BULL. AUSTRAL. MATH. SOC.
VOL. 13 (1975), 255-260.

34K15, 34K99

Oscillatory behaviour of an equation arising from an industrial problem

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The oscillatory character of the solutions of a differential equation with retarded arguments, relevant to an industrial problem, is investigated. It is also proved that one can maintain the oscillatory properties of this equation under proper conditions, if a forcing term is added to it. The results obtained extend already known results on the subject.

1. Introduction

The scalar equation

$$y'(t) = ay(\lambda t) + by(t) \tag{1.1}$$

arises as the mathematical idealisation and simplification of an industrial problem, involving wave motion in the overhead supply line to an electrified railway system [2], [7]. This equation has been discussed in detail in [1], [3], and [4], and its oscillatory behaviour (only for $b = 0$) was mentioned in [6].

The oscillatory behaviour of the functional differential equation

$$y'(t) = p(t)y\big(g(t)\big) + q(t)y(t) , \tag{1.2}$$

which is of more general form than (1.1), was examined in [8], and results obtained for it were also extended there to the functional differential equation resulting from it by adding a forcing term, namely

$$y'(t) = p(t)y\big(g(t)\big) + q(t)y(t) + r(t) . \tag{1.3}$$

The purpose of the present paper is to extend further results,

Received 2 June 1975.

obtained in [8] for (1.2) and (1.3), by investigating the oscillatory behaviour of the functional differential equations

$$y'(t) = \sum_{i=1}^{n} p_i(t)y\big(g_i(t)\big) + q(t)y(t) \tag{1.4}$$

and

$$y'(t) = \sum_{i=1}^{n} p_i(t)y\big(g_i(t)\big) + q(t)y(t) + r(t) , \tag{1.5}$$

which are of more general form than (1.2) and (1.3) respectively.

In what follows, a solution $y(t)$ of (1.4) or (1.5) is said to be *oscillatory*, if it has arbitrarily large zeros, and *nonoscillatory* otherwise.

2. Unforced oscillation

Sufficient growth conditions on $p_i(t)$, $g_i(t)$, $i = 1, 2, \ldots, n$, and $q(t)$ are given here, to guarantee that every solution of (1.4) oscillates.

The following lemma will be needed, which is a modification of a corollary in [9].

LEMMA 2.1. *Consider the functional differential equation*

$$x'(t) + \sum_{i=1}^{n} f_i(t)x\big(F_i(t)\big) = 0 , \tag{2.1}$$

subject to the following conditions:

(C1) $f_i(t), F_i(t) \in C\big[[0, \infty), R\big]$, $f_i(t) \geq 0$, $i = 1, 2, \ldots, n$;

(C2) $F_i(t) \leq t$, $\lim_{t\to\infty} F_i(t) = \infty$, *and* $F_i'(t) \geq 0$, $i = 1, 2, \ldots, n$.

If, in addition,

$$\limsup_{t\to\infty} \sum_{i=1}^{n} \int_{K(t)}^{t} f_i(s)ds > 1 , \tag{2.2}$$

where $K(t) = \max F_i(t)$, $i = 1, 2, \ldots, n$, then every solution of (2.1) is oscillatory.

Now make the transformation $y(t)\exp\left(-\int q(t)dt\right) = z(t)$, to get (1.4) in the form

$$(2.3) \qquad z'(t) + \sum_{i=1}^{n} l_i(t)z\big(g_i(t)\big) = 0 ,$$

– where $l_i(t) = -p_i(t) \exp \int_t^{g_i(t)} q(T)dT$, $i = 1, 2, \ldots, n$ – which is of the same form as (2.1), and for which Lemma 2.1 is true under proper modifications. Then, observing that if $z(t)$ oscillates, so does $y(t)$, we apply Lemma 2.1 to (2.3) to obtain the following result for (1.4).

THEOREM 2.1. *Consider the functional differential equation* (1.4), *subject to the following conditions:*

(i) $p_i(t), g_i(t) \in C\big[[0, \infty), R\big]$, $-p_i(t) \geq 0$, $i = 1, 2, \ldots, n$;

(ii) $g_i(t) \leq t$, $\lim_{t\to\infty} g_i(t) = \infty$, *and* $g_i'(t) \geq 0$, $i = 1, 2, \ldots, n$;

(iii) $q(t)$ *is continuous for any* $t \neq 0$.

