GENERATING GROUPS OF CERTAIN SOLUBLE VARIETIES

Dedicated to the memory of Hanna Neumann

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(Received 21 June 1972, revised 20 December 1972)

Communicated by M. F. Newman

1. Introduction

Any variety of groups is generated by its free group of countably infinite rank. A problem that appears in various forms in Hanna Neumann's book [7] (see, for intance, sections 2.4, 2.5, 3.5, 3.6) is that of determining if a given variety $\mathfrak{V}$ can be generated by $F_k(\mathfrak{V})$, one of its free groups of finite rank; and if so, if $F_n(\mathfrak{V})$ is residually a k-generator group for all $n \geqq k$. (Here, as in the sequel, all unexplained notation follows [7].)

To any variety $\mathfrak{V}$ generated by a finitely generated group one can associate the number $d(\mathfrak{V})$, the least positive integer such that $\mathfrak{V}$ is generated by its free group of rank $d(\mathfrak{V})$. For example, for the variety $\mathfrak{O}$ of all groups, $d(\mathfrak{O}) = 2$ (in fact every free group is residually free of rank 2 [8]); for $\mathfrak{A}$, the variety of abelian groups, $d(\mathfrak{A}) = 1$ and $d(\mathfrak{A}^l) = 2$ $(l \geqq 2)$ ([7] 16.35 and 25.34); for $\mathfrak{N}_c$, the variety of nilpotent groups of class at most c, $d(\mathfrak{N}_c) = c-1$ $(c \geqq 3)$ ([6], [9]); and more generally for $\mathfrak{V} \leqq \mathfrak{N}_c, d(\mathfrak{V}) \leqq c$ ([7] 35.12). Further examples may be found in [7] where, in addition, for two varieties $\mathfrak{U}$ and $\mathfrak{V}$, the dependence of $d(\mathfrak{U}\mathfrak{V})$ on $d(\mathfrak{U})$ and $d(\mathfrak{V})$ is discussed. Also, Baumslag [2] has shown that for arbitrary $\mathfrak{U}$, the non-cyclic free groups of $\mathfrak{U}\mathfrak{A}$ are residually free of rank 2 so that, in particular, $d(\mathfrak{U}\mathfrak{A}) \leqq 2$ (cf. [7] 25.33).

Corresponding results for $[\mathfrak{U}, \mathfrak{V}]$ are more isolated even for $\mathfrak{V} = \mathfrak{E}$, especially since $[\mathfrak{U}, \mathfrak{E}]$ is indecomposable for any $\mathfrak{U} \neq \mathfrak{O}$ ([7] 24.32). In the present paper we shall consider such problems for $\mathfrak{M}_{(1)} = [\mathfrak{A}^2, \mathfrak{E}]$, the variety of centre-by-metabelian groups; and more generally for $\mathfrak{M}_{(c)}$, defined inductively by $\mathfrak{M}_{(c)} = [\mathfrak{M}_{(c-1)}, \mathfrak{E}]$ $(c \geqq 2)$. In addition, we obtain information regarding the ascending chains

$$(1) \qquad \operatorname{Var} F_2(\mathfrak{M}_{(c)}) \leqq \operatorname{Var} F_3(\mathfrak{M}_{(c)}) \leqq \cdots$$

Research supported by grants from N. R. C. and N. S. F. respectively.

https://doi.org/10.1017/S1446788700016785 Published online by Cambridge University Press

(see [7] Section 1.6 for a general discussion of such chains).

Our results for $c = 1, 2$ rely heavily on a 3×3 matrix representation of $F_\infty(\mathfrak{M}_{(1)})$ found by Gupta [4] and a corresponding 4×4 matrix representation of $F_\infty(\mathfrak{M}_{(2)})$ (Section 4). These representations are generalizations of the well-known faithful 2×2 matrix representation of $F_\infty(\mathfrak{M})$ found by Magnus (see [7], 36.12), where $\mathfrak{M} = \mathfrak{A}^2$. In Section 2 we divert from our discussion to illustrate how the Magnus representation can be used to give an alternate and rather elementary proof of the result that $F_k(\mathfrak{M})$ is residually $F_2(\mathfrak{M})$ for $k \geqq 2$. On the whole, Section 2 serves the purpose of introducing the terminology and the computational techniques required in our discussion of $\mathfrak{M}_{(1)}$ and $\mathfrak{M}_{(2)}$.

In Section 3 we show that $d(\mathfrak{M}_{(1)}) = 4$ (Theorem 3.7) and establish that

$$\text{(2)} \qquad \operatorname{Var} F_2(\mathfrak{M}_{(1)}) = \operatorname{Var} F_3(\mathfrak{M}_{(1)}) < \operatorname{Var} F_4(\mathfrak{M}_{(1)}) = \mathfrak{M}_{(1)}.$$

The inequality in (2) is a result of Gupta [5] who shows that the laws of $F_3(\mathfrak{M}_{(1)})$ are consequences of those of $\mathfrak{M}_{(1)}$ plus an additional law u with the property that u^2 is a law of $\mathfrak{M}_{(1)}$. Further she shows that if U is the subgroup generated by the values of u in $F_\infty(\mathfrak{M}_{(1)})$, then $F_\infty(\mathfrak{M}_{(1)})/U$ is isomorphic to the group M_3 of 3×3 matrices mentioned above. Thus it follows from (2) that $d(\operatorname{Var}(M_3)) = 2$.

