Canad. J. Math. Vol. 75 (4), 2023 pp. 1272–1292
http://dx.doi.org/10.4153/S0008414X22000359

Gelfand–Naimark theorems for ordered *-algebras

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Abstract. The classical Gelfand–Naimark theorems provide important insight into the structure of general and of commutative C^*-algebras. It is shown that these can be generalized to certain ordered *-algebras. More precisely, for σ-bounded closed ordered *-algebras, a faithful representation as operators is constructed. Similarly, for commutative such algebras, a faithful representation as complex-valued functions is constructed if an additional necessary regularity condition is fulfilled. These results generalize the Gelfand–Naimark representation theorems to classes of *-algebras larger than C^*-algebras, and which especially contain *-algebras of unbounded operators. The key to these representation theorems is a new result for Archimedean ordered vector spaces V: If V is σ-bounded, then the order of V is induced by the extremal positive linear functionals on V.

1 Introduction

A *-algebra is a unital associative algebra $\mathcal{A}$ over the field of complex numbers that is endowed with an antilinear involution $\cdot^*$ of $\mathcal{A}$ fulfilling $(ab)^* = b^* a^*$ for all $a, b \in \mathcal{A}$. Note that *-algebras are always assumed to have a unit which is denoted by $\mathbb{1}$. A C^*-algebra is a *-algebra that is complete with respect to a norm $\|\cdot\|$ on $\mathcal{A}$ that fulfils $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$ and $\|a^* a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

The Gelfand–Naimark representation theorems [6, Theorem 1 and Lemma 1] are cornerstones of the theory of C^*-algebras and—together with the well-behaved spectral theory—make C^*-algebras important tools in mathematical physics. In their simplest form, these two theorems state that all C^*-algebras have a faithful representation as *-algebras of bounded operators on a Hilbert space, and that all commutative C^*-algebras have a faithful representation as *-algebras of bounded complex-valued functions. This allows to interpret C^*-algebras as algebras of observables of physical systems: of quantum systems in general, and in the commutative case of classical systems. From this point of view, the perplexing differences between the description of quantum systems by means of operators on a Hilbert space, and of classical systems by functions on a smooth manifold, are just artifacts of the choice of two different ways to represent the observable algebras. Consequently, the problem of quantization, i.e., of finding a somehow suitable quantum system to a given classical one, can be formulated

Received by the editors March 22, 2022; revised June 21, 2022; accepted July 4, 2022.
Published online on Cambridge Core July 7, 2022.
This work was supported by the Fonds de la Recherche Scientifique (FNRS) and the Fonds Wetenschappelijk Onderzoek—Vlaaderen (FWO) under EOS Project no. 30950721.
AMS subject classification: **47L60**, 06F25.
Keywords: *-algebra, quadratic module, operator algebra, representation theory.

in a mathematically precise way, e.g., as finding deformations of commutative C^*-algebras to noncommutative ones in the sense of [10].

However, the restriction of the Gelfand–Naimark theorems to C^*-algebras is unfortunate. While there obviously exist many interesting examples of *-algebras of functions or operators that are not C^*-algebras, and for which an abstract description might be desirable, the main motivation for generalizing the Gelfand–Naimark theorems might again come from physics: It is well known that some of the most basic *-algebras of observables that a (physics-)student gets to know in a course on quantum mechanics are far away from being C^*-algebras: If $\mathcal{A}$ is a complex associative algebra with unit $\mathbb{1}$ and $P, Q \in \mathcal{A}$ fulfil the canonical commutation relation $[P, Q] := PQ - QP = \lambda\mathbb{1}$ with $\lambda \in \mathbb{C}\backslash\{0\}$, then the n-fold commutator of P with Q^n fulfils the identity $[P, [P, \dots [P, Q^n] \dots]] = n!\lambda^n\mathbb{1}$ for all $n \in \mathbb{N}$. Consequently, there cannot exist a nontrivial submultiplicative seminorm $\|\cdot\|$ on $\mathcal{A}$, because submultiplicativity would imply at most exponential growth with n of $\|[P, [P, \dots [P, Q^n] \dots]]\|$. This rules out any possibility to embed $\mathcal{A}$ in a C^*-algebra, and also in many weaker types of topological *-algebras like pro-C^*-algebras, for which one can prove rather direct generalizations of the Gelfand–Naimark theorems. Note that faithful representations of *-algebras of canonical commutation relations are well known and can be given, e.g., by differential operators. The problem is not to find faithful representations, but to find a sufficiently large class of *-algebras for which the existence of such faithful representations can be proved by general arguments.

This note gives a solution by focusing not so much on topological properties, but on order properties of *-algebras. This is motivated by [5], where it was shown that a suitable order on the Hermitian elements of a *-algebra $\mathcal{A}$ allows to construct a C^*-seminorm on the "bounded" elements of $\mathcal{A}$, and by [13], where the continuous calculus and the spectral theory of C^*-algebras have been extended to certain ordered *-algebras. In Section 2, some basic definitions and results, mainly from locally convex analysis, are recapitulated. The general idea then is to follow the classical approach from [8]: Section 3 develops the main result for ordered vector spaces, namely Theorem 3.7, which guarantees that on σ-bounded Archimedean ordered vector spaces, there exist many (extremal) positive linear functionals, and which considerably generalizes the result from [8]. Here, an ordered vector space is called "σ-bounded" essentially if it contains an increasing sequence of positive elements that eventually becomes greater than every fixed element. Theorem 4.9 shows that every σ-bounded closed ordered *-algebra can be represented faithfully as operators on a (pre-) Hilbert space, and Theorem 4.24 shows that in the commutative case, such algebras also admit a faithful representation as functions if an additional regularity condition (which is clearly necessary) is fulfilled.

2 Preliminaries

The natural numbers are $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and the fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. If X and Y are partially ordered sets (i.e., sets together with a reflexive, transitive, and antisymmetric relation $\leq$), then a map $\Phi: X \to Y$ is called *increasing* if $\Phi(x) \leq \Phi(x')$ holds for all $x, x' \in X$ with $x \leq x'$. It is called an *order embedding* if it is increasing and if additionally $x \leq$

x' holds for all $x, x' \in X$ for which $\Phi(x) \le \Phi(x')$. Note that an order embedding is injective. A partially ordered set X is called *directed* if, for all $x, x' \in X$, there exists a $y \in X$ such that $x \le y$ and $x' \le y$. Similarly, a subset S of a partially ordered set X is called directed if it is directed with respect to the order inherited from X. For two vector spaces V and W over the same field of scalars $\mathbb{F}$, the vector space of all linear maps from V to W is denoted by $\mathcal{L}(V, W)$. In the special case that $W = \mathbb{F}$, we write $V^* := \mathcal{L}(V, \mathbb{F})$, the elements of V^* are called *linear functionals* on V, and the evaluation of a linear functional $\omega \in V^*$ on a vector $v \in V$ is denoted by means of the bilinear *dual pairing* $\langle \cdot\,, \cdot \rangle : V^* \times V \to \mathbb{F}$, $(\omega, v) \mapsto \langle \omega\,, v \rangle$.

The main technical tools needed in the following are some basic theorems from locally convex analysis. A *filter* on a set X is a nonempty set $\mathcal{F}$ of subsets of X with the following two properties:

- If $S, T \in \mathcal{F}$, then $S \cap T \in \mathcal{F}$.
- If $S \in \mathcal{F}$ and if a subset T of X fulfils $T \supseteq S$, then $T \in \mathcal{F}$.

Similarly, a *basis of a filter* on X is a nonempty set $\mathcal{E}$ of subsets of X such that for all $S, T \in \mathcal{E}$, there exists an $R \in \mathcal{E}$ with $R \subseteq S \cap T$. In this case,

$$\langle\!\langle\, \mathcal{E} \,\rangle\!\rangle_{\mathrm{fltr}} := \left\{ T \subseteq X \;\middle|\; \exists_{S \in \mathcal{E}} : S \subseteq T \right\} \tag{2.1}$$

is a filter on X, called *the filter generated by* $\mathcal{E}$. A (real) *topological vector space* is a real vector space V endowed with a (not necessarily Hausdorff) topology under which addition $V \times V \to V$ as well as scalar multiplication $\mathbb{R} \times V \to V$ are continuous. Then it follows from the continuity of addition that a subset U of V is a neighborhood of a vector $v \in V$ if and only if $U - v := \{ u - v \mid u \in U \}$ is a neighborhood of 0, so the topology of V is completely described by the 0-neighborhoods. The set $\mathcal{N}_0$ of all 0-neighborhoods of V is a filter on V, and $\mathcal{N}_{0,\mathrm{conv}}$, the set of all convex 0-neighborhoods of V, is a basis of a filter on V. In general, $\langle\!\langle\, \mathcal{N}_{0,\mathrm{conv}} \,\rangle\!\rangle_{\mathrm{fltr}} \subseteq \mathcal{N}_0$ holds, and V is called a (real) *locally convex vector space* if even $\langle\!\langle\, \mathcal{N}_{0,\mathrm{conv}} \,\rangle\!\rangle_{\mathrm{fltr}} = \mathcal{N}_0$, i.e., if the filter of 0-neighborhoods of V is generated by the convex 0-neighborhoods. A *locally convex topology* on V is a topology with which V becomes a locally convex vector space.

There is an easy procedure to construct locally convex topologies on a real vector space V: A subset S of V is called *absorbing* if, for every $v \in V$, there is a $\lambda \in \left]0, \infty\right[$ such that $\lambda v \in S$, and it is called *balanced* if $\lambda s \in S$ for all $s \in S$ and all $\lambda \in [-1, 1]$. If $\mathcal{N}_{0,\mathrm{abc}}$ is a basis of a filter on V consisting of only absorbing balanced convex subsets of V, then its generated filter $\langle\!\langle\, \mathcal{N}_{0,\mathrm{abc}} \,\rangle\!\rangle_{\mathrm{fltr}}$ is the filter of 0-neighborhoods of a locally convex topology on V. Note that every absorbing balanced convex subset S of V yields a seminorm $\|\cdot\|_S$ on V by setting $\|v\|_S := \inf\{ \lambda \in \left]0, \infty\right[\mid \lambda^{-1} v \in S \}$ for all $v \in V$, which is the *Minkowski functional* of S. Conversely, for every seminorm $\|\cdot\|$ on V, the unit ball $B_{\|\cdot\|} := \{ v \in V \mid \|v\| \le 1 \}$ is an absorbing balanced convex subset of V whose Minkowski functional is again the original seminorm $\|\cdot\|$. However, the map from absorbing balanced convex subsets of V to seminorms on V is only surjective, but not injective in general. Because of this, there is more freedom in describing locally convex vector spaces via a basis of the filter of 0-neighborhoods than via the corresponding seminorms.

An important example of Hausdorff locally convex vector spaces is the dual space V^* of an arbitrary real vector space V with the *weak-*-topology*, i.e., with the weakest topology on V^* under which all the evaluation maps $V^* \ni \omega \mapsto \langle \omega , v \rangle \in \mathbb{R}$ with vectors $v \in V$ are continuous. This is the locally convex topology whose filter of 0-neighborhoods is generated by the intersections of unit balls of finitely many seminorms of the form $V^* \ni \omega \mapsto |\langle \omega , v \rangle| \in [0, \infty[$ with $v \in V$.

The following classical theorems will be crucial for the various representation theorems. As usual, for such results, their proofs typically make use of the axiom of choice.