If, in addition,

$$(2.4) \qquad \limsup_{t\to\infty} \sum_{i=1}^{n} \int_{G(t)}^{t} -p_i(s)\left[\exp \int_s^{g_i(s)} q(T)dT\right]ds > 1 ,$$

where $G(t) = \max g_i(t)$, $i = 1, 2, \ldots, n$, *then every solution of* (1.4) *is oscillatory.*

REMARKS. The transformation used in [8] was also used here, and worked as well. Thus, "qualitatively", the theory in [4, paragraphs 4, 5, and 6] is completed further and results obtained in [6, p. 222] are also included in this presentation.

It can be easily verified on the other hand that Theorem 2.1 in [8] is a special case of Theorem 2.1 here, for $n = 1$; particularly

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interesting is the extension of the relationship (2.4).

3. Forced oscillation

Sufficient growth conditions on $p_i(t)$, $g_i(t)$, $i = 1, 2, \ldots, n$, $q(t)$, and $r(t)$ are given here, to assure that every solution of (1.5) oscillates. For this purpose, in addition to the transformations made in Section 2, set $r(t)\exp\left[-\int q(t)dt\right] = m(t)$, to obtain

$$z'(t) + \sum_{i=1}^{n} l_i(t)z\left(g_i(t)\right) = m(t) . \tag{3.1}$$

Now, taking account of a theorem in [5, p. 274] and of Lemma 2.1, given in Section 2, we modify properly Theorem 3.1 in [8, p. 429] by the same technique used there, to establish the following result for (1.5).

THEOREM 3.1. *Consider the functional differential equation* (1.5), *subject to the following hypotheses:*

(H1) $p_i(t), r(t) \in C\left[[0, \infty), R\right]$, $-p_i(t) \ge 0$, $i = 1, 2, \ldots, n$;

(H2) $g_i(t) \in C^1\left[[0, \infty), R\right]$, $g_i(t) \le t$, $\lim_{t\to\infty} g_i(t) = \infty$, $g_i'(t) \ge 0$, $i = 1, 2, \ldots, n$;

(H3) $q(t)$ *is continuous for any* $t \ne 0$;

(H4) $\limsup_{t\to\infty} \sum_{i=1}^{n} \int_{G(t)}^{t} -p_i(s)\left[\exp\int_s^{g_i(s)} q(T)dT\right]ds > 1$;

(H5) *there exists a function* $Q(t) \in C^1\left[[0, \infty), R\right]$, *such that* $Q'(t) = r(t)\exp\left[-\int q(t)dt\right]$, $t \ge 0$, *and either*

(I) $\lim_{t\to\infty} Q(t) = 0$, *or*

(II) *there exist constants* q_1, q_2 *and sequences* $\{t_m'\}$, $\{t_m''\}$, *such that* $\lim_{m\to\infty} t_m' = \lim_{m\to\infty} t_m'' = \infty$ *and*

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$$Q(t_m') = q_1 \ , \quad Q(t_m'') = q_2 \ , \quad q_1 \le Q(t) \le q_2 \ , \quad t \ge 0 \ .$$

Then, if (I) *holds, every solution* $y(t)$ *of* (1.5) *oscillates or* $\lim_{t\to\infty} y(t) = 0$, *while if* (II) *holds, every solution* $y(t)$ *of* (1.5) *oscillates.*

This result includes Theorem 3.1 in [8, p. 429] as special case for $n = 1$, and accomplishes the purpose of this paper.

Finally, the open question in [8, pp. 429-430] can be posed now for the higher-order retarded differential equations

$$y^{(n)}(t) = \sum_{i=1}^{n} p_i(t)y\left(g_i(t)\right) + q(t)y(t) \tag{3.2}$$

and

$$y^{(n)}(t) = \sum_{i=1}^{n} p_i(t)y\left(g_i(t)\right) + q(t)y(t) + r(t) \tag{3.3}$$

respectively, for $n > 1$.

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