In Section 4 we show that not only is $d(\mathfrak{M}_{(2)} = 4$ (Theorem 4.6), but that

$$\text{(3)} \qquad \operatorname{Var} F_2(\mathfrak{M}_{(2)}) = \operatorname{Var} F_3(\mathfrak{M})_{(2)} < \operatorname{Var} F_4(\mathfrak{M}_{(2)}) = \mathfrak{M}_{(2)}$$

which is the chain (2) with $\mathfrak{M}_{(1)}$ replaced by $\mathfrak{M}_{(2)}$. Our investigations, in Section 5, regarding $\mathfrak{M}_{(c)} (c \geqq 3)$ are not so complete. However, while we have not determined the precise chain (1) for these cases, we are able to verify that for $c \geqq 3$ $\operatorname{Var} F_{k-1}(\mathfrak{M}_{(c)})$ is properly contained in $\operatorname{Var} F_k(\mathfrak{M}_{(c)})$ for $k = 2, \cdots, c-1$. The proof uses methods similar to those in Levin [6].

Another problem for the varieties of the form $[\mathfrak{U}, \mathfrak{E}]$ is that of deciding when the centre of $F/[\mathrm{U}(F), F]$ is precisely $\mathrm{U}(F)/[\mathrm{U}(F), F]$, where F is a free group of finite or countably infinite rank. This, for instance, is the case if $\mathfrak{U} = \mathfrak{N}_c$(Witt, cf. [7] 31.63) or $\mathfrak{U} = \mathfrak{M}$ (follows from the fact that the centre of $F(\mathfrak{M})$ is trivial [1]). On the other hand, Cossey [3] has shown that this is not the case for $\mathfrak{U} = \operatorname{Var} SL(2, 5)$. As a by-product of our results, we show in Section 6 that the centre of $F/[F'', F, F]$ is precisely $[F'', F]/[F'', F, F]$.

We are thankful to Dr. M. F. Newman for his comments on an earlier draft of the paper.

2. The Variety $\mathfrak{M}$

Let ZG be the integral group ring of a free abelian group G freely generated by $x_1, x_2, \cdots$, and let $T_2 = ZG[\Lambda_2]$ be the ZG-algebra in the set $\Lambda_2 = \{\lambda_{21}^{(k)}; k = 1, 2, \cdots\}$ of commuting indeterminates. Let M_2 be the multiplicative group of 2×2 matrices (over T_2) generated by

(4) $$\begin{bmatrix} 1 & 0 \\ \lambda_{21}^{(k)} & x_k \end{bmatrix} = X_k^{(2)},$$

for $k = 1, 2, \cdots$. Let F be the (absolutely) free group freely generated by $x_1, x_2, \cdots$ and let $\phi_2 \colon F \to M_2$ be the homomorphism of F onto M_2 defined by $\phi_2(x_k) = X_k^{(2)}$. Define a mapping $\alpha_{21} : F \to T_2$ by $\alpha_{21}(w) =$ 21-entry of the matrix $\phi_2(w)$ for all $w \in F$.

LEMMA 2.1. (Magnus, *cf.* [7] 36.12). $\alpha_{21}(w) = 0$ *if and only if* $w \in F''$. *In particular* F/F'' *is isomorphic to* M_2 *under the natural mapping* $x_k F'' \to X_k^{(2)}$.

Since $G \cong F/F'$, we may identify G with F/F' (and ZG with $Z(F/F')$ correspondingly) by identifying x_k with $x_k F'$. Thus if $w = w(x_1, \cdots, x_n) \in F'$, then we may write

(5) $$w \equiv \prod_{n \geqq i > j \geqq 1} [x_i, x_j]^{q_{ij}} (\text{mod } F''),$$

where $q_{ij} = q_{ij}(x_1, \cdots, x_n) \in ZG$.

For each $l, k \in \{1, 2, \cdots\}$ and each $t \in Z$ we define an endomorphism $\theta_{l,k,t}$ of F and an endomorphism $\bar{\theta}_{l,k,t}$ of ZG as follows:

(6) $$\theta_{l,k,t}(x_l) = x_k^t, \theta_{l,k,t}(x_i) = x_i \text{ for } i \neq l,$$

(7) $$\bar{\theta}_{l,k,t}(x_l) = x_k^t, \bar{\theta}_{l,k,t}(x_i) = x_i \text{ for } i \neq l.$$

LEMMA 2.2. *If* $w \in F'$, *then* $\alpha_{21}(w) = \sum_i p_i \lambda_{21}^{(i)}$, *where each* p_i *is a uniquely determined element of* ZG. *Further if* $\alpha_{21}(\theta_{l,k,t}(w)) = \sum_i q_i \lambda_{21}^{(i)}$, *then for all* $i \notin \{l, k\}$, $q_i = \bar{\theta}_{l,k,t}(p_i)$. *(Here* θ, $\bar{\theta}$ *are as defined in* (6), (7)).

PROOF. It is clear that $\alpha_{21}(w)$ will be an expression of the form $\sum_i p_i \lambda_{21}^{(i)}$ and since $\lambda_{21}^{(i)}$ are linearly independent over ZG, p_i are unique. Replacing x_k by x_l^t in w has the effect of changing the corresponding matrix expression $\phi_2(w)$ by replacing $X_k^{(2)}$ by $(X_l^{(2)})^t$. Thus if $i \notin \{k, l\}$, the coefficient of $\lambda_{21}^{(i)}$ in $\alpha_{21}(\theta_{l,k,t(w)})$ is precisely $\bar{\theta}_{l,k,t}(p_i)$.

For any $p \in ZG$ let $e_k(p)$ denote the maximum of the absolute values of the exponents of x_k occurring in p. The following lemma will have repeated applications in the sequel.

LEMMA 2.3. *Let* $p \in ZG, p \neq 0$. *For any integers* l, k, $\bar{\theta}_{l,k,t}(p) \neq 0$ *whenever* $|t| \geqq 2e_k(p) + 1$.