Theorem 2.1 (Hahn–Banach) *Let V be a (real) locally convex vector space, let C be a closed and convex subset of V, and let $v \in V \setminus C$. Then there exists a continuous linear functional ω on V such that the inequality $\langle \omega , c \rangle \geq 1 + \langle \omega , v \rangle$ holds for all $c \in C$.*

Theorem 2.2 (Banach–Alaoglu) *Let V be a real vector space, and let U be an absorbing subset of V, then*

$$U^\circ := \left\{ \omega \in V^* \;\middle|\; \forall_{u \in U} : |\langle \omega , u \rangle| \leq 1 \right\} \tag{2.2}$$

is a convex subset of V^* and compact in the weak-*-topology.

If C is a convex subset of a real vector space V, then an *extreme point* of C is an element $e \in C$ with the following property: Whenever $e = \lambda c_1 + (1 - \lambda) c_2$ holds for some $c_1, c_2 \in C$ and $\lambda \in]0, 1[$, then $e = c_1 = c_2$. The set of all extreme points of C will be denoted by $\mathrm{ex}(C)$. Moreover, if V is a locally convex vector space and S a nonempty subset of V, then $\langle\!\langle S \rangle\!\rangle_{\text{cl-conv}}$ will denote the *closed convex hull* of S, i.e., the closure of the convex hull of S, or equivalently, the intersection of all closed convex subsets of V which contain S. In the special case that V is a real vector space and S a nonempty subset of V^*, then $\langle\!\langle S \rangle\!\rangle_{\text{cl-conv}}$ is understood to be the closed convex hull of S with respect to the weak-*-topology and

$$\langle\!\langle S \rangle\!\rangle_{\text{cl-conv}} = \left\{ \omega \in V^* \;\middle|\; \forall_{v \in V} : \langle \omega , v \rangle \in \langle\!\langle \{ \langle \rho , v \rangle \mid \rho \in S \} \rangle\!\rangle_{\text{cl-conv}} \right\} \tag{2.3}$$

holds. This can be seen either by elementary linear algebra or as a consequence of the Hahn–Banach Theorem and the fact that all weak-*-continuous linear functionals on V^* can be expressed as maps $V^* \ni \omega \mapsto \langle \omega , v \rangle \in \mathbb{R}$ with a suitable vector $v \in V$.

Theorem 2.3 (Krein–Milman) *Let V be a real vector space, and let K be a weak-*-compact and convex subset of V^*, then $K = \langle\!\langle \mathrm{ex}(K) \rangle\!\rangle_{\text{cl-conv}}$.*

The tools needed to prove the generalized Gelfand–Naimark theorems are essentially results about the existence of many (extremal) positive linear functionals on ordered vector spaces.

An *ordered vector space* is a real vector space V endowed with a partial order $\leq$ such that the conditions

$$u + v \leq u + w \qquad \text{and} \qquad \lambda v \leq \lambda w \tag{2.4}$$

hold for all $u, v, w \in V$ with $v \leq w$ and all $\lambda \in [0, \infty[$. Then $V^+ := \{ v \in V \mid 0 \leq v \}$ is the set of *positive elements* of V, which uniquely determines the order on V because $v \leq w$ is equivalent to $w - v \in V^+$ for all $v, w \in V$. An ordered vector space V is called *Archimedean* if it has the following property: Whenever $v \leq \varepsilon w$ holds for two vectors $v \in V$, $w \in V^+$, and all $\varepsilon \in]0, \infty[$, then $v \leq 0$. Note also that an ordered vector

https://doi.org/10.4153/S0008414X22000359 Published online by Cambridge University Press

space V is directed if and only if every $v \in V$ can be decomposed as $v = v_{(+)} - v_{(-)}$ with $v_{(+)}, v_{(-)} \in V^+$. Of course, such a decomposition is not uniquely determined in general.

If V and W are ordered vector spaces, then a linear map $\Phi\colon V \to W$ is increasing if and only if $\Phi(v) \in W^+$ for all $v \in V^+$. We write $\mathcal{L}(V, W)^+$ for the set of all increasing linear maps from V to W. If V is directed, then there actually exists a (unique) partial order on $\mathcal{L}(V, W)$ such that $\mathcal{L}(V, W)$ becomes an ordered vector space whose positive elements are precisely the increasing linear functions. However, for simplicity, a linear map $\Phi\colon V \to W$ will always be called *positive* if it is increasing, also in the case that V is not necessarily directed. Note that a positive linear map $\Phi\colon V \to W$ is an order embedding if and only if $\Phi(v) \in W \backslash W^+$ for all $v \in V \backslash V^+$. The case that $W = \mathbb{R}$ will be especially interesting: Then the set of positive linear functionals is denoted by

$$V^{*,+} := \mathcal{L}(V, \mathbb{R})^+ = \left\{ \omega \in V^* \;\middle|\; \forall_{v \in V^+} : \langle \omega , v \rangle \geq 0 \right\}. \tag{2.5}$$

If V is directed, then V^* is an ordered vector space itself and a positive linear functional ω on V is said to be *extremal* if, for every $\rho \in V^{*,+}$ with $\rho \leq \omega$, there is an $\mu \in [0, 1]$ such that $\rho = \mu\omega$. The set of all extremal positive linear functionals on V will then be denoted by $V^{*,+,\mathrm{ex}}$, and one can check that $0 \in V^{*,+,\mathrm{ex}}$ and $\lambda\omega \in V^{*,+,\mathrm{ex}}$ for all $\omega \in V^{*,+,\mathrm{ex}}$ and all $\lambda \in \,]0, \infty[$. Note that this definition of extremal positive linear functionals only makes sense on a directed ordered vector space V because it refers to the partial order on V^*. There is an extension theorem for (extremal) positive linear functionals from a sufficiently large linear subspace of a directed ordered vector space to the whole space (see [12, Lemma 1.3.2] for details)

Theorem 2.4 (Extension theorem) *Let V be a directed ordered vector space, and let S be a linear subspace of V with the property that for every $v \in V$, there exists an $s \in S$ such that $0 \leq s$ and $v \leq s$. Then S with the order inherited from V is a directed ordered vector space, and for every $\tilde{\omega} \in S^{*,+}$, there exists an $\omega \in V^{*,+}$ that extends $\tilde{\omega}$, i.e., that fulfils $\langle \omega , s \rangle = \langle \tilde{\omega} , s \rangle$ for all $s \in S$. Moreover, in the case that $\tilde{\omega} \in S^{*,+,\mathrm{ex}}$, there even exists $\omega \in V^{*,+,\mathrm{ex}}$ that extends $\tilde{\omega}$.*

The question of existence of many (extremal) positive linear functionals is nontrivial in general. More precisely, one asks whether or not the following two properties are fulfilled.

Definition 2.1 Let V be an ordered vector space, then we say that the order on V is *induced by its positive linear functionals* if, for all $v \in V \backslash V^+$, there is an $\omega \in V^{*,+}$ such that $\langle \omega , v \rangle < 0$.

Definition 2.2 Let V be a directed ordered vector space, then we say that the order on V is *induced by its extremal positive linear functionals* if, for all $v \in V \backslash V^+$, there is an $\omega \in V^{*,+,\mathrm{ex}}$ such that $\langle \omega , v \rangle < 0$.

These two conditions are equivalent to demanding that, for every $v \in V$, the inequality $0 \leq v$ holds if and only if $0 \leq \langle \omega , v \rangle$ for all $\omega \in V^{*,+}$ or for all $\omega \in V^{*,+,\mathrm{ex}}$, respectively.

We will see shortly that the above conditions of existence of "many" (extremal) positive linear functionals are closely related to the properties of certain locally convex topologies: Given two elements ℓ and u of an ordered vector space V, then the *order* *interval between ℓ and u* is defined as $[\ell, u] := \{ v \in V \mid \ell \le v \le u \}$. Moreover, a subset S of V is called *saturated* if $[\ell, u] \subseteq S$ is fulfilled for all $\ell, u \in S$ with $\ell \le u$. For example, every order interval is saturated. It is not hard to see that the intersection of finitely many (or arbitrarily many) saturated subsets of an ordered vector space V is again saturated. As a consequence, the set of all absorbing balanced convex and saturated subsets of V is a basis of the filter of 0-neighborhoods of a locally convex topology.

Definition 2.3 Let V be an ordered vector space, then the *normal topology* on V is the locally convex topology whose filter of 0-neighborhoods is generated by the absorbing balanced convex and saturated subsets of V.

This topology is not unknown in the theory of ordered vector spaces and *-algebras. For example, the normal topology is essentially the topology τ_n in [12, Section 1.5].

3 Existence of extremal positive linear functionals

Lemma 3.1 is a standard application of the Hahn–Banach theorem, and a proof is given for convenience of the reader.

Lemma 3.1 *Let V be an ordered vector space, and assume that there is a locally convex topology τ on V such that V^+ is closed with respect to τ. Then, for every $v \in V \backslash V^+$, there exists a positive linear functional ω on V that fulfils $\langle \omega, v \rangle < 0$ and that is continuous with respect to τ.*

Proof As V^+ is convex and closed with respect to τ, the Hahn-Banach theorem implies that, for every $v \in V \backslash V^+$, there exists a linear functional $\omega \in V^*$ that fulfils $\langle \omega, c \rangle \ge 1 + \langle \omega, v \rangle$ for all $c \in V^+$ and that is continuous with respect to τ. From $0 \in V^+$, it follows that $-1 \ge \langle \omega, v \rangle$. Moreover, $\langle \omega, c \rangle \ge 0$ holds for all $c \in V^+$, and hence $\omega \in V^{*,+}$: Indeed, if there was some $c \in V^+$ with $\langle \omega, c \rangle < 0$, then $\langle \omega, \lambda c \rangle = \langle \omega, v \rangle$ with $\lambda := \langle \omega, v \rangle / \langle \omega, c \rangle$, which yields a contradiction because $\lambda \in]0, \infty[$ by construction, and hence $\lambda c \in V^+$. ∎

Proposition 3.2 *Let V be an ordered vector space, then the order on V is induced by its positive linear functionals if and only if V^+ is closed in V with respect to the normal topology.*

Proof If V^+ is closed in V with respect to the normal topology, then the order on V is induced by its positive linear functionals as an immediate consequence of Lemma 3.1. Conversely, assume that the order on V is induced by its positive linear functionals, and let $v \in V \backslash V^+$ be given. Then there exists $\omega \in V^{*,+}$ such that $\langle \omega, v \rangle = -1$ and $U := \{ u \in V \mid \langle \omega, u \rangle \in]-1, 1[\}$ is an absorbing balanced convex and saturated subset of V, so $v + U$ is a neighborhood of v with respect to the normal topology. Moreover, application of ω shows that $(v + U) \cap V^+ = \emptyset$ holds, so V^+ is closed. ∎

For finding a sufficient condition under which the order of a directed ordered vector space is induced by its extremal positive linear functionals, it will be helpful to be able to decompose a positive linear functional into extremal ones. We follow essentially the argument of [12, Lemmas 12.4.3 and 12.4.4].

Proposition 3.3 *Let V be a directed ordered vector space, and let U be an absorbing balanced and directed subset of V. Write K for the set of all those $\omega \in V^{*,+}$ that fulfil $\langle \omega , u \rangle \leq 1$ for all $u \in U$, then K is weak-*-compact and $K = \langle\!\langle K \cap V^{*,+,\mathrm{ex}} \rangle\!\rangle_{\text{cl-conv}}$.*

Proof First, note that $K = U^\circ \cap V^{*,+}$ with U° like in (2.2) because U is balanced. As U° is convex and weak-*-compact by the Banach–Alaoglu theorem, and as $V^{*,+}$ is convex and weak-*-closed in V^*, it follows that K is also convex and weak-*-compact. The Krein–Milman theorem then shows that $K = \langle\!\langle \operatorname{ex}(K) \rangle\!\rangle_{\text{cl-conv}}$.

