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SOME FUNCTIONAL STABLE LIMIT THEOREMS

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1. **Introduction.** Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables which belong to the domain of attraction of a stable law of index $\alpha \neq 1$. That is,

$$\frac{S_n - nC_\alpha}{B_n} \xrightarrow{d} Y_\alpha(1), \tag{1}$$

where

$$S_n = \sum_{i=1}^{n} X_i, \qquad C_\alpha = \begin{Bmatrix} E(X_i), & 1 < \alpha \leq 2 \\ 0, & 0 < \alpha < 1 \end{Bmatrix},$$

and

$$B_n = n^{1/\alpha} L(n),$$

where $L(n)$ is a function of slow variation; also take $S_0 = 0$, $B_0 = 1$.

In §2, we are concerned with the weak convergence of the partial sum process to a stable process and the question of centering for stable laws and drift for stable processes. §3 deals with Cesaro sums of i.i.d. random variables. The final section is concerned with several related problems. I would like to thank Dr. C. C. Heyde for his help in this work.

2. **Stable processes and centering.**

LEMMA. *Under the conditions summarized by equation (1),*

$$\gamma_n(t) = \frac{S_{[nt]} - [nt] C_\alpha}{B_n} \xrightarrow{d} Y_\alpha(t), \tag{2}$$

where $Y_\alpha(t)$ is a stable process whose one-dimensional distributions are characterized by

$$Y_\alpha(t) \stackrel{d}{=} t^{1/\alpha} Y_\alpha(1) \tag{3}$$

Proof. By Theorem 2.7 of Skorokhod [9], this follows immediately from the convergence of the one-dimensional distributions, which we obtain as follows.

$$\frac{1}{B_n} = \frac{1}{n^{1/\alpha} L(n)} \sim \frac{t^{1/\alpha}}{(nt)^{1/\alpha} L(nt)} \sim \frac{t^{1/\alpha}}{B_{[nt]}} \tag{4}$$

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Then, from (1) and (4), we have

$$\frac{S_{[nt]-[nt]}C_\alpha}{B_n} \sim \frac{S_{[nt]-[nt]}C_\alpha}{B_{[nt]}} \cdot t^{1/\alpha} \to t^{1/\alpha} Y_\alpha \quad (1)$$

Q.E.D.

We have centered S_n with nc_α; a more general centering is with A_n. Since $(S_n - nc_\alpha)/B_n$ converges in distribution $S_n - A_n/B_n$ will converge in distribution if and only if

(5) $$A_n = nc_\alpha - dB_n + o(B_n).$$

If we write $(S_n - A_n)/B_n \xrightarrow{d} Z_\alpha(1)$ and $Z_\alpha(t)$ is the corresponding stable process, then we will see that the constant d is the centering parameter for the stable law $Z_\alpha(1)$ and $Z_\alpha(t)$ has drift dt.

THEOREM 1. *Let $Y_\alpha(t)$ be as in the Lemma. Then*

(6) $$\log E\{\exp(iuY_\alpha(t))\} = \lambda(u)\,|u|^\alpha\, t,$$

where $\lambda(u)$ is linear in sgn(u).

REMARKS. For an exact representation of $\lambda(u)$, see Lukacs [7]; the above is sufficient for our purposes. The content of this result is that, for nc_α centering, the corresponding stable process has zero drift.

Proof. From Proposition 14.18 of Breiman [4], we have

(7) $$\log E\{\exp(iuY_\alpha(t))\} = t \log E\{\exp(iuY_\alpha(1))\}.$$

From (3), we have

(8) $$\log E\{\exp(iuY_\alpha(t))\} = \log E\{\exp(iut^{1/\alpha}Y_\alpha(1))\}.$$

From Theorem 5.7.3, Lukacs [7], we also have

(9) $$\log E\{\exp(iuY_\alpha(1))\} = \lambda(u)\,|u|^\alpha + i\,du.$$

Now, (7), (8), and (9) imply

$$\lambda(u)\,|u|^\alpha\, t + i\,dut = \lambda(u)\,|u|^\alpha\, t + i\,dut^{1/\alpha},$$

which, in turn, implies that $d=0$; hence (6) holds. Q.E.D.

The following is immediate. This result about stable laws comes from consideration of the corresponding stable processes.

COROLLARY. *If $(S_n - nc_\alpha)/B_n \xrightarrow{d} Y_\alpha(1)$, then*

$$\log E\{\exp(iuY_\alpha(1))\} = \lambda(u)\,|u|^\alpha.$$

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In other words, if we center with $A_n = nc_\alpha$, we obtain as limit distributions stable laws in the restricted sense (Cf. [7, pp. 102–103]), with characteristic functions (ch.f.) $\exp\{\lambda(u)\,|u|^\alpha\}$.

For a random variable Y with ch.f. $\exp\{i\,du+\lambda(u)\,|u|^\alpha\}$, where $\alpha>1$, it is easily seen that $E\{Y\}=d$; for a process with ch.f. $\exp\{i\,dut+\lambda(u)\,|u|^\alpha\,t\}$, the drift is dt. Since

$$\frac{S_n-nc_\alpha}{B_n}\xrightarrow{d} Y_\alpha(1) \quad\text{and}\quad E\left(\frac{S_n-nc_\alpha}{B_n}\right)=0=E(Y_\alpha(1)),$$

one might suppose that the above results on centering follow from convergence of moments. Two counter-examples are instructive in this connection.