PROOF. The lemma follows immediately from the observation that if $|s_1| < |s|$ $(s_1 \neq s)$ and $s_2 \geqq 2|s| + 1$ then the equation $is_2 + s_1 = js_2 + s$ has no integral solution, from which it follows that if $s_2 \geqq 2|s| + 1$, then $x_l^i x_k^{s_1} - x_l^j x_k^s$ will not vanish for any replacement of x_l by $x_k^{s_2}$.

We conclude this section by giving an alternate proof of the following result.

THEOREM 2.4. (*cf.* [2]). *For $n \geqq 2$ $F_n(\mathfrak{M})$ is residually $F_2(\mathfrak{M})$.*

PROOF. It is enough to show that for $n \geqq 3, F_n(\mathfrak{M})$ is residually $F_{n-1}(\mathfrak{M})$. Let $w = w(x_1, \cdots, x_n)$ be an element of $F \backslash F''$. If $w \notin F'$ then $\theta_{i,i,0}(w) \notin F'$ for some $i \in \{1, \cdots, n\}$. Thus we may assume that $w \in F' \backslash F''$. By Lemmas 2.2 and 2.1, $\alpha_{21}(w) = p_1\lambda_{21}^{(1)} + \cdots + p_n\lambda_{21}^{(n)} \neq 0$, and we may assume, without loss of generality, that $p_1 = p_1(x_1, \cdots, x_n) \neq 0$. Since $n \geqq 3$, by Lemma 2.2 the coefficient of λ_{21} in the expansion of $\alpha_{21}(\theta_{n,n-1,t}(w))$ is precisely $\bar{\theta}_{n,n-1,t}(p_1)$ which by Lemma 2.3 is non-zero for a large enough t. It follows by Lemma 2.1 that $\theta_{n,n-1,t}(w) \notin F''$. This completes the proof of the theorem.

3. The variety $\mathfrak{M}_{(1)}$ ($= [\mathfrak{M}, \mathfrak{E}]$)

As in Section 2 let $\Lambda_3 = \{\lambda_{i,i-1}^{(k)}; i = 2,3; k = 1,2,\cdots\}$ and let $T_3 = ZG[\Lambda_3]$. Let M_3 be the group of 3×3 matrices (over T_3) generated by

$$\begin{bmatrix} 1 & 0 & 0 \\ \lambda_{21}^{(k)} & x_k & 0 \\ 0 & \lambda_{32}^{(k)} & 1 \end{bmatrix} = X_k^{(3)}, \tag{8}$$

for $k = 1,2,\cdots$ and let ϕ_3 be the homomorphism of F onto M_3 defined by $\phi_3(x_k) = X_k^{(3)}$ for $k = 1,2,\cdots$. Further let α_{ij} $(3 \geqq i > j \geqq 1)$ be the mapping of F into T_3 defined by $\alpha_{ij}(w) =$ ij-entry of the matrix $\phi_3(w)$ for all $w \in F$.

LEMMA 3.1. (Gupta [4]). *Let $w \in F''$. Then $\alpha_{31}(w) = 0$ if and only if $w = w_1w_2$, where w_1 is a product of values of the word*

$$u_{1234}(x) = [x_1^{-1}, x_2^{-1}; x_3, x_4][x_1^{-1}, x_3^{-1}; x_4, x_2][x_1^{-1}, x_4^{-1}; x_2, x_3] \\ [x_3^{-1}, x_4^{-1}; x_1, x_2][x_4^{-1}, x_2^{-1}; x_1, x_3][x_2^{-1}, x_3^{-1}, x_1, x_4], \tag{9}$$

and $w_2 \in [F'', F]$.

LEMMA 3.2. (Gupta [4]). *$F_3(\mathfrak{M}_{(1)})$ is isomorphic to the subgroup of M_3 generated by $X_1^{(3)}, X_2^{(3)}, X_3^{(3)}$.*

LEMMA 3.3. (Gupta [5]). *$u_{1234}(x) \notin [F'', F]$ but $u_{1234}^2(x) \in [F'', F]$, where $u_{1234}(x)$ is defined by (9). Further if $w = w(x_1, \cdots, x_n)$ $(n \geqq 4)$ is an n-variable word in F'' such that $\alpha_{31}(w) = 0$, then*

$$w = \prod_{1 \leqq i < j < k < l \leqq n} u_{ijkl}^{\varepsilon(ijkl)}(x) \pmod{[F'', F]}, \tag{10}$$

where $u_{ijkl}(x)$ is defined as in (9) and $\varepsilon(ijkl) \in \{0, 1\}$.

The next lemma is analogous to Lemma 2.2 and the proof is essentially the same.

LEMMA 3.4. *If $w \in F''$*, then *$\alpha_{31}(w) = \sum_{i,j} p_{ij}\lambda_{32}^{(i)}\lambda_{21}^{(j)}$, where each p_{ij} is a uniquely*

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determined element of ZG. Further if $\alpha_{31}(\theta_{l,k,t}(w)) = \sum_{i,j} q_{ij}\lambda_{32}^{(i)}\lambda_{21}^{(j)}$, *then for all* $i,j \notin \{l,k\}, q_{ij} = \bar{\theta}_{l,k,t}(p_{ij})$, *where* $\theta, \bar{\theta}$ *are defined in* (6), (7).