In order to complete the proof, it is sufficient to show that $\operatorname{ex}(K) \subseteq V^{*,+,\mathrm{ex}}$. Denote the linear span of K in V^* by W. Then the map $h\colon W \to [0,\infty[$

$$\omega \mapsto h(\omega) := \sup\left\{ |\langle \omega , u \rangle| \;\middle|\; u \in U \right\}$$

is a seminorm on W and $K = \{ \omega \in W \cap V^{*,+} \mid h(\omega) \leq 1 \}$. Moreover, as U is balanced and directed, $h(\omega + \omega') \geq h(\omega) + h(\omega')$, hence $h(\omega + \omega') = h(\omega) + h(\omega')$, holds for all $\omega, \omega' \in W \cap V^{*,+}$.

Now, let $\omega \in \operatorname{ex}(K)$ be given. If $h(\omega) = 0$, then $\omega = 0$ because U is absorbing, so $\omega \in V^{*,+,\mathrm{ex}}$ is trivially fulfilled. Otherwise, $h(\omega) = 1$ because on the one hand, $h(\omega) \leq 1$ is clear, and on the other, $\omega = h(\omega)\big(h(\omega)^{-1}\omega\big) + \big(1 - h(\omega)\big)0$ is a representation of ω as a convex combination of the two elements $h(\omega)^{-1}\omega$ and 0 of K, which excludes the possibility that $h(\omega) \in \,]0,1[$. In this second case that $h(\omega) = 1$, consider some $\rho \in V^{*,+}$ that fulfils $\rho \leq \omega$. Then, for all $u \in U$, there exists $v \in U \cap V^+$ such that $u \leq v$ because U is directed, and hence $\langle \rho , u \rangle \leq \langle \rho , v \rangle \leq \langle \omega , v \rangle \leq 1$. This means $\rho \in K$, and the same estimate with $\omega - \rho$ in place of ρ shows that $\omega - \rho \in K$ as well. If $h(\rho) = 0$ or $h(\omega - \rho) = 0$, then $\rho = \mu\omega$ with $\mu = 0$ or $\mu = 1$, respectively. Otherwise, $\omega = h(\rho)\big(h(\rho)^{-1}\rho\big) + h(\omega - \rho)\big(h(\omega - \rho)^{-1}(\omega - \rho)\big)$ is a representation of ω as a nontrivial convex combination of the two elements $h(\rho)^{-1}\rho$ and $h(\omega - \rho)^{-1}(\omega - \rho)$ of K; therefore, $\rho = \mu\omega$ with $\mu = h(\rho)$. We conclude that $\omega \in V^{*,+,\mathrm{ex}}$ in this case as well. ∎

Corollary 3.4 *Let V be a directed ordered vector space. If there exists a locally convex topology τ on V whose filter of 0-neighborhoods has a basis consisting of absorbing balanced convex and saturated as well as directed subsets of V and with respect to which V^+ is closed, then the order on V is induced by its extremal positive linear functionals.*

Proof Given $v \in V \backslash V^+$, then Lemma 3.1 shows that there exists a positive linear functional ω on V such that $\langle \omega , v \rangle < 0$ and that is continuous with respect to τ. Continuity of ω implies that there exists an absorbing balanced convex and saturated as well as directed subset U of V such that $|\langle \omega , u \rangle| \leq 1$ for all $u \in U$. Define the set $K := \{ \rho \in V^{*,+} \mid \forall_{u \in U} : \langle \rho , u \rangle \leq 1 \}$ like in Proposition 3.3, then $\omega \in K = \langle\!\langle K \cap V^{*,+,\mathrm{ex}} \rangle\!\rangle_{\text{cl-conv}}$, and equation (2.3) shows that there necessarily exists a $\rho \in K \cap V^{*,+,\mathrm{ex}}$ that also fulfils $\langle \rho , v \rangle < 0$. ∎

There is a rather large class of ordered vector spaces for which the normal topology cannot only be described explicitly, but also allows to apply Corollary 3.4.

Definition 3.1 An ordered vector space V is said to be *σ-bounded* if there exists an increasing sequence $(\hat{v}_n)_{n \in \mathbb{N}}$ in V^+ with the property that, for all $v \in V$, there is an $n \in \mathbb{N}$ such that $-\hat{v}_n \leq v \leq \hat{v}_n$ holds. Such a sequence will be called a *dominating sequence*.

Note that being σ-bounded can be seen as the combination of two properties: First, it is required that V is directed, and then, additionally, that there exists an increasing sequence $(\hat{v}_n)_{n\in\mathbb{N}}$ in V^+ which has the property that, for every $v \in V^+$, there exists an $n \in \mathbb{N}$ such that $v \leq \hat{v}_n$.

Definition 3.2 Let V be a σ-bounded ordered vector space with a dominating sequence $(\hat{v}_n)_{n\in\mathbb{N}}$, and let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence in $]0,\infty[$. Then the subset U_δ of V is defined as the union of increasing order intervals

$$U_\delta := \bigcup_{N\in\mathbb{N}} \Big[-\textstyle\sum_{n=1}^N \delta_n \hat{v}_n \,,\, \sum_{n=1}^N \delta_n \hat{v}_n \Big]. \tag{3.1}$$

Of course, U_δ depends not only on the sequence $(\delta_n)_{n\in\mathbb{N}}$, but also on the choice of the dominating sequence $(\hat{v}_n)_{n\in\mathbb{N}}$ and of V itself, which will always be clear from the context.

Proposition 3.5 *Let V be a σ-bounded ordered vector space with a dominating sequence $(\hat{v}_n)_{n\in\mathbb{N}}$, and let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence in $]0,\infty[$. Then U_δ is an absorbing balanced convex and saturated as well as directed subset of V.*

Proof Given $v \in V$, then there exists $n \in \mathbb{N}$ such that $-\hat{v}_n \leq v \leq \hat{v}_n$ and therefore $\delta_n v \in U_\delta$. So U_δ is absorbing. Every order interval in V of the form $[-w, w]$ with $w \in V^+$ clearly is balanced, convex, saturated, and directed. As U_δ is the union of an increasing sequence of such sets, it is balanced, convex, saturated, and directed itself. ∎

Proposition 3.6 *Let V be a σ-bounded ordered vector space, and let $(\hat{v}_n)_{n\in\mathbb{N}}$ be a dominating sequence in V. Then the set of all U_δ with $(\delta_n)_{n\in\mathbb{N}}$ a sequence in $]0,\infty[$ is a basis of the filter of 0-neighborhoods of the normal topology on V.*

Proof Proposition 3.5 already shows that such a set U_δ is a 0-neighborhood of the normal topology on V for every sequence $(\delta_n)_{n\in\mathbb{N}}$ in $]0,\infty[$.

Conversely, if S is a 0-neighborhood of the normal topology on V, then there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $]0,\infty[$ such that $U_\delta \subseteq S$, which can be constructed as follows: Let S' be an absorbing balanced convex and saturated subset of V such that $S' \subseteq S$. For every $n \in \mathbb{N}$, there exists $\epsilon_n \in]0,\infty[$ such that $\epsilon_n \hat{v}_n \in S'$ because S' is absorbing, and we can define $\delta_n := 2^{-n}\epsilon_n \in]0,\infty[$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^N \delta_n \hat{v}_n = \sum_{n=1}^N 2^{-n}\epsilon_n \hat{v}_n \in S'$ for all $N \in \mathbb{N}$ because S' is balanced and convex. Consequently, $\big[-\sum_{n=1}^N \delta_n \hat{v}_n, \sum_{n=1}^N \delta_n \hat{v}_n\big] \subseteq S' \subseteq S$ for all $N \in \mathbb{N}$ because S' is also saturated, so $U_\delta \subseteq S$. ∎

The above description of the normal topology on a σ-bounded ordered vector space has essentially already been given in [12, Propositions 4.1.2 and 4.1.3] and has also been applied indirectly to the decomposition of positive linear functionals into extremal ones in [12, Theorem 12.4.7]. Even though these results where stated for O^*-algebras or *-algebras, their above generalization to ordered vector spaces does not require any new techniques. Combining them with a completely order-theoretic characterization of those σ-bounded ordered vector spaces, whose set of positive elements is closed, gives our first important result.

https://doi.org/10.4153/S0008414X22000359 Published online by Cambridge University Press

Theorem 3.7 *Let V be a σ-bounded ordered vector space, then the following are equivalent:*

(1) *V is Archimedean.*
(2) *V^+ is closed in V with respect to the normal topology.*
(3) *The order on V is induced by its extremal positive linear functionals.*
(4) *The order on V is induced by its positive linear functionals.*

Proof Let $(\hat{v}_n)_{n\in\mathbb{N}}$ be a dominating sequence of V, which exists by assumption.

First, consider the case that V is Archimedean. In order to prove the implication (1) $\Longrightarrow$ (2), we have to show that $V\backslash V^+$ is open with respect to the normal topology on V. Given $v \in V\backslash V^+$, then construct recursively a sequence $(w_n)_{n\in\mathbb{N}_0}$ in $V\backslash V^+$ and a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $]0,\infty[$ as follows: Set $w_0 \coloneqq v$. If w_{n-1} has been defined for some $n \in \mathbb{N}$, then choose $\delta_n \in]0,\infty[$ such that $-w_{n-1} \le \delta_n \hat{v}_n$ does not hold, i.e., such that $w_{n-1} + \delta_n \hat{v}_n \in V\backslash V^+$, and set $w_n \coloneqq w_{n-1} + \delta_n \hat{v}_n$. Note that such a δ_n exists because $w_{n-1} \in V\backslash V^+$ and because V is Archimedean by assumption. From the construction of the sequence $(\delta_n)_{n\in\mathbb{N}}$, it follows that $v + U_\delta \subseteq V\backslash V^+$: Indeed, for every $x \in v + U_\delta$, there exists $N \in \mathbb{N}$ such that $x \le v + \sum_{n=1}^N \delta_n \hat{v}_n = w_N \in V\backslash V^+$, and hence $x \in V\backslash V^+$. Proposition 3.6 now shows that $V\backslash V^+$ is a neighborhood of v with respect to the normal topology on V.

The implication (2) $\Longrightarrow$ (3) is just an application of Corollary 3.4 using that all the subsets U_δ of V with $(\delta_n)_{n\in\mathbb{N}}$ a sequence in $]0,\infty[$ are absorbing balanced convex and saturated as well as directed by Proposition 3.5 and form a basis of the filter of 0-neighborhoods of the normal topology on the σ-bounded ordered vector space V by Proposition 3.6.

Finally, the implication (3) $\Longrightarrow$ (4) is trivial, and in order to prove (4) $\Longrightarrow$ (1), assume that the order on V is induced by its positive linear functionals and let $v \in V$ and $w \in V^+$ be given such that $v \le \varepsilon w$ for all $\varepsilon \in]0,\infty[$. Then it follows that $\langle \rho, v \rangle \le 0$ holds for all $\rho \in V^{*,+}$ because $\mathbb{R}$ is Archimedean, and therefore $v \le 0$. ∎

One special class of σ-bounded ordered vector spaces are ordered vector spaces V with a *strong order unit* e, i.e., an element $e \in V^+$ with the property that, for all $v \in V$, there exists $\lambda \in [0,\infty[$ such that $v \le \lambda e$. In this case, $(ne)_{n\in\mathbb{N}}$ is a dominating sequence, and the existence of positive linear functionals on Archimedean ordered vector spaces with a strong order unit was already proved in [8, Lemma 2.5]. Theorem 3.7 generalizes this classical result to σ-bounded ordered vector spaces and applies the decomposition of positive linear functionals into extremal ones from [12, Theorem 12.4.7].