Even if $X_n\xrightarrow{d}X$ and $E(X_n)=0$ we cannot conclude that $E(X_n)\to E(X)$, as the following example illustrates. Let X_n have probability density function

$$f_n(x)=\begin{Bmatrix} \dfrac{f(x)}{A(n)}, & -n<x<n \\ 0, & |x|\geq n \end{Bmatrix},$$

where $f(x)$ is a stable density with ch.f.$=\exp\{-|u|^\alpha\}$, $\alpha\leq 1$, and $A(n)=\int_{-n}^{n} f(x)\,dx$. If we suppose that X has the density $f(x)$ then $X_n\xrightarrow{d}X$, $E(X_n)=0$, but $E(X)$ fails to exist.

Even when $E(X)$ exists, $E(X_n)=0$, $X_n\xrightarrow{d}X$ does not imply $E(X_n)\to E(X)$, as the following example illustrates. Take

$$P\{X_n=a\}=\begin{cases} \dfrac{n-d}{n}, & a=d \\ \dfrac{d}{n}, & a=d-n \\ 0, & \text{otherwise} \end{cases}$$

where $0<|d|<\infty$. Then $X_n\xrightarrow{d}X$, concentrated at d, $E(X_n)=0$ and $E(X)=d$. So, clearly, $E(X_n)\to E(X)$ fails to hold.

We henceforth assume that the X_i have been properly centered; i.e., X_i-c_α is used beyond this point. Thus we may write

$$\frac{S_n}{B_n}\xrightarrow{d} Y_\alpha(1) \tag{10}$$

and

$$\frac{S_{[nt]}}{B_n}\xrightarrow{d} Y_\alpha(t) \tag{11}$$

3. **Cesaro sums of random variables.** The Cesaro sums corresponding to S_n are

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given by

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} S_k. \tag{12}$$

(Cf. [1, p. 378])

THEOREM 2. *Under the conditions summarized by* (10),

$$\frac{\sigma_n}{B_n} \xrightarrow{d} Y_\alpha\left(\frac{1}{1+\alpha}\right). \tag{13}$$

Proof. We first obtain the invariance principle

$$\frac{\sigma_n}{B_n} \xrightarrow{d} \int_0^1 Y_\alpha(t)\, dt \tag{14}$$

by applying the measurable mapping theorem [3, Theorem 5.1] with the continuous function h given by

$$h(x) = \int_0^1 x(t)\, dt.$$

and noting that

$$h(S_{[nt]}) = \sum_{k=1}^{n} S_{k-1} \int_{k-1/n}^{k/n} dt = \frac{n-1}{n}\,\sigma_{n-1}$$

By the invariance principle (14), we can take $B_n = n^{1/\alpha}$ and

$$\log E\{\exp(iuX_k)\} = \lambda(u)\, |u|^\alpha, \tag{15}$$

without loss of generality. Now, observe that

$$n\sigma_n = \sum_{k=1}^{n} S_k = \sum_{k=1}^{n} (n+1-k)X_k \stackrel{d}{=} \sum_{k=1}^{n} kX_k. \tag{16}$$

From (15) and (16), the ch.f. of $\sigma_n/n^{1/\alpha}$ is given by

$$\exp\left\{\sum_{k=1}^{n} \lambda(u)\, |u|^\alpha\, k^\alpha n^{-(1+\alpha)}\right\} = \exp\left\{\lambda(u)\, |u|^\alpha \cdot \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^\alpha\right\} \tag{17}$$

Now, we note that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^\alpha \to \int_0^1 x^\alpha\, dx = \frac{1}{1+\alpha} \tag{18}$$

Hence, from (17) and (18), the limiting ch.f. of $\sigma_n/n^{1/\alpha}$ is

$$\exp\{\lambda(u)\, |u|^\alpha\, (1+\alpha)^{-1}\}, \tag{19}$$

the ch.f. of $Y_\alpha(1/1+\alpha)$. By invariance, the result follows. Q.E.D.

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4. **Some related results.** From (11) and the measurable mapping theorem, we have for any continuous functional h:

$$h\left(\frac{S_{[nt]}}{B_n}\right) \xrightarrow{d} h(Y_\alpha(t)). \tag{20}$$

In addition to the above result on Cesaro sums, some examples are

(i) $h(x)=\sup_{0\le t\le 1} x(t)-\inf_{0\le t\le 1} x(t)$,

which yields

$$B_n^{-1}\left(\max_{0\le k\le n} S_k-\min_{0\le k\le n} S_k\right) \xrightarrow{d} \sup_{0\le t\le 1} Y_\alpha(t)-\inf_{0\le t\le 1} Y_\alpha(t)$$

(ii) $(h(x))(t)=X(t)-tX(1)$,

which yields

$$B_n^{-1}(S_{[nt]-[nt]}\bar{X}) \xrightarrow{d} Y_\alpha(t)-tY_\alpha(1) \equiv Y_\alpha^0(t),$$

and

(iii) $h(x)=\sup_{\delta\le t\le 1}(X(t)/t^\beta)$, with δ and β positive, which yields

$$n^\beta B_n^{-1}\max_{\delta n\le k\le n}\frac{S_k}{k^\beta} \xrightarrow{d} \sup_{\delta\le t\le 1}\frac{Y_\alpha(t)}{t^\beta}.$$

The first of these is related to the extent of a random walk ([5], [6]), the second to adjusted range and tied-down stable process [6] and the last to some problems relating to the time of first passage over a curvilinear boundary ([2], [8]).

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