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Moduli of Vector Bundles on Curves in Positive Characteristics

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Abstract. Let X be a projective curve of genus 2 over an algebraically closed field of characteristic 2. The Frobenius map on X induces a rational map on the moduli scheme of rank-2 bundles. We show that up to isomorphism, there is only one (up to tensoring by an order two line bundle) semi-stable vector bundle of rank 2 (with determinant equal to a theta characteristic) whose Frobenius pull-back is not semi-stable. The indeterminacy of the Frobenius map at this point can be resolved by introducing Higgs bundles.

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1. Introduction and Results

Let X be a smooth projective curve of genus 2 over an algebraically closed field k of characteristic $p > 0$. Let Ω be its canonical bundle. Define the (absolute) Frobenius morphism [3, 4] $F: X \longrightarrow X$ which maps local sections $f \in \mathcal{O}_X$ to f^p. As X is smooth, F is a (finite) flat map.

Let J^0, J^1 be the moduli schemes of isomorphism classes of line bundles of degree 0 and 1, respectively. Choose a theta characteristic $L_\theta \in J^1$. Denote by S_O (resp. S_θ) the moduli scheme of S-equivalence classes of semi-stable vector bundles of rank 2 and determinant O_X (resp. L_θ) on X [8]. We study the Frobenius pull-backs of the bundles in S_O and S_θ. The geometry of S_θ has been studied extensively by Bhosle [1].

The operation of Frobenius pull-back has a tendency to destabilize bundles [9]. In particular, the map $V \longmapsto F^*(V)$ is rational on the moduli scheme.

The Frobenius destabilizes only finite many bundles in S_O (see Theorem 3.2). For any $V \in S_O$, Proposition 3.3 gives a necessary and sufficient criterion for $F^*(V)$ to be non-semi-stable in terms of theta characteristic.

For a given vector bundle V on X, let

$$J_2(V) = \{V \otimes L : L \in J^0, L^2 = O_X\}$$

THEOREM 1.1 *Suppose $p = 2$. Then there exists a bundle $V_1 \in Ext^1(L_\theta, \mathcal{O}_X)$ such that if $V \in S_\theta \setminus J_2(V_1)$, then $F^*(V)$ is semi-stable. Hence, the Frobenius map induces a map $\Omega^{-1} \otimes F^* : S_\theta \setminus J_2(V_1) \longrightarrow S_O$.*

We show that there is a natural way of resolving the indeterminacy of the Frobenius map at the points in $J_2(V_1)$, by replacing S_O, S_θ with moduli schemes of suitable Higgs bundles. Denote by $S_O(\Omega)$ (resp. $S_\theta(L_\theta)$) the moduli scheme of semi-stable Higgs bundles with associated line bundle O_X (resp. L_θ) [7]. For any Higgs bundle on X, one may also consider its Frobenius pull-back.

THEOREM 1.2. *Suppose $p = 2$.*

(1) *If $(V, \phi) \in S_\theta(L_\theta)$, then either $V \in S_\theta$ or $V \in J_2(O_X \oplus L_\theta)$.*
(2) *There exist Higgs fields ϕ_0 and ϕ_1 such that $(F^*(W), F^*(\phi_0))$ and $(F^*(V), F^*(\phi_1))$ are semi-stable for all $W \in J_2(O_X \oplus L_\theta)$ and $V \in J_2(V_1)$.*

Hence, the Frobenius defines a map on a Zariski open set $U \subset S_\theta(L_\theta)$

$$\Omega^{-1} \otimes F^* : U \longrightarrow S_O(\mathcal{O}_X),$$

where U contains the scheme $S_\theta \setminus J_2(V_1)$ and the points $(W, \phi_0), (V, \phi_1)$ for any $W \in J_2(O_X \oplus L_\theta)$ and $V \in J_2(V_1)$.