It should be unnecessary to point out that there are many examples of important σ-bounded ordered vector spaces which do not have a strong order unit. These can be as ordinary as the space of real-valued polynomial functions on $\mathbb{R}$ with the pointwise order.

Before closing this section, we prove a new representation theorem for ordered vector spaces. Given any set X, then $\mathbb{R}^X$ denotes the ordered vector space of all $\mathbb{R}$-valued functions on X with the pointwise operations and the pointwise order. As a consequence of Theorem 3.7, we obtain the following corollary.

Corollary 3.8 *Let V be an Archimedean σ-bounded ordered vector space, then the map $\pi\colon V \to \mathbb{R}^{V^{*,+,\mathrm{ex}}}$, $v \mapsto \pi(v)$ with $\pi(v)\colon V^{*,+,\mathrm{ex}} \to \mathbb{R}$,*

$$\omega \mapsto \pi(v)(\omega) := \langle\, \omega\,,\, v\,\rangle, \tag{3.2}$$

is linear and an order embedding.

Proof It is easily checked that π is linear and increasing. Given $v \in V\backslash V^+$, then there exists $\omega \in V^{*,+,\mathrm{ex}}$ such that $\langle\, \omega\,,\, v\,\rangle < 0$ because the order on V is induced by its extremal positive linear functionals by Theorem 3.7, so $\pi(v)(\omega) < 0$, which means that $\pi(v) \in \mathbb{R}^{V^{*,+,\mathrm{ex}}}\backslash(\mathbb{R}^{V^{*,+,\mathrm{ex}}})^+$. This shows that π is an order embedding. ■

If V is a Riesz space, i.e., an ordered vector space in which all finite infima and suprema exist, then Corollary 3.8 essentially is the representation theorem for Riesz spaces from [4]: One can show that, in this case, the extremal positive linear functionals on V are Riesz morphisms, i.e., compatible with finite infima and suprema, so that the representation π from (3.2) is also a Riesz morphism. We will not discuss the details here, but rather turn our attention in the next section to the case that V is an ordered vector space that carries a multiplication.

4 Representation of ordered *-algebras

We now come to the main section, which develops the generalized Gelfand–Naimark theorems for ordered *-algebras. The definition of *-algebras has already been given in the introduction. An element a of a *-algebra $\mathcal{A}$ is called *Hermitian* if $a = a^*$ and the real linear subspace of all Hermitian elements in $\mathcal{A}$ is denoted by $\mathcal{A}_{\mathrm{H}}$. An *ordered* **-algebra* is a *-algebra $\mathcal{A}$ together with a partial order $\le$ on $\mathcal{A}_{\mathrm{H}}$ such that $\mathcal{A}_{\mathrm{H}}$ becomes an ordered vector space fulfilling $0 \le \mathbb{1}$ and $a^* b\, a \in \mathcal{A}_{\mathrm{H}}^+$ for all $a \in \mathcal{A}$, $b \in \mathcal{A}_{\mathrm{H}}^+$. This ordered vector space of Hermitian elements is automatically directed because $4a = (a+\mathbb{1})^2 - (a-\mathbb{1})^2$ and $(a \pm \mathbb{1})^2 \in \mathcal{A}_{\mathrm{H}}^+$ hold for all $a \in \mathcal{A}_{\mathrm{H}}$. A *unital* **-subalgebra* $\mathcal{B}$ of an ordered *-algebra $\mathcal{A}$, i.e., a linear subspace $\mathcal{B}$ of $\mathcal{A}$ fulfilling $\mathbb{1} \in \mathcal{B}$ as well as $b^* \in \mathcal{B}$ and $bb' \in \mathcal{B}$ for all $b, b' \in \mathcal{B}$, is again an ordered *-algebra with the order on $\mathcal{B}_{\mathrm{H}}$ inherited from $\mathcal{A}_{\mathrm{H}}$. Ordered *-algebras have already been used for understanding representations of *-algebras, e.g., in [9] or [12]. The set of positive Hermitian elements of an ordered *-algebra is an "m-admissible cone" in the language of [12], or a "quadratic module" in the language of (noncommutative) real algebraic geometry.

An ordered *-algebra $\mathcal{A}$ is called σ*-bounded* if the ordered vector space $\mathcal{A}_{\mathrm{H}}$ is σ-bounded. Similarly, an ordered *-algebra $\mathcal{A}$ is said to be *closed* if $\mathcal{A}_{\mathrm{H}}$ is an Archimedean ordered vector space. Note that this is not a topological property but rather an order property; only for σ-bounded ordered *-algebras, it follows from Theorem 3.7 that $\mathcal{A}$ is closed as an ordered *-algebra if and only if the quadratic module $\mathcal{A}_{\mathrm{H}}^+$ is closed in $\mathcal{A}_{\mathrm{H}}$ with respect to a certain locally convex topology. Note also that this property is not at all related to the notion of Archimedean quadratic modules in real algebraic geometry. It is unfortunate that the term "Archimedean" is being used in several different ways. Because of this, the term "Archimedean" should be avoided in the context of ordered *-algebras. There are some important examples of ordered *-algebras that will be relevant in the following, and these also illustrate that the notion of ordered *-algebras used here is rather minimalistic.

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Example 4.1 For every set X, the space $\mathbb{C}^X$ of all complex-valued functions on X with the pointwise operations and the pointwise order on the Hermitian (i.e., real-valued) functions is a commutative, closed, and ordered *-algebra.

Example 4.2 If $\mathcal{D}$ is a pre-Hilbert space, i.e., a complex vector space endowed with an inner product $\langle \cdot \mid \cdot \rangle$, antilinear in the first and linear in the second argument, then we write $\mathcal{L}^*(\mathcal{D})$ for the *-algebra of all linear endomorphisms a of $\mathcal{D}$ that are adjointable in the algebraic sense, i.e., for which there exists a (necessarily unique) linear endomorphism a^* of $\mathcal{D}$ such that $\langle \phi \mid a(\psi) \rangle = \langle a^*(\phi) \mid \psi \rangle$ holds for all $\phi, \psi \in \mathcal{D}$. An element a of $\mathcal{L}^*(\mathcal{D})$ is Hermitian if and only if $\langle \phi \mid a(\phi) \rangle \in \mathbb{R}$ for all $\phi \in \mathcal{D}$ and $\mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$ will always be endowed with the usual partial order of operators, i.e., given $a \in \mathcal{L}^*(\mathcal{D})_{\mathrm{H}}$, then a is positive if and only if $\langle \phi \mid a(\phi) \rangle \geq 0$ for all $\phi \in \mathcal{D}$. This way, $\mathcal{L}^*(\mathcal{D})$ becomes a closed ordered *-algebra which is not commutative in general. Unital *-subalgebras of $\mathcal{L}^*(\mathcal{D})$ are called *O^*-algebras* (see, e.g., [12]).

Example 4.3 A *Φ-algebra* is an Archimedean ordered vector space $\mathcal{R}$ in which the supremum $r \vee s \coloneqq \sup\{r, s\}$ and the infimum $r \wedge s \coloneqq \inf\{r, s\}$ of any two elements $r, s \in \mathcal{R}$ exist, and that is endowed with a bilinear product that turns $\mathcal{R}$ into an associative, commutative, and unital real algebra with the properties that $rs \in \mathcal{R}^+$ for all $r, s \in \mathcal{R}^+$ and $rs = 0$ for all $r, s \in \mathcal{R}$ with $r \wedge s = 0$. With $|r| \coloneqq r \vee (-r) \in \mathcal{R}^+$, it follows from the properties of $\vee$ and $\wedge$ that $(|r| - r) \wedge (|r| + r) = |r| + ((-r) \wedge r) = |r| - |r| = 0$, so $|r|^2 - r^2 = (|r| - r)(|r| + r) = 0$, i.e., $r^2 = |r|^2 \geq 0$ for all $r \in \mathcal{R}$. The complexification $\mathcal{A} \coloneqq \mathcal{R} \otimes \mathbb{C}$ of such a Φ-algebra $\mathcal{R}$ is therefore a commutative, closed, and ordered *-algebra whose real linear subspace of Hermitian elements is $\mathcal{A}_{\mathrm{H}} \cong \mathcal{R}$. For Φ-algebras, there exists, e.g., a representation theorem by means of functions with values in the extended real line $[-\infty, \infty]$ (see [7]). In Theorem 4.24, we will prove a representation theorem by means of $\mathbb{R}$-valued functions under the additional assumption of σ-boundedness.

Example 4.4 If $\mathcal{A}$ is a C^*-algebra, then its Hermitian elements can be endowed with a partial order that turns $\mathcal{A}$ into a closed ordered *-algebra in which the positive Hermitian elements are precisely those Hermitian ones whose spectrum is a subset of $[0, \infty[$. This is a well known, but nontrivial result in the theory of C^*-algebras. Showing, e.g., that the sum of two positive Hermitian elements is again a positive Hermitian element required some considerable effort in the original proof of the (noncommutative) representation theorem for C^*-algebras in [6]. Moreover, in a C^*-algebra $\mathcal{A}$, the unit $\mathbb{1}$ is a strong order unit of $\mathcal{A}_{\mathrm{H}}$, i.e., for every $a \in \mathcal{A}_{\mathrm{H}}$, there exists a $\lambda \in [0, \infty[$ such that $a \leq \lambda \mathbb{1}$ (see also the discussion under Theorem 3.7).

Example 4.5 Let $\mathcal{A}$ be a *-algebra, and define the set

$$\mathcal{A}_{\mathrm{H}}^{++} \coloneqq \left\{ \sum_{n=1}^{N} a_n^* a_n \;\middle|\; N \in \mathbb{N}; a_1, \dots, a_N \in \mathcal{A} \right\}$$

of *algebraically positive elements* in $\mathcal{A}$. If $\mathcal{A}_{\mathrm{H}}^{++}$ does not contain a real linear subspace of $\mathcal{A}_{\mathrm{H}}$ in addition to the trivial one $\{0\}$, then $\mathcal{A}$ can be turned into an ordered *-algebra such that $\mathcal{A}_{\mathrm{H}}^{+} = \mathcal{A}_{\mathrm{H}}^{++}$. Otherwise, i.e., if there is $a \in \mathcal{A}_{\mathrm{H}} \setminus \{0\}$ such that both a and $-a$ are algebraically positive, there is no possibility to turn $\mathcal{A}$ into an ordered *-algebra. So the existence of a suitable order on a *-algebra (especially the antisymmetry of the order) is a nontrivial condition. We will see that this, together with two or three further conditions, allows to prove representation theorems similar to, but more general than those known for C^*-algebras.

Note also that there are many different types of ordered algebras that have been examined in the last decades. Motivations to do so can be rather divers, e.g., generalizing results for ordered vector spaces (especially Riesz spaces) to ordered algebras, generalizing results from (commutative) real algebraic geometry to the noncommutative case, or studying operator algebras by purely algebraic means. Consequently, one can find conflicting definitions in the literature.