LEMMA 3.5. *Let* $w = w(x_1, \cdots, x_n)$ $(n \geqq 2) \in F''$ *be an n-variable word such that* $\alpha_{31}(w) \neq 0$. *Then there is an automorphism* ξ *of* F *such that for some* $i \in \{1, \cdots, n\}$ *the coefficient of* $\lambda_{32}^{(i)}\lambda_{21}^{(i)}$ *in the expansion of* $\alpha_{31}(\xi(w))$ *is non-zero.*

PROOF. Let $\alpha_{31}(w) = \sum_{k,l} p_k \lambda_{32}^{(k)}\lambda_{21}^{(l)}$. If for some $i, p_{ii} \neq 0$ then we take ξ to be the identity automorphism of F. Otherwise, we may assume that for some i, $j \in \{1, \cdots, n\} (i \neq j), p_{ii} = 0 = p_{jj}$ and one of p_{ij}, p_{ji} is non-zero. Let ξ_1 be the automorphism of F which maps x_j to $x_i x_j$ and x_k to x_k for $k \neq j$, and ξ_2 be the corresponding automorphism mapping x_j to $x_j x_i$ and x_k to x_k for $k \neq j$. Let $\alpha_{31}(\xi_1(w)) = \sum_{k,l} q_{kl} \lambda_{32}^{(k)}\lambda_{21}^{(l)}$ and $\alpha_{31}(\xi_2(w)) = \sum_{k,l} r_{kl} \lambda_{32}^{(k)}\lambda_{21}^{(l)}$. One verifies that

$$q_{ii} = \bar{p}_{ij} + x_j \bar{p}_{ji} \text{ and } r_{ii} = x_j \bar{p}_{ij} + \bar{p}_{ji},$$

where $\bar{p}$ is obtained from p on replacing x_j by $x_j x_i$. If both q_{ii} and r_{ii} are zero then both $\bar{p}_{ij}$ and $\bar{p}_{ji}$ must be zero and equivalently both p_{ij} and p_{ji} must be zero, contrary to the assumption.

We can now prove,

THEOREM 3.6. *Let* $\mathfrak{G} = \mathrm{Var}(M_3)$. *Then for all* $n \geqq 2, F_n(\mathfrak{G})$ *is residually* $F_2(\mathfrak{G})$. *In particular* $F_3(\mathfrak{M}_{(1)})$ *is residually* $F_2(\mathfrak{M}_{(1)})$.

PROOF. Let $w = w(x_1, \cdots, x_n)$ $(n \geqq 3)$ be an n-variable word in F such that $w \notin \mathfrak{G}(F) < F''$. If $w \notin F''$ then, by Theorem 2.4, $\theta_{l,k,t}(w) \notin F''$ for some $l, k \in \{1, \cdots, n\}$ $(l \neq k)$ and some $t \in Z$. Thus we may assume that $w \in F''$. Using an automorphism ξ of F, if necessary, we may, by Lemma 3.5, assume that the coefficient p_{ii} of $\lambda_{32}^{(i)}\lambda_{21}^{(i)}$ in the expansion of $\alpha_{31}(w)$ is non-zero for some $i \in \{1, \cdots, n\}$. It follows, by Lemma 2.3, that $\bar{\theta}_{l,k,t}(p_{ii}) \neq 0$ for $i \notin \{l,k\}$ $(l \neq k)$. Thus by Lemma 3.4, $\theta_{l,k,t}(w) \notin \mathfrak{G}(F)$. The second part of the theorem uses Lemma 3.2.

We conclude this section by proving the following result.

THEOREM 3.7. *For* $n \geqq 4, F_n(\mathfrak{M}_{(1)})$ *is residually* $F_4(\mathfrak{M}_{(1)})$.

PROOF. Let $w = w(x_1, \cdots, x_n)$ $(n \geqq 5)$ be an n-variable word in $F \backslash [F'', F]$. As in Theorem 3.6 we may assume that $w \in F'' \backslash [F'', F]$. Further, if $\alpha_{31}(w) \neq 0$ then, as in Theorem 3.6, $\theta_{l,k,t}(\xi(w)) \notin [F'', F]$. Thus we may assume that $\alpha_{31}(w) = 0$ so that, by Lemma 3.3,

$$w = \prod_{1 \leqq i < j < k < l \leqq n} u_{ijkl}^{\varepsilon(ijkl)} \pmod{[F'', F]}$$

and for some $i < j < k < l, \varepsilon(ijkl) \neq 0$. Since $n \geqq 5$, we can choose $r \notin \{i, j, k, l\}$. By Lemma 3.3, $\theta_{r,r,0}(w) \notin [F'', F]$. This completes the proof of the theorem.

4. The variety $\mathfrak{M}_{(2)}$ $(= [\mathfrak{M}, \mathfrak{E}, \mathfrak{E}])$

In this section we first of all show that $d(\mathfrak{M}_{(2)}) > 3$. We do this by exhibiting a 4-variable word which is not a law in $F_4(\mathfrak{M}_{(2)})$ but is a law in $F_3(\mathfrak{M}_{(2)})$.

THEOREM 4.1. *Let* $w = [u_{1234}(x), x_4]$, *where* $u_{1234}(x)$ *is defined by* (10). *Then* w *is a law in* $F_3(\mathfrak{M}_{(2)})$ *but not a law in* $F_4(\mathfrak{M}_{(2)})$. *In particular* $\operatorname{Var} F_3(\mathfrak{M}_{(2)}) < \operatorname{Var} F_4(\mathfrak{M}_{(2)})$.

PROOF. Since $u_{1234}(x)$ is a law in $F_3(\mathfrak{M}_{(1)})$ (Gupta [4]), it follows that w is a law in $F_3(\mathfrak{M}_{(2)})$. To complete the proof it suffices to show that w is not a law in $F_4(\mathfrak{M}_{(2)} \wedge \mathfrak{N}_7)$.