Cartier's theorem gives a criterion for descent under Frobenius [3]. Higgs bundles appear naturally in characteristic $p > 0$ context. To see this, let (V, ∇) be a vector bundle with a (flat) connection, $\nabla : V \longrightarrow \Omega \otimes_{O_X} V$. One associates to the pair (V, ∇) its p-curvature which is a homomorphism of O_X-modules [3, 4]: $\psi : V \longrightarrow F^*(\Omega) \otimes_{O_X} V$. Thus the pair (V, ∇) gives a Higgs bundle with associated line bundle $F^*(\Omega)$.

2. Bundle Extensions and the Frobenius Morphism

Suppose L is a line bundle on X. Then $F^*(L) = L^p$. The push-forward, $F_*(O_X)$, is a vector bundle of rank p and one has the exact sequence of vector bundles [9]

$$0 \longrightarrow O_X \longrightarrow F_*(O_X) \longrightarrow B_1 \longrightarrow 0.$$

Tensoring the sequence with a line bundle L and using the projection formula, we obtain

$$0 \longrightarrow L \longrightarrow F_*(L^p) \longrightarrow B_1 \otimes L \longrightarrow 0.$$

The associated long cohomology sequence is

$$\cdots \longrightarrow H^0(B_1 \otimes L) \longrightarrow H^1(L) f_L \longrightarrow H^1(F_*(L^p)) \longrightarrow \cdots$$

Since F is an affine morphism, the Leray spectral sequence for F degenerates at E_2. Hence $H^i(F_*(L^p)) \cong H^i(L^p)$. Substituting this into the long exact sequence, one obtains

$$\cdots \longrightarrow H^0(B_1 \otimes L) \longrightarrow H^1(L) \xrightarrow{f_L} H^1(L^p) \longrightarrow \cdots \tag{1}$$

Suppose $V \in \mathrm{Ext}^1(L_2, L_1) \cong H^1(L_2^{-1} \otimes L_1)$, i.e.

$$0 \longrightarrow L_1 \longrightarrow V \longrightarrow L_2 \longrightarrow 0,$$

where L_1, L_2 are line bundles. Since F is a flat morphism, we have

$$0 \longrightarrow F^*(L_1) \longrightarrow F^*(V) \longrightarrow F^*(L_2) \longrightarrow 0.$$

This gives a map

$$F^* : \mathrm{Ext}^1(L_2, L_1) \longrightarrow \mathrm{Ext}^1(F^*(L_2), F^*(L_1)) \cong \mathrm{Ext}^1(L_2^p, L_1^p).$$

Take $L = L_2^{-1} \otimes L_1$ in (1).

PROPOSITION 2.1. *$F^*(V) = L_1^p \oplus L_2^p$ if and only if V is in the image of the connecting homomorphism $H^0(B_1 \otimes L_2^{-1} \otimes L_1) \longrightarrow H^1(L_2^{-1} \otimes L_1)$.*

Proof. Since the functors $\Gamma(X, .)$ and $\mathrm{Hom}(O_X, .)$ are equivalent, the diagram

$$\begin{array}{ccc} H^1(L_2^{-1} \otimes L_1) & \xrightarrow{f_{L_2^{-1} \otimes L_1}} & H^1(L_2^{-p} \otimes L_1^p) \\ \downarrow \mathrm{id}\, V & & \downarrow \mathrm{id} \\ \mathrm{Ext}^1(L_2, L_1) & \xrightarrow{F^*} & \mathrm{Ext}^1(L_2^p, L_1^p) \end{array}$$

commutes. Now the proposition follows directly from the long exact sequence

$$\cdots \longrightarrow H^0(B_1 \otimes L_2^{-1} \otimes L_1) \longrightarrow H^1(L_2^{-1} \otimes L_1) \longrightarrow H^1(L_2^{-p} \otimes L_1^p) \longrightarrow \cdots \qquad \square$$

3. The Moduli of Semi-Stable Vector and Higgs Bundles

Suppose V is a vector bundle on X. The slope of V is defined as

$$\mu(V) = \deg(V)/\mathrm{rank}(V).$$

A vector bundle V is semi-stable (resp. stable) if for every proper subbundle W of V, $\mu(W) \leqslant \mu(V)$ (resp. $\mu(W) < \mu(V)$). The schemes S_O and S_θ are defined to be the moduli schemes of all S-equivalence classes [8] of rank 2 semi-stable vector bundles with determinant equal to O_X and L_θ, respectively.