Example 4.6 The definition of O^*-algebras used in [1] is different from the one of Example 4.2. In fact, the notion of complex O^*-algebras used there is a special case of the ordered *-algebras here, requiring also existence and good behavior of certain suprema and infima of algebra elements (the stability of the set of positive Hermitian elements under conjugations $a \mapsto b^* a\, b$ with arbitrary algebra elements b is a nontrivial consequence of their axioms of O^*-algebras; see the remark under Theorem 2.2). In [1, Theorem 5.5], a representation theorem is proved that makes use of these infima and suprema. In contrast to this, Theorems 4.24 and 4.9 here are based on Theorem 3.7, and therefore require σ-boundedness instead.

If $\mathcal{A}$ is a *-algebra, then its complex dual vector space $\mathcal{A}^*$ carries an antilinear involution $\cdot^* : \mathcal{A}^* \to \mathcal{A}^*$, $\omega \mapsto \omega^*$, given by $\langle\, \omega^*\, ,\, a \,\rangle := \overline{\langle\, \omega\, ,\, a^* \,\rangle}$ for all $a \in \mathcal{A}$. We say again that an element $\omega \in \mathcal{A}^*$ is *Hermitian* if $\omega^* = \omega$ and write $\mathcal{A}^*_{\mathrm{H}}$ for the set of all Hermitian linear functionals on $\mathcal{A}$, which is a real linear subspace of $\mathcal{A}^*$. Note that a linear functional ω on $\mathcal{A}$ is Hermitian if and only if $\langle\, \omega\, ,\, a \,\rangle \in \mathbb{R}$ holds for all $a \in \mathcal{A}_{\mathrm{H}}$. Thus, every $\omega \in \mathcal{A}^*_{\mathrm{H}}$ can be restricted to an $\mathbb{R}$-linear functional on $\mathcal{A}_{\mathrm{H}}$, and one can check that this restriction describes an $\mathbb{R}$-linear isomorphism between the vector spaces $\mathcal{A}^*_{\mathrm{H}}$ and $(\mathcal{A}_{\mathrm{H}})^*$. An *(extremal) positive Hermitian linear functional* on an ordered *-algebra $\mathcal{A}$ is then defined as a Hermitian linear functional on $\mathcal{A}$ whose restriction to a (real) linear functional on the ordered vector space $\mathcal{A}_{\mathrm{H}}$ is an (extremal) positive linear functional. The sets of these (extremal) positive Hermitian linear functionals are denoted by $\mathcal{A}^{*,+}_{\mathrm{H}}$ and $\mathcal{A}^{*,+,\mathrm{ex}}_{\mathrm{H}}$, respectively, and we say that the order on $\mathcal{A}$ is *induced by its (extremal) positive Hermitian linear functionals* if the order on $\mathcal{A}_{\mathrm{H}}$ is induced by its (extremal) positive linear functionals. Note that positivity of a Hermitian linear functional ω on an ordered *-algebra $\mathcal{A}$ is in general a stronger condition than just the requirement that $\langle\, \omega\, ,\, a^* a \,\rangle \geq 0$ for all $a \in \mathcal{A}$, which is used quite often in the literature when linear functionals on general *-algebras are discussed. However, if $\mathcal{A}$ is an ordered *-algebra in which only the algebraically positive Hermitian linear functionals are positive, i.e., if $\mathcal{A}$ is of the type of Example 4.5, then $\langle\, \omega\, ,\, a^* a \,\rangle \geq 0$ for all $a \in \mathcal{A}$ is also sufficient for a Hermitian linear functional ω on $\mathcal{A}$ to be positive.

Positive Hermitian linear functionals on an ordered *-algebra $\mathcal{A}$ have some nice properties: Given $\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}$ and $a, b \in \mathcal{A}$, then the *Cauchy–Schwarz inequality*

$$\big|\langle\, \omega\, ,\, b^* a \,\rangle\big|^2 \leq \langle\, \omega\, ,\, b^* b \,\rangle \langle\, \omega\, ,\, a^* a \,\rangle \tag{4.1}$$

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holds, as well as

$$|\langle \omega , a \rangle|^2 \le \langle \omega , \mathbb{1} \rangle \langle \omega , a^* a \rangle \tag{4.2}$$

in the special case that $b = \mathbb{1}$. This has an important consequence: If a positive Hermitian linear functional ω on $\mathcal{A}$ fulfils $\langle \omega , \mathbb{1} \rangle = 0$, then $\omega = 0$. A *state* on $\mathcal{A}$ is a positive Hermitian linear functional ω on $\mathcal{A}$ that fulfils $\langle \omega , \mathbb{1} \rangle = 1$. So every $\tilde{\omega} \in \mathcal{A}_{\mathrm{H}}^{*,+} \setminus \{0\}$ is a multiple of a unique state ω on $\mathcal{A}$, namely of $\omega = \langle \tilde{\omega} , \mathbb{1} \rangle^{-1} \tilde{\omega}$. The set of all states on $\mathcal{A}$ will be denoted by $\mathcal{S}(\mathcal{A})$ and is clearly a convex (possibly empty) subset of the real vector space $\mathcal{A}_{\mathrm{H}}^*$. Again, note that by this definition, states are positive on whole $\mathcal{A}_{\mathrm{H}}^+$, not just on squares.

A map $\Phi: \mathcal{A} \to \mathcal{B}$ between two *-algebras is said to be *multiplicative* if $\Phi(aa') = \Phi(a)\Phi(a')$ holds for all $a, a' \in \mathcal{A}$. Furthermore, it is called a *unital*-homomorphism* if it is linear and multiplicative, maps the unit of $\mathcal{A}$ to the unit of $\mathcal{B}$ and fulfils $\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathcal{A}$. This last condition is equivalent to $\Phi(a) \in \mathcal{B}_{\mathrm{H}}$ for all $a \in \mathcal{A}_{\mathrm{H}}$. If $\mathcal{A}$ and $\mathcal{B}$ are ordered *-algebras, then such a unital *-homomorphism Φ is called *positive* or an *order embedding* if its restriction to an $\mathbb{R}$-linear map between the ordered vector spaces $\mathcal{A}_{\mathrm{H}}$ and $\mathcal{B}_{\mathrm{H}}$ is positive or an order embedding, respectively.

For ordered *-algebras, we are going to discuss two different types of representations, which correspond to the first two examples mentioned above.

Definition 4.1 Let $\mathcal{A}$ be an ordered *-algebra. Then a *representation as functions* of $\mathcal{A}$ is a tuple (X, π) of a set X and a positive unital *-homomorphism $\pi: \mathcal{A} \to \mathbb{C}^X$. Similarly, a *representation as operators* of $\mathcal{A}$ is a tuple $(\mathcal{D}, \pi)$ of a pre-Hilbert space $\mathcal{D}$ and a positive unital *-homomorphism $\pi: \mathcal{A} \to \mathcal{L}^*(\mathcal{D})$. Moreover, such a representation (as functions or as operators) is called *faithful* if π is an order embedding.

Of course, representations as functions are especially interesting for commutative ordered *-algebras. As we will see, the existence of faithful representations of an ordered *-algebra $\mathcal{A}$ is closely linked to the question of whether or not the order on $\mathcal{A}$ is induced by its (extremal) positive Hermitian linear functionals.

4.1 Representation as operators

The well-known construction of the Gelfand–Naimark–Segal (GNS) representation yields a representation as operators of an ordered *-algebra $\mathcal{A}$ out of any positive Hermitian linear functional.

Proposition 4.7 *Let $\mathcal{A}$ be an ordered *-algebra, and let $\omega \in \mathcal{A}_{\mathrm{H}}^{*,+}$. Then the map $\langle \cdot \,|\, \cdot \rangle_\omega : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$,*

$$(a, b) \mapsto \langle a \,|\, b \rangle_\omega := \langle \omega , a^* b \rangle, \tag{4.3}$$

is sesquilinear (antilinear in the first, and linear in the second argument) and fulfils $\overline{\langle a \,|\, b \rangle_\omega} = \langle b \,|\, a \rangle_\omega$ as well as $\langle a \,|\, a \rangle_\omega \in [0, \infty[$ for all $a, b \in \mathcal{A}$. Write $\|\cdot\|_\omega$ for the corresponding seminorm on $\mathcal{A}$, defined as $\|a\|_\omega := \langle a \,|\, a \rangle_\omega^{1/2}$ for all $a \in \mathcal{A}$, and write $\ker\|\cdot\|_\omega := \{ a \in \mathcal{A} \mid \|a\|_\omega = 0 \}$ for its kernel. Then $\langle \cdot \,|\, \cdot \rangle_\omega$ remains well defined on the quotient vector space $\mathcal{A}/\ker\|\cdot\|_\omega$ on which it describes an inner product. Now, write $\mathcal{D}_\omega$ for the pre-Hilbert space of $\mathcal{A}/\ker\|\cdot\|_\omega$ with inner product $\langle \cdot \,|\, \cdot \rangle_\omega$, and

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$[b]_\omega \in \mathcal{A}/\ker\|\cdot\|_\omega$ for the equivalence class of an element $b \in \mathcal{A}$. Then, for every $a \in \mathcal{A}$, the map $\pi_\omega(a): \mathcal{D}_\omega \to \mathcal{D}_\omega$,

$$[b]_\omega \mapsto \pi_\omega(a)\big([b]_\omega\big) := [ab]_\omega, \tag{4.4}$$

is a well-defined linear endomorphism of $\mathcal{D}_\omega$, it is adjointable with adjoint $\pi_\omega(a^*)$, and the resulting map $\mathcal{A} \ni a \mapsto \pi_\omega(a) \in \mathcal{L}^*(\mathcal{D}_\omega)$ is a positive unital *-homomorphism. Altogether, $(\mathcal{D}_\omega, \pi_\omega)$ is a representation as operators of the ordered *-algebra $\mathcal{A}$.

Proof The only detail which is not completely part of the classical GNS construction for *-algebras as described, e.g., in [12, Section 8.6], is the observation that π_ω is not only a unital *-homomorphism, but also positive, because $\langle\, [b]_\omega \,|\, \pi_\omega(a)\,[b]_\omega \,\rangle = \langle\, \omega\,,\, b^* a\, b \,\rangle \geq 0$ for all $a \in \mathcal{A}^+_{\mathrm{H}}$. ∎

Definition 4.2 Let $\mathcal{A}$ be an ordered *-algebra, and let $\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}$, then the representation as operators $(\mathcal{D}_\omega, \pi_\omega)$ from Proposition 4.7 is called the *GNS representation of $\mathcal{A}$ with respect to ω.*

The problem of existence of representations as operators of ordered *-algebras can be treated completely analogous to the case of general *-algebras:

Proposition 4.8 *Let $\mathcal{A}$ be an ordered *-algebra, then there exists a faithful representation as operators of $\mathcal{A}$ if and only if the order on $\mathcal{A}$ is induced by its positive Hermitian linear functionals.*

Proof Assume that there exists a faithful representation as operators $(\mathcal{D}, \pi)$ of $\mathcal{A}$. Given $a \in \mathcal{A}_{\mathrm{H}} \setminus \mathcal{A}^+_{\mathrm{H}}$, then there exists $\phi \in \mathcal{D}$ such that $\langle\, \phi \,|\, \pi(a)(\phi) \,\rangle < 0$. However, $\mathcal{A} \ni b \mapsto \langle\, \phi \,|\, \pi(b)(\phi) \,\rangle \in \mathbb{C}$ is a positive Hermitian linear functional. So we conclude that the order on $\mathcal{A}$ is induced by its positive Hermitian linear functionals.