Expanding w modulo $\gamma_8(F)[F'', F, F]$ shows that

$$w = [x_1, x_2; x_1, x_2, x_3, x_4; x_4][x_1, x_3; x_1, x_3, x_2, x_4; x_4][x_1, x_4; x_1, x_4, x_2, x_3; x_4]$$
$$[x_2, x_3; x_2, x_3, x_1, x_4; x_4][x_2, x_4; x_2, x_4, x_1, x_3; x_4][x_3, x_4; x_3, x_4, x_1, x_2; x_4]$$

(cf. [5]). Since the frequency of generators is different in each factor, as words in $\gamma_7(F)$ the factors of w are independent of each other modulo $\gamma_8(F)$. However, it is readily verified that modulo $\gamma_8(F), [F'', F, F]$ is generated by all commutators of the form $[x_{i1}, x_{i2}; x_{i3}, x_{i4}; x_{i5}; x_{i6}]$ plus those of the forms $[x_{i1}, x_{i2}; x_{i3}, x_{i4}; x_{i5}; x_{i6}; x_{i7}]$, $[x_{i1}, x_{i2}; x_{i3}, x_{i4}; x_{i5}, x_{i6}; x_{i7}]$, $[x_{i1}, x_{i2}, x_{i3}; x_{i4}, x_{i5}; x_{i6}; x_{i7}]$ and $[x_{i1}, x_{i2}; x_{i3}, x_{i4}; x_{i5}; x_{i6}, x_{i7}]$ where $i1, i2, \cdots, i7 \in \{1, 2, \cdots\}$. In particular if $w \equiv 1$ (modulo $\gamma_8(F)[F'', F, F]$) then it is not difficult to verify that in fact $w \in [\gamma_3(F), \gamma_2(F), F, F]\gamma_8(F)$. Since the generators of weight 7 cannot alter the frequency pattern of any factor of w, it follows that if w lies in $\gamma_8(F)[F'', F, F]$ then each factor of w lies in $\gamma_8(F)[F'', F, F]$, and in particular, $[x_1, x_2; x_1, x_2, x_1, x_2; x_2] \in \gamma_8(F)[F'', F, F]$. In what follows we shall show that $[x_1, x_2; x_1, x_2, x_1, x_2; x_2]$ is in fact non-trivial modulo $\gamma_8(F)[F'', F, F]$.

Let H be the free group of class 7 freely generated by a, b and let N_1 be the normal subgroup of H generated by all basic commutators ([7], 31.51) of weight 7 other than the following three commutators:

$$c_1 = [b, a, a, b, b; b, a], \quad c_2 = [b, a, a, b; b, a, b] \text{ and } c_3 = [b, a, b, b; b, a, a].$$

Let N_2 be the normal subgroup of H generated by $N_1, c_1^2, c_2^2, c_3^2, c_1c_3^{-1}$. Then

$$d = [b, a, a, b; b, a; b] = [b, a, a, b, b; b, a][b, a, a, b; b, a, b] \text{ by the Witt identity ([7], 33.34)}$$
$$= c_1c_2 \notin N_2.$$

We next observe that modulo N_2,

$$[b, a, b; b, a; b; a] \equiv [b, a, b; b, a; a; b] \equiv [b, a, b, a; b, a; b][b, a, b; b, a, a; b]$$
$$\equiv dc_3c_2^{-1} = c_1c_2c_3c_2^{-1} = c_1c_3 \equiv 1, \text{ and } [b, a, a; b, a; b; b]$$
$$\equiv [b, a, a, b; b, a; b]\ [b, a, a; b, a, b; b] \equiv dc_2c_3^{-1} = c_1c_2c_2c_3^{-1} = c_1c_3^{-1} \equiv 1.$$

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Thus H/N_2 is a centre-by-centre-by-metabelian group of class 7 in which $d = [b, a, a, b; b, a; b]$ is non-trivial. This completes the proof of the theorem.

We now construct an $\mathfrak{M}_{(2)}$-group which will be useful in the sequel.

As in Sections 2 and 3 let $\Lambda_4 = \{\lambda_{i,i-1}^{(k)}; i = 2, 3, 4; k = 1, 2, \cdots\}$ and $T_4 = ZG[\Lambda_4]$. Let M_4 be the group of 4×4 matrices (over T_4) generated by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda_{21}^{(k)} & x_k & 0 & 0 \\ 0 & \lambda_{32}^{(k)} & 1 & 0 \\ 0 & 0 & \lambda_{43}^{(k)} & 1 \end{bmatrix} = X_k^{(4)}$$

for $k = 1, 2, \cdots$. Let ϕ_4 be the homomorphism of F onto M_4 defined by $\phi_4(x_k) = X_k^{(4)}$ for $k = 1, 2, \cdots$ and let α_{ij} $(4 \geqq i > j \geqq 1)$ be the mapping of F into T_4 defined by $\alpha_{ij}(w) = ij$-entry of the matrix $\phi_4(w)$ for all $w \in F$. Using matrix multiplication the following lemma is routinely verified.

LEMMA 4.2. (i) *If* $w \in [F'', F, F]$, *then* $w \in$ *kernel of* ϕ_4;

(ii) $[u_{1234}(x), x_5] \in$ *kernel of* ϕ_4;

(iii) *If* $w \in F''$, then $\alpha_{41}[w, x_k] = -\lambda_{43}^{(k)}\alpha_{31}(w)$.

We now establish the following useful analogue of Lemma 3.5.

LEMMA 4.3. *Let* $w = w(x_1, \cdots, x_n)$ $(n \geqq 2)$ *be an n-variable word in* $[F'', F]$ *such that* $\alpha_{41}(w) \neq 0$. *Then there is an automorphism* ξ *of* F *such that for some* $i \in \{1, \cdots, n\}$, *the coefficient of* $\lambda_{43}^{(i)}\lambda_{32}^{(i)}\lambda_{21}^{(i)}$ *is non-zero in the expansion of* $\alpha_{41}\xi(w)$).