A Higgs bundle (V, ϕ) with an associated line bundle L on X consists of a vector bundle V and a Higgs field which is a morphism of bundles $\phi : V \longrightarrow V \otimes L$. Frobenius pulls back Higgs fields $F^*(V) \xrightarrow{F^*(\phi)} F^*(V) \otimes F^*(L)$, hence, pulls back Higgs bundles.

A Higgs bundle (V, ϕ) is said to be semi-stable (resp. stable) if for every proper subbundle W of V, satisfying $\phi(W) \subset W \otimes L$, one has $\mu(W) \leqslant \mu(V)$ (resp. $\mu(W) < \mu(V)$). The scheme $S_O(\Omega)$ (resp. $S_\theta(L_\theta)$) is defined to be the moduli scheme of all S-equivalence classes of rank 2 semi-stable Higgs bundles on X with determinant O_X (resp. L_θ) and with associated line bundle Ω (resp. L_θ) [7].

Let

$$K = \{V \in S_O : V \text{ is semi-stable but not stable}\}.$$

Suppose $V \in K$. Then there exists $L \in J^0$ such that

$$0 \longrightarrow L^{-1} \longrightarrow V \longrightarrow L \longrightarrow 0.$$

The pull-back of V by Frobenius then fits into the following sequence

$$0 \longrightarrow L^{-p} \xrightarrow{f_1} F^*(V) \xrightarrow{f_2} L^p \longrightarrow 0.$$

PROPOSITION 3.1 *$F^* : K \longrightarrow K$ is a well-defined morphism.*

Proof. Let $H \subset V$ be a subbundle of maximum degree. If $f_2|_H = 0$, then $H = L^{-p}$ and $\deg(H) = \deg(L^{-p}) = 0$. If $f_2|_H \neq 0$, then $\deg(H) \leqslant \deg(L^p) = 0$. □

In general, $F^*(V)$ may not be semi-stable. For example, a theorem of Raynaud states that the bundle B_1 is always semi-stable while $F^*(B_1)$ is never semi-stable for all $p > 2$ [9]. The following theorem was communicated to Joshi by V. B. Mehta:

THEOREM 3.2. *Let X be a curve of genus 2 over an algebraically closed field of characteristic $p > 2$. Then there exists a finite set S, such that $F^*(V)$ is semi-stable for all $V \in S_O \setminus S$. In other words, F^* induces a morphism $F^* : S_O \setminus S \longrightarrow S_O$.*

Proof. By a theorem of Narasimhan–Ramanan, when $p = 0$ [6], $S_O \cong \mathbb{P}^3$. Moreover, as was remarked to one of us by Ramanan, the proof given there works in all characteristic $p \neq 2$. The Frobenius morphism is defined on a nonempty Zariski open set U in $S_O \cong \mathbb{P}^3$. By Proposition 3.1, U contains K which is an ample divisor in $\mathbb{P}^3$. Therefore $S_O \setminus U$ is of co-dimension 3, hence, is a finite set. Note that K can also be identified with the Kummer surface of J^0 in $\mathbb{P}^3$ [6]. □

When X is ordinary, F^* is étale on a nonempty Zariski open set of S_O [5]. Although unable to identify explicitly this finite set upon which the Frobenius is not defined, we provide the following criterion.

PROPOSITION 3.3. *Let X be a curve of genus 2 over an algebraically closed field of characteristic $p > 0$. Suppose $V \in S_O$ Then $F^*(V)$ is not semi-stable if and only if $F^*(V)$ is an extension*

$$0 \longrightarrow M \longrightarrow F^*(V) \longrightarrow M^{-1} \longrightarrow 0,$$

where $M \in J_2(L_\theta)$.

Proof. One direction is clear. We use inseparable descent to prove the other direction. Suppose $F^*(V)$ is not semi-stable. Then we have an exact sequence

$$0 \longrightarrow M \longrightarrow F^*(V) \longrightarrow M^{-1} \to 0,$$

where $\deg(M) > 0$.

