup S^2(x) \qquad \sigma'(x) = S^{-2}(x) \leftharpoonup \widehat{\delta}^{-1}$$

for all $x \in A$. If we use that $\delta\sigma(x) = \sigma'(x)\delta$ and then replace x by $S^2(a)$, we find

$$(a \leftharpoonup \widehat{\delta}^{-1})\delta = \delta(\widehat{\delta}^{-1} \rightharpoonup S^4(a))$$

for all a. Now let $\widehat{\delta}$ act on this equation from the left. Because $\Delta(\delta) = \delta \otimes \delta$ and $\varepsilon(\sigma^{-1}(\delta)) = \tau$ (the scaling constant), this action commutes, up to τ, with both multiplication from left or right by δ. Then we get Radford's formula. ∎

The first thing we would like to remark is that Radford's formula is *self-dual*, in the following sense. If we pair the formula with any element $b \in \widehat{A}$, we can move S^4 to the other side (because $\langle S(a), b\rangle = \langle a, S(b)\rangle$), we can move δ^{-1} and δ to the other side to get $\delta \rightharpoonup b \leftharpoonup \delta^{-1}$ and finally, we can move also $\widehat{\delta}$ and $\widehat{\delta}^{-1}$ and obtain $\widehat{\delta}^{-1}(\delta \rightharpoonup b \leftharpoonup \widehat{\delta}^{-1})\widehat{\delta}$. So we again get Radford's formula, now for the dual $\widehat{A}$.

In [1] the formula is obtained for a Hopf algebra with integrals. Before, the formula was obtained for general multiplier Hopf algebras with integrals. This was mentioned at a talk of the second author at the AMS meeting in Bowling Green in 2005. In [3] however, Radford's formula is obtained for any algebraic quantum hypergroup, including the case of any algebraic quantum group.

In the case of a multiplier Hopf *-algebra with positive integrals, we have analytic forms of all the formulas in Proposition 2.1 and Proposition 2.2 thanks to the work of

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Kustermans [8]. This means that there is also an analytic form of Radford's formula in this case. In fact, in [18], we proved an analytic version of Radford's formula for any locally compact quantum group. It comes under the form

$$P^{-2it} = \delta^{it}(J\delta^{it}J)\widehat{\delta}^{it}(\widehat{J}\,\widehat{\delta}^{it}\widehat{J})$$

for all $t \in \mathbb{R}$ ([18, Theorem 4.20]). Here again, we have an equation with unitary operators on a Hilbert space. It is not completely obvious, but with $t = i$, we recover a formula (for unbounded operators) that is essentially Radford's formula. The above formula is not found in the original papers by Kustermans and Vaes [9, 10], but it follows quickly from formulas found in [11] (see an earlier remark).

Finally, we also want to refer to the work of C. Voigt [20]. He introduced the notion of a bornological quantum group. Basically, it is a bornological vector space, endowed with a suitable product and coproduct. Regular multiplier Hopf algebras with integrals are a special case. In fact, the theory of algebraic quantum groups served as a model for the development of these bornological quantum groups. In [21, §4], a proof was given of Radford's formula for a bornological quantum group. The formulation is just a for algebraic quantum groups (as in Theorem 2.3 of this note) and also the proof is very similar. As bornological quantum groups are generalizing the algebraic quantum groups, the result of C. Voigt is more general than our Theorem 2.3.

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