Proof of the Conditions for a Turning Value of $f(x, y)$ without the use of Taylor's Theorem.

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The standard method of establishing rigidly the tests for a maximum or minimum value of a function of two independent variables, depending as it does on the use of Taylor's Theorem and on a very critical consideration of the Remainder in that theorem, presents difficulties so considerable that it is not surprising that most text-books on the Calculus frankly decline to enter on the discussion, and assume the necessity and sufficiency of the well known Lagrange's Condition. It is the object of this paper to show that a satisfactory proof of the tests may be given, from the purely geometrical standpoint, without recourse to Taylor's Theorem. The method requires only an elementary knowledge of the process of changing the independent variables in partial derivatives, and may therefore be introduced comparatively early in the Calculus course.

We may replace the usual definition of a turning value $f(x_1, y_1)$ of the function $f(x, y)$—that it is a value such that

$$\{f(x_1, y_1) - f(x_1 + \Delta x_1, y_1 + \Delta y_1)\}$$

is either always positive (Maximum $T.V.$) or always negative (Minimum $T.V.$) for any values of Δx_1, Δy_1, however small, which are less than certain finite small quantities—by its geometrical equivalent:

> "If $x = x_1$, $y = y_1$ give a turning value z_1 of $z \equiv f(x, y)$, then *any* plane through the line $x = x_1$, $y = y_1$ must cut the surface $z = f(x, y)$ in a curve which has a turning point of the same kind at (x_1, y_1, z_1)"

I. Let us first suppose that z has a *maximum* turning value at $P(x_1, y_1, z_1)$; let $NP \equiv z_1$ be that maximum ordinate. Draw any plane at all through NP; and now, keeping the same origin and z-axis, rotate the x- and y-axes through such an angle θ, into the new positions $\xi' 0 \xi$, $\eta' 0 \eta$, that $\xi' 0 \xi$ is parallel to the drawn plane. Let the new coordinates of P be (ξ_1, η_1, z_1). The equation of the plane through NP is now $\eta = \eta_1 =$ constant; it is "any" plane through NP, and the curve in which it cuts the surface will have a maximum turning point at P if $\dfrac{\partial z}{\partial \xi} = 0$ and $\dfrac{\partial^2 z}{\partial \xi^2} = -ve$, at (ξ_1, η_1, z_1).

But $x = \xi \cos\theta - \eta \sin\theta$,

$y = \xi \sin\theta + \eta \cos\theta$,

so
$$\frac{\partial z}{\partial \xi} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \xi}$$
$$= \cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y} = 0.$$

Since θ is any angle we have the first conditions
$$\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i})$$

Again,
$$\frac{\partial^2 z}{\partial \xi^2} = \frac{\partial z_\xi}{\partial \xi} = \cos\theta \frac{\partial z_\xi}{\partial x} + \sin\theta \frac{\partial z_\xi}{\partial y}$$
$$= \cos^2\theta \frac{\partial^2 z}{\partial x^2} + 2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 z}{\partial y^2}$$
$$= \sin^2\theta \left\{ \cot^2\theta \frac{\partial^2 z}{\partial x^2} + 2\cot\theta \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} \right\} \quad ..(\text{ii})$$

which is a quadratic in $\cot\theta$, and will have the invariable negative sign if
$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{iii})$$

and if $\dfrac{\partial^2 z}{\partial x^2} = -ve$, and therefore also, from (iii),
$$\frac{\partial^2 z}{\partial y^2} = -ve. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{iv})$$

The conditions (i), (iii) and (iv) together are accordingly the necessary and sufficient conditions that *any* section of $z = f(x, y)$ by a plane through NP shall be a curve having a maximum turning point at P, that is, that z_1 shall be a maximum turning value of $f(x, y)$.

The conditions may be derived in exactly the same way from the criteria
$$\frac{\partial z}{\partial \eta} = 0, \quad \frac{\partial^2 z}{\partial \eta^2} = -ve.,$$

but it is clearly unnecessary to discuss these, as the "ξ-section" considered was any section at all through NP.

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II. Similarly the conditions for a *minimum* turning value of z, viz.,

"Every $\frac{\partial z}{\partial \xi} = 0$; every $\frac{\partial^2 z}{\partial \xi^2} = +ve$" lead at once to conditions (i) and (iii) again, while (iv) is replaced by

$$\frac{\partial^2 z}{\partial x^2} = +ve, \quad \frac{\partial^2 z}{\partial y^2} = +ve. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

III. The interpretation of the cases when condition (iii) is not satisfied may now be treated in the same fashion.

Thus if

$$\left(\frac{\partial^2 z}{\partial x \, \partial y}\right)^2 > \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2}$$

the quadratic (ii) has real factors in $\cot\theta$, say $(\cot\theta - \cot\alpha)$ and $(\cot\theta - \cot\beta)$, where $0 < \alpha < \beta < \pi$. Its sign is accordingly no longer invariable; and so $\frac{\partial^2 z}{\partial \xi^2}$ has one sign when $\alpha < \theta < \beta$ and when $\pi + \alpha < \theta < \pi + \beta$, is zero for $\theta = \alpha, \beta, \pi + \alpha, \pi + \beta$, and has the opposite sign for other values of θ between 0 and 2π. The sections through NP therefore show maximum or minimum turning points, or neither, according to the value of θ; the value z_1 is not a true turning value of $f(x, y)$, and we need go no further.

If however

$$\left(\frac{\partial^2 z}{\partial x \, \partial y}\right)^2 = \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2},$$

the quadratic (ii) is a complete square, and vanishes for two values of θ between 0 and 2π, which differ by π and so give the same section through NP,—a section in which $\frac{\partial^2 z}{\partial \xi^2} = 0$ at P. The nature of the point in this case is uncertain, and we shall require to test the sign given by this value of θ to

$$\frac{\partial^3 z}{\partial \xi^3} \equiv \cos^3\theta \frac{\partial^3 z}{\partial x^3} + 3 \sin\theta \cos^2\theta \frac{\partial^3 z}{\partial x^2 \, \partial y} + 3 \sin^2\theta \cos\theta \frac{\partial^3 z}{\partial x \, \partial y^2} + \sin^3\theta \frac{\partial^3 z}{\partial y^3},$$

or to a still higher derivative, if this also should prove to be zero.