Following [3], consider the natural connection on $F^*(V)$ with zero p-curvature. Then the second fundamental form of this connection is a morphism

$$T_X \longrightarrow \mathrm{Hom}(M, M^{-1}) = M^{-2}.$$

As V is semi-stable, this morphism must not be the zero morphism. In other words, $M^{-2} \otimes \Omega$ has a nonzero section. Since $\deg(M) > 0$ and $\deg(\Omega) = 2$, we must have $\Omega = M^2$. Hence $M \in J_2(L_\theta)$. □

4. The Moduli Spaces in Characteristic 2

In this section, we assume $p = 2$. Then B_1 is a line bundle and equal to a theta characteristic [9]. Choose L_θ to be B_1.

4.1. THE MODULI OF SEMI-STABLE BUNDLES

Suppose $V \in S_\theta$. By a theorem in [6], there exist $L_1 \in J^0$, $L_2 \in J^1$ with $L_1 \otimes L_2 = L_\theta$ such that $V \in \mathrm{Ext}^1(L_2, L_1)$. Since $L_\theta = B_1$, $h^0(B_1 \otimes L_2^{-1} \otimes L_1)$ is 1 if $L_\theta = L_2 \otimes L_1^{-1}$ and 0 otherwise. Hence, by Proposition 2.1, there is a unique (up to a scalar) V_1 not isomorphic to $O_X \oplus L_\theta$ and

$$0 \longrightarrow O_X \longrightarrow V_1 \longrightarrow L_\theta \longrightarrow 0$$

such that $F^*(V_1) = O_X \oplus \Omega$. It is immediate that V_1 is stable [6].

Suppose $V \notin J_2(V_1)$. Then by Proposition 2.1,

$$F^*(V) \neq F^*(L_1) \oplus F^*(L_2) = L_1^2 \oplus L_2^2.$$

If $M \subset F^*(V)$ is a destabilizing subbundle, i.e. $\deg(M) \geqslant 2$, then $M^{-1} \otimes L_2^2$ has a global section implying $\deg(M) \leqslant \deg(L_2^2) = 2$. Moreover, if $\deg(M) = 2$, then $M = L_2^2$ implying that $F^*(V)$ contains L_2^2 as a subbundle. Then the sequence

$$0 \longrightarrow L_1^2 \longrightarrow F^*(V) \longrightarrow L_2^2 \longrightarrow 0,$$

splits. This is a contradiction. This proves Theorem 1.1.

4.2. RESTORING FROBENIUS STABILITY: HIGGS BUNDLES

The scheme S_θ embeds in $S_\theta(L_\theta)$ by the map $V \longmapsto (V, 0)$. If $(V, \phi) \in S_\theta(L_\theta)$ and V is not semi-stable, then V is an extension

$$0 \longrightarrow L_1 \longrightarrow V \xrightarrow{f} L_2 \longrightarrow 0, \tag{2}$$

where $\deg(L_1) \geqslant 1 > \deg(L_2)$. Moreover, $\phi(L_1)$ is not contained in $L_1 \otimes L_\theta$ (otherwise $\phi(L_1) \subset L_1 \otimes L_\theta$ implying (V, ϕ) is not semi-stable). This implies that there exists a line bundle $H \subset V$ such that $H \neq L_1$ and $\phi(L_1) \subset H \otimes L_\theta$. Then

$$\deg(L_1) \leqslant \deg(H) + \deg(L_\theta).$$

Since $L_1 \neq H$, $0 \neq f(H) \subset L_2$ implies that $\deg(H) \leqslant \deg(L_2)$. To summarize, we have the following inequalities:

$$\deg(L_2) + \deg(L_\theta) \geqslant \deg(H) + \deg(L_\theta) \geqslant \deg(L_1) > \deg(L_2).$$

Since $\deg(L_\theta) = 1$, $\deg(L_1) = \deg(H) + 1 = \deg(L_2) + 1 = 1$. The degree of H is thus zero implying that $f(H) = L_2$, so the exact sequence (2) splits. In addition, since $0 \neq \phi(L_1) \subset H \otimes L_\theta$, $\phi|_{L_1}$ must be a nonzero constant morphism and $L_1 = L_2 \otimes L_\theta$. Since $L_1 \otimes L_2 = L_\theta$, $V \in J_2(O_X \oplus L_\theta)$. This proves the first part of Theorem 1.2.