PROOF. By Lemmas 4.2 and 3.4 we write

$$\alpha_{41}(w) = \sum_{i,j,k} p_{ijk}\lambda_{43}^{(i)}\lambda_{32}^{(j)}\lambda_{21}^{(k)}, \tag{11}$$

where p_{ijk} are uniquely determined elements of ZG and for some $i, j, k \in \{1, \cdots, n\}$, $p_{ijk} \neq 0$. Let $i, j \in \{1, \cdots, n\}$ $(i \neq j)$ be fixed and let ξ_1, ξ_2, ξ_3 be automorphisms of F defined as follows: $\xi_1(x_j) = x_i x_j, \xi_1(x_k) = x_k$ for $k \neq j$; $\xi_2(x_j) = x_j x_i, \xi_2(x_k) = x_k$ for $k \neq j$; $\xi_3(x_i) = x_i^{-1}, \xi_3(x_k) = x_k$ for $k \neq i$. Let

$$\alpha_{41}(\xi_1(w)) = \sum_{i,j,k} q_{ijk}\lambda_{43}^{(i)}\lambda_{32}^{(j)}\lambda_{21}^{(k)}; \quad \alpha_{41}(\xi_2(w)) = \sum_{i,j,k} r_{ijk}\lambda_{43}^{(i)}\lambda_{32}^{(j)}\lambda_{21}^{(k)}; \text{ and} \tag{12}$$

$$\alpha_{41}(\xi_3(w)) = \sum_{i,j,k} s_{ijk}\lambda_{43}^{(i)}\lambda_{32}^{(j)}\lambda_{21}^{(k)}.$$

For $p \in ZG$, let p^* be the element of ZG obtained from p on replacing x_i by x_i^{-1} and $\bar{p}$ be the element of ZG obtained from p on replacing x_j by $x_j x_i$. If $p_{iii} = p_{jjj} = 0$, then using matrix multiplication the following can be verified:

$$q_{iii} = \bar{p}_{iij} + x_j\bar{p}_{iji} + \bar{p}_{jii} + x_j\bar{p}_{jji} + \bar{p}_{jij} + x_j\bar{p}_{ijj}; \tag{13}$$

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(14) $r_{iii} = x_j\bar{p}_{iij} + \bar{p}_{iji} + \bar{p}_{jii} + \bar{p}_{jji} + x_j\bar{p}_{jij} + x_j\bar{p}_{ijj};$

(15) $s_{iij} = x_i^{-1}p^*_{iij}, s_{iji} = x_i^{-1}p^*_{iji}, s_{jji} = -x_i^{-1}p^*_{jji}$ and $s_{jij} = -x_i^{-1}p^*_{jij};$

(16) $q_{jji} = \bar{p}_{jji}, q_{jij} = x_i\bar{p}_{jij}$ and $q_{iji} = \bar{p}_{iji} + \bar{p}_{jji} + \bar{p}_{ijj};$

(17) $q_{kii} = \bar{p}_{kij} + x_j\bar{p}_{kji} + \bar{p}_{kii} + x_j\bar{p}_{kjj};$ and

(18) $r_{kii} = x_j\bar{p}_{kij} + \bar{p}_{kji} + \bar{p}_{kii} + x_j\bar{p}_{kjj}.$

To complete the proof of the lemma, let us assume that,

(19) $p_{iii} = p_{jjj} = q_{iii} = q_{jjj} = r_{iii} = r_{jjj} = s_{iii} = s_{jjj} = 0.$

Then from (13) and (14) we conclude that

(20) $p_{iij} - p_{iji} = p_{jji} - p_{jij}$, and hence also $s_{iij} - s_{iji} = s_{jji} - s_{jij}$.

Using (15), this last equation yields

$$p_{iij} - p_{iji} = -p_{jji} + p_{jij},$$

which together with the first equation in (20) gives

(21) $p_{iij} = p_{iji}$ and $p_{jji} = p_{jij}$, and hence also $q_{iij} = q_{iji}$ and $q_{jji} = q_{jij}$.

Using (16) the last equation in (21) together with the second equation in (21) gives $p_{jji} = 0 = p_{jij}$; by symmetry,

(22) $p_{iij} = p_{iji} = p_{jji} = p_{jij} = 0$, and similarly $q_{iij} = q_{iji} = q_{jji} = q_{jij} = 0$.

Using (22) in the last equation in (16) yields

(23) $p_{ijj} = 0$ and (by symmetry) $p_{jii} = 0$.

Thus we have shown that if (19) holds for any $i, j \in \{1, \cdots, n\}$, then

(24) $p_{iij} = p_{iji} = p_{jii} = p_{jji} = p_{jij} = p_{ijj} = 0,$

and the same for the corresponding q, r, s terms. Assuming (19) for every pair $i, j \in \{1, \cdots, n\} (i \neq j)$, if $k \notin \{i, j\}$, then $q_{kii} = r_{kii} = \bar{p}_{kii} = \bar{p}_{kjj} = 0$, so that from (17) and (18) we get as in the proof of Lemma 3.5, $p_{kij} = 0$, which implies by (11) that $\alpha_{41}(w) = 0$, contrary to the hypothesis. This completes the proof of the lemma.

As an immediate consequence of Lemma 4.3, we prove the following.