Conversely, assume that the order on $\mathcal{A}$ is induced by its positive Hermitian linear functionals. Using the GNS representations of $\mathcal{A}$, define the orthogonal sum of pre-Hilbert spaces

$$\mathcal{D}_{\mathrm{tot}} := \bigoplus_{\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}} \mathcal{D}_\omega$$

with inner product denoted by $\langle\, \cdot \,|\, \cdot \,\rangle_{\mathrm{tot}}$, as well as for every element $a \in \mathcal{A}$ the linear endomorphism

$$\pi_{\mathrm{tot}}(a) := \bigoplus_{\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}} \pi_\omega(a)$$

of $\mathcal{D}_{\mathrm{tot}}$, i.e., $\sum_{\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}} [b]_\omega \mapsto \pi_{\mathrm{tot}}(a)\big(\sum_{\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}} [b]_\omega\big) := \sum_{\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}} \pi_\omega(a)\big([b]_\omega\big)$. Then it is easy to check that $\pi_{\mathrm{tot}}(a)$ is adjointable with adjoint $\pi_{\mathrm{tot}}(a^*)$ and that the resulting map $\pi_{\mathrm{tot}}: \mathcal{A} \to \mathcal{L}^*(\mathcal{D}_{\mathrm{tot}})$, $a \mapsto \pi_{\mathrm{tot}}(a)$ is a positive unital *-homomorphism. Moreover, π_{tot} is an order embedding: Indeed, for every $a \in \mathcal{A}_{\mathrm{H}} \setminus \mathcal{A}^+_{\mathrm{H}}$, there exists an $\omega \in \mathcal{A}^{*,+}_{\mathrm{H}}$ such that $\langle\, \omega\,,\, a \,\rangle < 0$ and thus $\langle\, [\mathbb{1}]_\omega \,|\, \pi_{\mathrm{tot}}(a)([\mathbb{1}]_\omega) \,\rangle_{\mathrm{tot}} = \langle\, \omega\,,\, a \,\rangle < 0$. It follows that $(\mathcal{D}_{\mathrm{tot}}, \pi_{\mathrm{tot}})$ is a faithful representation as operators. ∎

Application of Theorem 3.7 to Proposition 4.8 immediately yields the following generalization of the (noncommutative) Gelfand–Naimark theorem.

Theorem 4.9 *Let $\mathcal{A}$ be a σ-bounded ordered *-algebra, then $\mathcal{A}$ has a faithful representation as operators if and only if $\mathcal{A}$ is closed.*

The original (noncommutative) Gelfand–Naimark theorem is essentially the special case of this Theorem 4.9 for ordered *-algebras $\mathcal{A}$ in which the multiplicative unit $\mathbb{1}$ is also a strong order unit. As discussed under Theorem 3.7, $\mathcal{A}$ is automatically σ-bounded in this case. Note also that the image of a σ-bounded ordered *-algebra $\mathcal{A}$ under a faithful representation as operators is an O^*-algebra with metrizable graph topology in the language of [12], which, conversely, are always σ-bounded. So Theorem 4.9 yields an order-theoretic characterization of the O^*-algebras with metrizable graph topology. This goes in a similar direction as [11], where a topological characterization of a large class of O^*-algebras has been given.

Despite being valid only in the σ-bounded case, Theorem 4.9 still implies that, heuristically, every closed ordered *-algebra "behaves essentially like an O^*-algebra". In order to make this more precise, we need the following lemma.

Lemma 4.10 *Let $\mathcal{A}$ be a *-algebra that is generated by a countable subset of $\mathcal{A}$. If $\mathcal{A}_{\mathrm{H}}$ is endowed with any partial order $\leq$ such that $\mathcal{A}$ becomes an ordered *-algebra, then $\mathcal{A}$ with this order is σ-bounded.*

Proof As $\mathcal{A}$ is generated by a countable subset, it has at most countable dimension, so there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_{\mathrm{H}}$ such that the $\mathbb{R}$-linear span of $\{ a_n \mid n \in \mathbb{N} \}$ is whole $\mathcal{A}_{\mathrm{H}}$. Moreover, from $(\mathbb{1} \pm a_n)^2 \in \mathcal{A}^+_{\mathrm{H}}$, it follows that $\pm 2a_n \leq \mathbb{1} + a_n^2$ for all $n \in \mathbb{N}$. So define the increasing sequence $(\hat{v}_n)_{n\in\mathbb{N}}$ in $\mathcal{A}^+_{\mathrm{H}}$ as $\hat{v}_n := n\sum_{k=1}^{n}(\mathbb{1} + a_n^2)$, then $(\hat{v}_n)_{n\in\mathbb{N}}$ is a dominating sequence because for every $b \in \mathcal{A}_{\mathrm{H}}$ there exists an $n \in \mathbb{N}$ such that b can be expressed as $b = \sum_{k=1}^{n} 2\beta_n a_n$ with coefficients $\beta_1, \dots, \beta_n \in [-n, n]$, and hence $b \leq \hat{v}_n$. ∎

Proposition 4.11 *Let $\mathcal{A}$ be a closed ordered *-algebra, and let $(a_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{A}$. Then the unital *-subalgebra of $\mathcal{A}$ that is generated by $\{ a_n \mid n \in \mathbb{N} \}$ has a faithful representation as operators.*

Proof By Lemma 4.10, the unital *-subalgebra of $\mathcal{A}$ that is generated by $\{ a_n \mid n \in \mathbb{N} \}$ is σ-bounded, so Theorem 4.9 applies. ∎

4.2 Representation as functions

A slight modification of the well-known Gelfand transformation yields a representation as functions of any ordered *-algebra.

Definition 4.3 Let $\mathcal{A}$ be an ordered *-algebra, then the set of all multiplicative states on $\mathcal{A}$, i.e., of all positive unital *-homomorphisms from $\mathcal{A}$ to $\mathbb{C}$, will be denoted by $\mathcal{S}_{\mathrm{m}}(\mathcal{A})$.

Proposition 4.12 *Let $\mathcal{A}$ be an ordered *-algebra. Then the map $\pi_{\mathrm{Gelfand}}\colon \mathcal{A} \to \mathbb{C}^{\mathcal{S}_{\mathrm{m}}(\mathcal{A})}$, $a \mapsto \pi_{\mathrm{Gelfand}}(a)$ with $\pi_{\mathrm{Gelfand}}(a)\colon \mathcal{S}_{\mathrm{m}}(\mathcal{A}) \to \mathbb{C}$,*

$$\omega \mapsto \pi_{\mathrm{Gelfand}}(a)(\omega) := \langle \omega, a \rangle, \tag{4.5}$$

is a positive unital *-homomorphism and $\big(\mathcal{S}_{\mathrm{m}}(\mathcal{A}), \pi_{\mathrm{Gelfand}}\big)$ is a representation as functions of $\mathcal{A}$.

Proof Immediate consequence of the properties of the elements in $\mathcal{S}_{\mathrm{m}}(\mathcal{A})$. ∎

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In order to guarantee the existence of many multiplicative states, we have to examine states which are at the same time extremal positive Hermitian linear functionals.

Definition 4.4 Let $\mathcal{A}$ be an ordered *-algebra. Then a state ω on $\mathcal{A}$ is called *pure* if ω is also an extremal positive Hermitian linear functional on $\mathcal{A}$. The set of all pure states on $\mathcal{A}$ will be denoted by $\mathcal{S}_{\mathrm{p}}(\mathcal{A}) \coloneqq \mathcal{S}(\mathcal{A}) \cap \mathcal{A}_{\mathrm{H}}^{*,+,\mathrm{ex}}$.

The above definition of pure states is equivalent to the more common one as extreme points of the convex set of states.

Proposition 4.13 *Let $\mathcal{A}$ be an ordered *-algebra, then $\mathcal{S}_{\mathrm{p}}(\mathcal{A}) = \mathrm{ex}\big(\mathcal{S}(\mathcal{A})\big)$.*

Proof If ω is an extreme point of $\mathcal{S}(\mathcal{A})$, then it is also an extremal positive Hermitian linear functional, hence a pure state: Indeed, given $\rho \in \mathcal{A}_{\mathrm{H}}^{*,+}$ such that $\rho \le \omega$, then, as a consequence of the Cauchy–Schwarz inequality, either both $\langle \rho , \mathbb{1} \rangle$ and $\langle \omega - \rho , \mathbb{1} \rangle$ are in $]0,1[$, or $\rho = \mu\omega$ with $\mu \in \{0,1\}$. In the former case,

$$\omega = \langle \omega - \rho , \mathbb{1} \rangle \big(\langle \omega - \rho , \mathbb{1} \rangle^{-1} (\omega - \rho) \big) + \langle \rho , \mathbb{1} \rangle \big(\langle \rho , \mathbb{1} \rangle^{-1} \rho \big)$$

is a representation of ω as a nontrivial convex combination of two elements of $\mathcal{S}(\mathcal{A})$, which implies that $\rho = \mu\omega$ with $\mu = \langle \rho , \mathbb{1} \rangle$.

Conversely, if ω is pure state on $\mathcal{A}$, then it is an extreme point of $\mathcal{S}(\mathcal{A})$: If $\omega = \lambda\rho + (1-\lambda)\rho'$ with $\rho, \rho' \in \mathcal{S}(\mathcal{A})$ and $\lambda \in]0,1[$, then $\lambda\rho \le \omega$ and $(1-\lambda)\rho' \le \omega$. Consequently, there are $\mu, \mu' \in [0,1]$ such that $\lambda\rho = \mu\omega$ and $(1-\lambda)\rho' = \mu'\omega$. Evaluation on $\mathbb{1}$ shows that $\lambda = \mu$ and $(1-\lambda) = \mu'$, and hence $\rho = \omega = \rho'$. ∎

The sets of pure states and of multiplicative states on an ordered *-algebra are closely related. In order to see this, the following concept will be helpful.

Definition 4.5 Let $\mathcal{A}$ be an ordered *-algebra, let ω be a state on $\mathcal{A}$, and let $a \in \mathcal{A}$. The *variance of* ω on a is defined as

$$\mathrm{Var}_\omega(a) \coloneqq \big\langle \omega , \big(a - \langle \omega , a \rangle \mathbb{1}\big)^* \big(a - \langle \omega , a \rangle \mathbb{1}\big) \big\rangle . \tag{4.6}$$

Note that $\mathrm{Var}_\omega(a) \in [0,\infty[$ and that $\mathrm{Var}_\omega(a) = \langle \omega , a^* a \rangle - |\langle \omega , a \rangle|^2$ holds for every state ω on every ordered *-algebra $\mathcal{A}$ and all $a \in \mathcal{A}$.

Proposition 4.14 *If $\mathcal{A}$ is an ordered *-algebra and ω is a multiplicative state on $\mathcal{A}$, then ω is a pure state on $\mathcal{A}$.*

Proof By Proposition 4.13, the pure states are precisely the extreme points of the set of all states. So assume that $\rho, \rho' \in \mathcal{S}(\mathcal{A})$ and $\lambda \in]0,1[$ fulfil $\omega = \lambda\rho + (1-\lambda)\rho'$, then one can check that the identity

$$\mathrm{Var}_\omega(a) = \mathrm{Var}_{\lambda\rho+(1-\lambda)\rho'}(a) = \lambda \mathrm{Var}_\rho(a) + (1-\lambda)\mathrm{Var}_{\rho'}(a) + \lambda(1-\lambda)|\langle \rho - \rho' , a \rangle|^2$$

holds for all $a \in \mathcal{A}$. Moreover, $\mathrm{Var}_\omega(a) = 0$ because ω is multiplicative. It follows that $|\langle \rho - \rho' , a \rangle|^2 = 0$ for all $a \in \mathcal{A}$ because $\mathrm{Var}_\rho(a)$ and $\mathrm{Var}_{\rho'}(a)$ are nonnegative, so $\rho = \rho' = \omega$. ∎

In order to be able to obtain a converse statement, we need some more assumptions.