Suppose $(V, \phi) \in S_\theta(L_\theta)$. If $V \in S_\theta \setminus J_2(V_1)$, then $F^*(V)$ is semi-stable by Theorem 1.1; hence, $(F^*(V), F^*(\phi))$ is semi-stable.

The split case: Suppose $W = L \oplus (L \otimes L_\theta)$, where $L \in J_2(O_X)$.

We take the Higgs field ϕ_0 to be the identity map:

$$1 = \phi_0 : L \otimes L_\theta \longrightarrow L \otimes L_\theta.$$

If $M \subset W$, then either $M = L \otimes L_\theta$ or $\mu(M) < \mu(W)$. Since $L \otimes L_\theta$ is not ϕ_0-invariant, (W, ϕ_0) is stable. The Frobenius pull-back $F^*(\phi_0)$ is again a constant map $F^*(\phi_0) : \Omega \longrightarrow O_X \otimes \Omega$. Now if $N \subset O_X \oplus \Omega$, then either $N = \Omega$ or $\mu(N) < \mu(O_X \oplus \Omega)$. Since Ω is not $F^*(\phi_0)$-invariant, $(F^*(W), F^*(\phi_0))$ is stable.

The non-split case: Suppose $V = L \otimes V_1$, where $L \in J_2(O_X)$.

The bundle V is a nontrivial extension:

$$0 \longrightarrow L \xrightarrow{f_1} V \xrightarrow{f_2} L \otimes L_\theta \longrightarrow 0. \tag{3}$$

Tensoring the sequence with L_θ gives

$$0 \longrightarrow L \otimes L_\theta \xrightarrow{g_1} V \otimes L_\theta \xrightarrow{g_2} L \otimes \Omega \longrightarrow 0. \tag{4}$$

Set

$$\phi_1 = g_1 \circ \phi_0 \circ f_2 : V \longrightarrow L_\theta \otimes V.$$

The Frobenius pull-back decomposes V: $F^*(V) = O_X \oplus \Omega$. Pulling back the exact

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sequences (3) and (4) by Frobenius gives

$$0\longrightarrow O_X \stackrel{F^*(f_1)}{\longrightarrow} O_X\oplus\Omega \stackrel{F^*(f_2)}{\longrightarrow} \Omega\longrightarrow 0$$

$$0\longrightarrow O_X\otimes\Omega \stackrel{F^*(g_1)}{\longrightarrow} (O_X\oplus\Omega)\otimes\Omega \stackrel{F^*(g_2)}{\longrightarrow} \Omega\otimes\Omega\longrightarrow 0$$

Suppose $N\subset O_X\oplus\Omega$. Then either $N=\Omega$ or $\mu(N)<\mu(O_X\oplus\Omega)$. The Frobenius pull-back of ϕ_1 is a composition:

$$F^*(\phi_1)=F^*(g_1)\circ F^*(\phi_0)\circ F^*(f_2).$$

Since the map $F^*(f_2)$ is surjective, the restriction map $F^*(f_2)|_\Omega$ is an isomorphism. The map ϕ_0 is an isomorphism and g_1 is injective; hence, $g_1\circ\phi_0$ is injective. This implies $F^*(g_1)\circ F^*(\phi_0)$ is injective. Therefore $F^*(\phi_1)|_\Omega$ is injective. Since $\deg(\Omega)<\deg(\Omega\otimes\Omega)$, $F^*(\phi_1)|_\Omega$ being injective implies

$$F^*(\phi_1)(\Omega)\not\subset\Omega\otimes\Omega\subset(O_X\oplus\Omega)\otimes\Omega.$$

In other words, $\Omega\subset O_X\oplus\Omega$ is not $F^*(\phi_1)$-invariant. Hence $(F^*(V),F^*(\phi_1))$ is stable. This proves Theorem 1.2.

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