THEOREM 4.4. *$F_3(\mathfrak{M}_{(2)})$ is residually $F_2(\mathfrak{M}_{(2)})$.*

PROOF. Let $w = w(x_1, x_2, x_3) \in F \backslash [F'', F, F]$. By Theorem 3.6, we may assume that $w \in [F'', F] \backslash [F'', F, F]$. If $\alpha_{41}(w) \neq 0$ then, as in the proof of Theorem 3.6, using Lemma 4.3 we can map w to a 2-variable word which does not belong to $[F'', F, F]$. If $\alpha_{41}(w) = 0$, then we may write $w \equiv [v_1, x_1][v_2, x_2][v_3, x_3]$

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mod $[F'', F, F]$), where $v_1, v_2, v_3 \in F''$ and $\alpha_{31}(v_i) = 0$ for $i = 1, 2, 3$ (by Lemma 4.2 (iii)). By Lemma 3.2, each $v_i \in [F'', F]$ and hence $w \in [F'', F, F]$, contrary to the assumption.

For the proof of our final result in this section, we need the following lemma.

LEMMA 4.5. *Let*
$$w = [u_{2345}(x), x_1][u_{1345}(x), x_2][u_{1245}(x), x_3][u_{1235}(x), x_4][u_{1234}(x), x_5],$$
where $u_{ijkl}(x)$ *is defined by* (9). *Then* $w \in [F'', F, F]$.

PROOF. If
$$v = [x_1^{-1}, x_2^{-1}; x_3, x_4, x_5][x_1^{-1}, x_3^{-1}; x_4, x_2, x_5][x_1^{-1}, x_4^{-1}; x_2, x_3, x_5][x_3^{-1}, x_4^{-1}; x_1, x_2, x_5][x_4^{-1}, x_2^{-1}; x_1, x_3, x_5][x_2^{-1}, x_3^{-1}; x_1, x_4, x_5]$$
then working modulo $[F'', F]$, it can be verified directly that

$$v \equiv 1. \tag{25}$$

Further, using the Witt identity

$[a, b, c^a][c, a, b^c][b, c, a^b] = 1$ with $a = [x_1^{-1}, x_2^{-1}]$, $b = [x_3, x_4]$, $c = x_5$ and working modulo $[F'', F, F]$ gives $[x_1^{-1}, x_2^{-1}; x_3, x_4; x_5][x_5, [x_1^{-1}, x_2^{-1}], [x_3, x_4]^{x_5}][x_3, x_4, x_5, [x_1^{-1}, x_2^{-1}]] \equiv 1$ and hence

$$[x_1^{-1}, x_2^{-1}; x_3, x_4; x_5][x_1^{-1}, x_2^{-1}, x_5^{-1}; x_3, x_4]^{x_5}[x_1^{-1}, x_2^{-1}; x_3, x_4, x_5]^{-1} \equiv 1. \tag{26}$$

To complete the proof of the lemma, we first expand w applying (26) to each factor. Next we note, using (25), that the 6-weight contributions of each $[u_{ijkl}(x), x_t]$ lie in $[F'', F, F]$. Finally, the remaining 5-weight commutators in w can be rearranged to form a product of elements of the form

$$[[x_i^{-1}, x_j^{-1}], [x_k, x_l, x_m][x_l, x_m, x_k][x_m, x_k, x_l]] \text{ and}$$
$$[[x_i^{-1}, x_j^{-1}, x_k^{-1}][x_j^{-1}, x_k^{-1}, x_i^{-1}][x_k^{-1}, x_i^{-1}, x_j^{-1}], [x_l, x_m]]^{-1}$$

which belong to $[F'', F, F]$.

We are now in a position to prove the following theorem.

THEOREM 4.6. *For each* $n \geqq 4$, $F_n(\mathfrak{M}_{(2)})$ *is residually* $F_4(\mathfrak{M}_{(2)})$.

PROOF. Let $w = w(x_1, \cdots, x_n)$ $(n \geqq 5)$ be an n-variable word in $F \backslash [F'', F, F]$. Then as in Theorem 3.6, we may assume that $w \in [F'', F]$, so that $w \equiv \Pi_{i=1}^n [v_i, x_i]$ (mod $[F'', F, F]$), where $v_i \in F''$. By Lemma 4.2 (iii), $\alpha_{41}(w) = \Sigma_{i=1}^n - \lambda_{43}^{(i)} \alpha_{31}(v_i)$. There are two cases to be considered.

CASE I. ($\alpha_{41}(w) \neq 0$). In this case, as in the proof of Theorem 4.4, we can use Lemma 4.3 to map w to an $(n-1)$-variable non-trivial word (mod $[F'', F, F]$).

CASE II. ($\alpha_{41}(w) = 0$). In this case $\alpha_{31}(v_i) = 0$ for each $i = 1, \cdots, n$. Thus by Lemma 3.3 each v_i is of the form (10). If $n > 5$ then $\theta_{k,k,0}(w) \notin [F'', F, F]$ for some k (by Theorem 4.1, $[u_{1234}(x), x_5] \notin [F'', F, F]$). If $n = 5$, then $w \equiv [u_{2345}(x), x_1]^{\beta_1} \cdots [u_{1234}(x), x_5]^{\beta_5}$ (mod $[F'', F, F]$), where $\beta_1, \cdots, \beta_5 \in \{0, 1\}$, by Lemma 3.3. Since $w \notin [F'', F, F]$, by Lemma 4.5 $\beta_i = 0$ and $\beta_j = 1$ for some i, j, and we may assume, without loss of generality, that $\beta_1 = 0$ and $\beta_2 = 1$. Then $\theta_{1\,2,1}(w) = [u_{2345}(x), x_2] \notin [F'', F, F]$. This completes the proof of the theorem.

5. The variety $\mathfrak{M}_{(c)}$ ($c \geqq 3$)

While we are unable to determine the precise chain (1) for $c \geqq 3$, our main result in this section goes some way towards the solution of this problem. Our method is similar to the one used in Levin [6].