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Definition 4.6 An ordered *-algebra $\mathcal{A}$ is called *radical* if it has the following property: Whenever $a, b \in \mathcal{A}_{\mathrm{H}}$ commute and fulfil $\mathbb{1} \le a$ and $0 \le ab$, then $0 \le b$.

There are some important examples of radical commutative ordered *-algebras.

Proposition 4.15 *Let $\mathcal{A}$ be a commutative ordered *-algebra. If $\mathcal{A}$ has a faithful representation as functions, then $\mathcal{A}$ is radical and closed.*

Proof Let (X, π) be a faithful represent as functions of $\mathcal{A}$. It is easy to check that $\mathbb{C}^X$ is radical and closed, essentially because $\mathbb{C}$ is radical and closed. Using that $\pi\colon \mathcal{A} \to \mathbb{C}^X$ is a positive unital *-homomorphism and an order embedding, it follows immediately that $\mathcal{A}$ has to be radical and closed as well. ■

Proposition 4.16 *Let $\mathcal{A} := \mathcal{R} \otimes \mathbb{C}$ be the complexification of a Φ-algebra $\mathcal{R}$, then $\mathcal{A}$ is a radical and closed commutative ordered *-algebra.*

Proof As discussed in Example 4.3, $\mathcal{A}$ is a closed commutative ordered *-algebra. Consider $a, b \in \mathcal{A}_{\mathrm{H}}$ such that $\mathbb{1} \le a$ and $0 \le ab$. We can express b as the difference $b = b_+ - b_-$ of its positive and negative components, $b_+ := b \vee 0$, and $b_- := (-b) \vee 0$. From $b_+ \wedge b_- = 0$, it follows that $b_+ b_- = 0$. As $\mathcal{A}_{\mathrm{H}}^+$ is closed under products by definition of Φ-algebras, multiplication with b_- yields $0 \le b_- ab = -b_- a\, b_- \le 0$, so $b_- a\, b_- = 0$. From $\mathbb{1} \le a$, it now follows that $0 \le (b_-)^2 \le b_- a\, b_- = 0$, so $(b_-)^2 = 0$. As a consequence, $2\varepsilon(\varepsilon\mathbb{1} - b_-) = (\varepsilon\mathbb{1} - b_-)^2 + \varepsilon^2\mathbb{1} \ge 0$, and hence $b_- \le \varepsilon\mathbb{1}$ for all $\varepsilon \in\,]0, \infty[$, which implies that $b_- \le 0$ because $\mathcal{A}$ is closed by assumption. This finally shows that $b = b_+ - b_- \ge 0$. ■

It is also worthwhile to mention an important nonexample.

Example 4.17 Consider the *-algebra $\mathbb{C}[x, y]$ of complex polynomials in two variables x and y with the *-involution given by complex conjugation of coefficients, and thus $\mathbb{C}[x, y]_{\mathrm{H}} \cong \mathbb{R}[x, y]$. On $\mathbb{C}[x, y]_{\mathrm{H}}$, choose the partial order that turns $\mathbb{C}[x, y]$ into an ordered *-algebra with cone of positive elements given by sums of Hermitian squares, i.e.,

$$\mathbb{C}[x, y]_{\mathrm{H}}^+ := \mathbb{C}[x, y]_{\mathrm{H}}^{++} = \left\{ \sum_{n=1}^N p_n^* p_n \;\middle|\; N \in \mathbb{N};\, p_1, \dots, p_N \in \mathbb{C}[x, y] \right\}.$$

Note that $\mathbb{C}[x, y]_{\mathrm{H}}^+ \cap (-\mathbb{C}[x, y]_{\mathrm{H}}^+) = \{0\}$ because every sum of Hermitian squares of polynomials is pointwise positive on $\mathbb{R}^2$. Moreover, the product of two elements of $\mathbb{C}[x, y]_{\mathrm{H}}^+$ is again in $\mathbb{C}[x, y]_{\mathrm{H}}^+$. It is well known that there exist polynomials $p \in \mathbb{C}[x, y]_{\mathrm{H}} \backslash \mathbb{C}[x, y]_{\mathrm{H}}^+$ that are pointwise positive on $\mathbb{R}^2$. An explicit example from [2] is

$$p := x^2 y^2 (x^2 + y^2 - \mathbb{1}) + \mathbb{1} = x^4 y^2 + x^2 y^4 - x^2 y^2 + \mathbb{1}.$$

Now, consider $q := x^2 + y^2 + \mathbb{1} \in \mathbb{C}[x, y]_{\mathrm{H}}^+$, then

$$pq = x^6 y^2 + x^4 y^4 + x^2 y^6 + x^2 y^2 + x^2 + y^2 + \left(x^2 y^2 - \mathbb{1}\right)^2 \in \mathbb{C}[x, y]_{\mathrm{H}}^+.$$

As $\mathbb{1} \le q$, we conclude that $\mathbb{C}[x, y]$ (with this choice for the order) is not a radical ordered *-algebra and especially does not have a faithful representation as functions due to Proposition 4.15. Closer inspection shows that this is indeed because of the

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existence of ill-behaved pure states (see, e.g., [12, Corollary 11.6.4] and the discussion there for details).

We proceed with examining the relation between pure states and multiplicative states.

Lemma 4.18 *Let $\mathcal{A}$ be an ordered *-algebra, let ω be a state on $\mathcal{A}$, and let $a \in \mathcal{A}$ with* $\mathrm{Var}_\omega(a) = 0$, *then*

$$\langle\, \omega\,,\, b^* a \,\rangle = \langle\, \omega\,,\, b^* \,\rangle\langle\, \omega\,,\, a \,\rangle \text{ and } \langle\, \omega\,,\, a^* b \,\rangle = \langle\, \omega\,,\, a^* \,\rangle\langle\, \omega\,,\, b \,\rangle \tag{4.7}$$

hold for all $b \in \mathcal{A}$. A state ω on $\mathcal{A}$ thus is multiplicative if (and only if) $\mathrm{Var}_\omega(a) = 0$ *for all $a \in \mathcal{A}$.*

Proof The Cauchy–Schwarz inequality yields

$$\begin{aligned}\left|\langle\, \omega\,,\, a^* b \,\rangle - \langle\, \omega\,,\, a^* \,\rangle\langle\, \omega\,,\, b \,\rangle\right|^2 &= \left|\left\langle\, \omega\,,\, \left(a - \langle\, \omega\,,\, a \,\rangle\mathbb{1}\right)^* \left(b - \langle\, \omega\,,\, b \,\rangle\mathbb{1}\right) \right\rangle\right|^2 \\ &\leq \mathrm{Var}_\omega(a)\,\mathrm{Var}_\omega(b)\end{aligned}$$

for all $a, b \in \mathcal{A}$. If $\mathrm{Var}_\omega(a) = 0$, then this implies that (4.7) holds. ∎

Lemma 4.19 *Let $\mathcal{A}$ be a commutative ordered *-algebra, and let $\omega \in \mathcal{S}(\mathcal{A})$, then the subset $\left\{\, a \in \mathcal{A} \;\middle|\; \mathrm{Var}_\omega(a) = 0 \,\right\}$ of $\mathcal{A}$ is a unital *-subalgebra and this is the largest (with respect to inclusion) unital *-subalgebra of $\mathcal{A}$ on which the restriction of ω is multiplicative. In the special case that* $\mathrm{Var}_\omega(\mathbb{1} + a^2) = 0$ *holds for all $a \in \mathcal{A}_{\mathrm{H}}$, it follows that ω is multiplicative on whole $\mathcal{A}$.*

Proof First assume that ω is an arbitrary state on $\mathcal{A}$. Then it is easy to check that $\mathrm{Var}_\omega(\lambda a) = |\lambda|^2 \mathrm{Var}_\omega(a)$ and also (using the commutativity of $\mathcal{A}$) $\mathrm{Var}_\omega(a^*) = \mathrm{Var}_\omega(a)$ hold for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. Moreover, if $a, b \in \mathcal{A}$ fulfil $\mathrm{Var}_\omega(a) = \mathrm{Var}_\omega(b) = 0$, then one can check with the help of Lemma 4.18 that

$$\mathrm{Var}_\omega(a+b) = \left\langle\, \omega\,,\, a^*a + a^*b + b^*a + b^*b \,\right\rangle - \left|\langle\, \omega\,,\, a \,\rangle + \langle\, \omega\,,\, b \,\rangle\right|^2 = 0$$

and

$$\mathrm{Var}_\omega(ab) = \left\langle\, \omega\,,\, b^*a^*a\,b \,\right\rangle - \left|\left\langle\, \omega\,,\, ab \,\right\rangle\right|^2 = 0\,.$$

As $\mathrm{Var}_\omega(\mathbb{1}) = 0$ is clearly fulfilled as well, one sees that $\{\, a \in \mathcal{A} \mid \mathrm{Var}_\omega(a) = 0 \,\}$ is a unital *-subalgebra of $\mathcal{A}$. From Lemma 4.18, it also follows that the restriction of ω to $\{\, a \in \mathcal{A} \mid \mathrm{Var}_\omega(a) = 0 \,\}$ is multiplicative. Conversely, if $\mathcal{B}$ is another unital *-subalgebra of $\mathcal{A}$ such that the restriction of ω to $\mathcal{B}$ is multiplicative, then it follows that $\mathrm{Var}_\omega(b) = \langle\, \omega\,,\, b^*b \,\rangle - |\langle\, \omega\,,\, b \,\rangle|^2 = 0$ for all $b \in \mathcal{B}$ and therefore the inclusion $\mathcal{B} \subseteq \{\, a \in \mathcal{A} \mid \mathrm{Var}_\omega(a) = 0 \,\}$ holds.

Now, assume that $\mathrm{Var}_\omega(\mathbb{1} + a^2) = 0$ holds for all $a \in \mathcal{A}_{\mathrm{H}}$. As $4b = \left(\mathbb{1} + (b+\mathbb{1})^2\right) - \left(\mathbb{1} + (b-\mathbb{1})^2\right)$ holds for all $b \in \mathcal{A}_{\mathrm{H}}$, it follows from $\mathrm{Var}_\omega\left(\mathbb{1} + (b \pm \mathbb{1})^2\right) = 0$ that $\mathrm{Var}_\omega(b) = 0$ for all $b \in \mathcal{A}_{\mathrm{H}}$, and hence $\mathrm{Var}_\omega(c) = 0$ for all $c \in \mathcal{A}$ because $c = c_r + \mathrm{i}c_i$ with $c_r \coloneqq (c + c^*)/2 \in \mathcal{A}_{\mathrm{H}}$ and $c_i \coloneqq (c - c^*)/(2\mathrm{i}) \in \mathcal{A}_{\mathrm{H}}$. Application of Lemma 4.18 shows that ω is multiplicative on whole $\mathcal{A}$. ∎

Theorem 4.20 *Let $\mathcal{A}$ be a radical commutative ordered *-algebra, then $\mathcal{S}_{\mathrm{p}}(\mathcal{A}) = \mathcal{S}_{\mathrm{m}}(\mathcal{A})$, i.e., a state on $\mathcal{A}$ is a pure state if and only if it is multiplicative.*

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Proof Let ω be a state on $\mathcal{A}$. If ω is multiplicative, then Proposition 4.14 shows that ω is also a pure state. Conversely, if ω is a pure state, then it remains to show that the identity $\operatorname{Var}_\omega(a^2 + \mathbb{1}) = 0$ holds for all $a \in \mathcal{A}_{\mathrm{H}}$, which, by Lemma 4.19, already implies that ω is multiplicative.