Let $Z[y_1, \cdots, y_m]$ ($m \geqq 3$) be the free associative Z-algebra in non-commuting indeterminates $y_1, \cdots, y_m$ and let I_{m+5} be the ideal generated by all monomials of length $m + 5$. Put $R = Z[y_1, \cdots, y_m]/I_{m+5}$. We first prove the following lemma.

LEMMA 5.1. *Let* $\rho_m = \sum_\sigma |\sigma| \langle y, y_{1\sigma}, \cdots, y_{m\sigma} \rangle$, *where* $y = \langle\langle y_1, y_2\rangle, \langle y_3, y_4\rangle\rangle$ *and* σ *runs through all permutation of* $\{1, 2, \cdots, m\}$ *with* $|\sigma| = 1$ *or* -1 *according as* σ *is even or odd. Then* $\rho_m \notin I_{m+5}$. *(Here* $\langle r_1, r_2 \rangle$ *denotes the Lie commutator* $r_1 r_2 - r_2 r_1$.)

PROOF. CASE I. (m odd).

If $\rho_m \equiv 0$ (mod I_{m+5}), then the sum of the terms with left factor y_1^2 in the expansion of ρ_m is in I_{m+5}. However, these occur precisely in the terms with left factor $y_1 y$ and a straight-forward computation shows that this sum is $-\, y_1 y \sum_{\sigma'} |\sigma'|\, y_{2\sigma'} \cdots y_{m\sigma'}$ (since m is odd), where σ' runs through all permutations of $\{2, \cdots, m\}$, and this is clearly non-zero modulo I_{m+5}.

CASE II. (m even).

In this case we proceed as in Case I, but this time we consider terms with left factors $\langle y_1, y_2 \rangle y$ and show that this sum is not in I_{m+5}. The computation in this case is simplified by making use of the identity

$$\langle y, \cdots, y_i, y_j, \cdots \rangle - \langle y, \cdots, y_j, y_i, \cdots \rangle = \langle y, \cdots, \langle y_i, y_j \rangle, \cdots \rangle$$

to rewrite ρ_m as

$$\rho_m = \sum_\mu \langle y, \langle y_{1\mu}, y_{2\mu} \rangle, \cdots, \langle y_{(m-1)\mu}, y_{m\mu} \rangle \rangle,$$

where μ runs through all *even* permutations of $\{1, \cdots, m\}$ satisfying $(2i-1)\mu < (2i)\mu$ for $i = 1, \cdots, m/2$. We omit the rest of the details.

THEOREM 5.2. Var $F_2(\mathfrak{M}_{(c)}) < \cdots <$ Var $F_{c-1}(\mathfrak{M}_{(c)})$ ($c \geqq 4$).

PROOF. To show that Var $F_{c-2}(\mathfrak{M}_{(c)}) <$ Var $F_{c-1}(\mathfrak{M}_{(c)})$, we consider the word

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$$w_{c-1} = \Pi_\sigma [x, x_{1\sigma}, \cdots, x_{(c-1)\sigma}]^{|\sigma|}$$

where $x = [x_1, x_2; x_1, x_3]$ and σ runs through all permutations of $\{1, \cdots, c-1\}$. It is immediate that w_{c-1} is a law in $F_{c-2}(\mathfrak{M}_{(c)})$. With $m = c-1$, the group $A(R)$ of units of R belongs to Var $F_{c-1}(\mathfrak{M}_{(c)})$. Thus to see that w_{c-1} is not a law in $F_{c-1}(\mathfrak{M}_{(c)})$, we note by Lemma 5.1 that $\rho_{c-1} \notin I_{c+4}$. Finally to see that Var $F_{k-2}(\mathfrak{M}_{(c)}) < Var\, F_{k-1}(\mathfrak{M}_{(c)}) (4 \leqq k \leqq c)$, observe by Lemma 5.1 again that

$$w_{c-1,k-1} = [w_{k-1}, x_k, \cdots, x_{c-1}]$$

is not a law in $F_{k-1}(\mathfrak{M}_{(c)})$ but is clearly a law in $F_{k-2}(\mathfrak{M}_{(c)})$. This completes the proof of the theorem.

6. Concluding remarks

Let F be a free group of finite or countably infinite rank and let W be a fully invariant subgroup of F. In general the centre of $F/[W,F]$ is not $W/[W,F]$ (see Cossey [3] for an example). If $W = F''$, then using Lemma 2.1, it is not difficult to see that the centre of $F/[F'',F]$ is precisely $F''/[F'',F]$. Here we are able to prove the corresponding result for $W = [F'',F]$.

THEOREM 6.1. *The centre of* $F/[F'',F,F]$ *is precisely* $[F'',F]/[F'',F,F]$.

PROOF. Since $F''/[F'',F]$ is the centre of $F/[F'',F]$, it follows that the centre of $F/[F'',F,F]$ is contained in $F''/[F'',F,F]$. Let $w \in F'' \backslash [F'',F]$ such that $[w,x_k] \in [F'',F,F]$ for all $k = 1,2,\cdots$. By Lemma 4.2 (iii), $0 = \alpha_{41}[w,x_k] = -\lambda_{43}^k \alpha_{31}(w)$. Thus $\alpha_{31}(w) = 0$ and by Lemma 3.3, w is a product of the form (10). Since $[w,x_k]$ is a law in $F/[F'',F,F]$, it follows by (proof of) Theorem 4.6 that $[u_{1234}(x), x_4]$ is a law in $F/[F'',F,F]$, contrary to Theorem 4.1. Thus $w \in F'' \backslash [F'',F]$ implies that $[w,x_k] \notin [F'',F,F]$, and hence the centre of $F/[F'',F,F]$ is precisely $[F'',F]/[F'',F,F]$.

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