So let $a \in \mathcal{A}_{\mathrm{H}}$ be given and define the subset $S_a := \{ (\mathbb{1} + a^2)b \mid b \in \mathcal{A}_{\mathrm{H}} \}$ of $\mathcal{A}_{\mathrm{H}}$. It is clear that S_a is a (real) linear subspace of $\mathcal{A}_{\mathrm{H}}$. If $(\mathbb{1} + a^2)b = (\mathbb{1} + a^2)b'$ with $b, b' \in \mathcal{A}_{\mathrm{H}}$, then $b - b' = 0$ because $0 = (\mathbb{1} + a^2)(b - b')$ and because $\mathcal{A}$ is radical. So every element of S_a is of the form $(\mathbb{1} + a^2)b$ with a unique $b \in \mathcal{A}_{\mathrm{H}}$ and the map $\tilde{\rho} \colon S_a \to \mathbb{R}$,

$$(\mathbb{1} + a^2)b \mapsto \left\langle \tilde{\rho}, (\mathbb{1} + a^2)b \right\rangle := \langle \omega, b \rangle,$$

is well defined and is clearly $\mathbb{R}$-linear. Moreover, for every $c \in \mathcal{A}_{\mathrm{H}}$, there exists an element $(\mathbb{1} + a^2)b \in S_a$ with $b \in \mathcal{A}_{\mathrm{H}}$ such that $0 \leq (\mathbb{1} + a^2)b$ and $c \leq (\mathbb{1} + a^2)b$, e.g., $(\mathbb{1} + a^2)b := (\mathbb{1} + a^2)(\mathbb{1} + c)^2/2$. Using again that $\mathcal{A}$ is radical one sees that $\tilde{\rho}$ is positive with respect to the order on S_a inherited from $\mathcal{A}_{\mathrm{H}}$, so the extension theorem for positive linear functionals applies and shows that there exists a positive linear functional ρ on $\mathcal{A}_{\mathrm{H}}$ fulfilling $\langle \rho, (\mathbb{1} + a^2)b \rangle = \langle \omega, b \rangle$ for all $b \in \mathcal{A}_{\mathrm{H}}$. Using the isomorphism between $(\mathcal{A}_{\mathrm{H}})^*$ and $\mathcal{A}^*_{\mathrm{H}}$, we can also treat ρ as a positive Hermitian linear functional on $\mathcal{A}$.

As $\langle \rho, b \rangle \leq \langle \rho, (\mathbb{1} + a^2)b \rangle = \langle \omega, b \rangle$ holds for all $b \in \mathcal{A}_{\mathrm{H}}$, it follows that $\rho \leq \omega$, and hence there exists $\mu \in [0,1]$ such that $\rho = \mu\omega$ because ω is a pure state by assumption. From evaluation on $\mathbb{1} + a^2$ and $(\mathbb{1} + a^2)^2$, one gets

$$\mu \left\langle \omega, \mathbb{1} + a^2 \right\rangle = \left\langle \rho, \mathbb{1} + a^2 \right\rangle = \langle \omega, \mathbb{1} \rangle = 1$$

and

$$\mu \left\langle \omega, (\mathbb{1} + a^2)^2 \right\rangle = \left\langle \rho, \left(\mathbb{1} + a^2\right)^2 \right\rangle = \left\langle \omega, \mathbb{1} + a^2 \right\rangle,$$

which yields $\mu \neq 0$ and $\left\langle \omega, \mathbb{1} + a^2 \right\rangle = \mu^{-1}$ as well as $\langle \omega, (\mathbb{1} + a^2)^2 \rangle = \mu^{-2}$, and thus $\operatorname{Var}_\omega(\mathbb{1} + a^2) = 0$. ∎

Similar results about the relation between pure and multiplicative states on certain *-algebras have already occurred before, e.g., [3, Theorem 2] for Banach *-algebras or [12, Proposition 11.3.9] for general *-algebras endowed with a special choice of a (pre-)order.

Corollary 4.21 *Let X be a set, and let $\mathcal{A}$ be a unital *-subalgebra of $\mathbb{C}^X$, endowed with the pointwise order inherited from $\mathbb{C}^X$, then the pure states on $\mathcal{A}$ are precisely the multiplicative ones.*

Note, however, that there might be more multiplicative states on $\mathcal{A}$ than just evaluation functionals at points of X.

Corollary 4.22 *Let $\mathcal{A}$ be a radical commutative ordered *-algebra, then*

$$\mathcal{A}^{*,+,\mathrm{ex}}_{\mathrm{H}} = \left\{ \omega \in \mathcal{A}^{*,+}_{\mathrm{H}} \;\middle|\; \forall_{a \in \mathcal{A}} : \langle \omega, \mathbb{1} \rangle \langle \omega, a^* a \rangle = |\langle \omega, a \rangle|^2 \right\} \tag{4.8}$$

and the set $\mathcal{A}^{*,+,\mathrm{ex}}_{\mathrm{H}}$ of extremal positive Hermitian linear functionals on $\mathcal{A}$ is weak-*-closed in $\mathcal{A}^*_{\mathrm{H}}$.

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Proof Let ω be a positive Hermitian linear functional on $\mathcal{A}$. Then either $\omega = 0$, in which case $\omega \in \mathcal{A}_{\mathrm{H}}^{*,+,\mathrm{ex}}$ and $\langle\,\omega\,,\,\mathbb{1}\,\rangle\langle\,\omega\,,\,a^*a\,\rangle = |\langle\,\omega\,,\,a\,\rangle|^2$ for all $a \in \mathcal{A}$ hold, or $\langle\,\omega\,,\,\mathbb{1}\,\rangle > 0$. In this second case, $\tilde{\omega} := \langle\,\omega\,,\,\mathbb{1}\,\rangle^{-1}\omega$ is a state on $\mathcal{A}$ and the following chain of equivalences holds: ω is an extremal positive Hermitian linear functional if and only if $\tilde{\omega}$ is a pure state, which is equivalent to $\tilde{\omega}$ being multiplicative by Theorem 4.20, and Lemma 4.18 shows that this holds if and only if $\mathrm{Var}_{\tilde{\omega}}(a) = 0$ for all $a \in \mathcal{A}$. As $\langle\,\omega\,,\,\mathbb{1}\,\rangle^2\mathrm{Var}_{\tilde{\omega}}(a) = \langle\,\omega\,,\,\mathbb{1}\,\rangle\langle\,\omega\,,\,a^*a\,\rangle - |\langle\,\omega\,,\,a\,\rangle|^2$, identity (4.8) is proved.

Finally, as $\mathcal{A}_{\mathrm{H}}^{*,+}$ is weak-*-closed in $\mathcal{A}_{\mathrm{H}}^{*}$ and as $\mathcal{A}_{\mathrm{H}}^{*} \ni \omega \mapsto \langle\,\omega\,,\,\mathbb{1}\,\rangle\langle\,\omega\,,\,a^*a\,\rangle - |\langle\,\omega\,,\,a\,\rangle|^2 \in \mathbb{R}$ is a weak-*-continuous function, one sees that the right-hand side of (4.8) is weak-*-closed in $\mathcal{A}_{\mathrm{H}}^{*}$. ∎

Corollary 4.23 *Let $\mathcal{A}$ be a radical commutative ordered *-algebra, then the following are equivalent:*

(1) *The Gelfand transformation $\big(\mathcal{S}_{\mathrm{m}}(\mathcal{A}), \pi_{\mathrm{Gelfand}}\big)$ of $\mathcal{A}$ discussed in Proposition 4.12 is a faithful representation as functions.*
(2) *There exists a faithful representation as functions of $\mathcal{A}$.*
(3) *The order on $\mathcal{A}$ is induced by its extremal positive Hermitian linear functionals.*

Proof The implication (1) $\Longrightarrow$ (2) is trivial.

Assume that there exists a faithful representation as functions (X, π) of $\mathcal{A}$, and let $a \in \mathcal{A}_{\mathrm{H}} \backslash \mathcal{A}_{\mathrm{H}}^{+}$ be given. Then there exists $x \in X$ fulfilling $\pi(a)(x) < 0$. However, the linear functional $\mathcal{A} \ni b \mapsto \pi(b)(x) \in \mathbb{C}$ is a multiplicative state, hence a pure state by Proposition 4.14, and thus especially an extremal positive Hermitian linear functional on $\mathcal{A}$. This proves the implication (2) $\Longrightarrow$ (3).

Finally, if the order on $\mathcal{A}$ is induced by its extremal positive Hermitian linear functionals, then for every $a \in \mathcal{A}_{\mathrm{H}} \backslash \mathcal{A}_{\mathrm{H}}^{+}$, there exists an extremal positive Hermitian linear functional $\tilde{\omega}$ on $\mathcal{A}$ such that $\langle\,\tilde{\omega}\,,\,a\,\rangle < 0$. From $\tilde{\omega} \neq 0$, it follows that $\langle\,\tilde{\omega}\,,\,\mathbb{1}\,\rangle > 0$ and therefore $\omega := \langle\,\tilde{\omega}\,,\,\mathbb{1}\,\rangle^{-1}\tilde{\omega}$ is a well-defined pure state on $\mathcal{A}$, hence also a multiplicative state by Theorem 4.20. So we see that $\pi_{\mathrm{Gelfand}}(a)$ cannot be positive because $\pi_{\mathrm{Gelfand}}(a)(\omega) < 0$, and conclude that the Gelfand transformation of $\mathcal{A}$ is faithful, i.e., (3) $\Longrightarrow$ (1) holds. ∎

There are some representation theorems for commutative *-algebras endowed with a locally convex topology defined by submultiplicative seminorms, i.e., seminorms fulfilling the estimate $\|ab\| \leq \|a\|\|b\|$ for all elements a and b of the algebra. One example is of course the commutative Gelfand–Naimark theorem. However, for radical commutative ordered *-algebras, Corollary 4.23 combined with Corollary 3.4 yields an approach using a rather different type of locally convex topologies.

Application of Theorem 3.7 to Corollary 4.23 immediately yields the following generalization of the commutative Gelfand–Naimark theorem.

Theorem 4.24 *Let $\mathcal{A}$ be a σ-bounded radical commutative ordered *-algebra, then $\mathcal{A}$ has a faithful representation as functions if and only if $\mathcal{A}$ is closed.*

Like before, this implies that, heuristically, every closed radical commutative ordered *-algebra "behaves essentially like a *-algebra of functions."

Corollary 4.25 *Let $\mathcal{A}$ be a closed radical commutative ordered *-algebra, and let $(a_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{A}$. Then the unital *-subalgebra of $\mathcal{A}$ that is generated by $\{ a_n \mid n \in \mathbb{N} \}$ has a faithful representation as functions.*

Proof By Lemma 4.10, the unital *-subalgebra of $\mathcal{A}$ that is generated by $\{ a_n \mid n \in \mathbb{N} \}$ is σ-bounded, so Theorem 4.24 applies. ■

Theorems 4.9 and 4.24 allow to develop a theory of *-algebras of (possibly unbounded) operators in a representation-independent way. While the assumption of being σ-bounded might be replaced by others, e.g., the existence of a well-behaved locally convex topology, the property of being radical is necessary for a commutative ordered *-algebra to have a faithful representation as functions (see Proposition 4.15).

Acknowledgment I would like to thank Profs. K. Schmüdgen and G. Buskes, and the anonymous referee, for some helpful discussions and comments on earlier versions of the manuscript.

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