ON FUNCTIONS OF BOUNDED BOUNDARY ROTATION I

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1. Introduction

Let V_k denote the class of functions

$$f(z) = z + a_2 z^2 + \ldots \tag{1.1}$$

which map $U = \{|z| < 1\}$ conformally onto an image domain $f(U)$ of boundary rotation at most $k\pi$ (see (7) for the definition and basic properties of the class V_k). In this note we discuss the valency of functions in V_k, and also their Maclaurin coefficients.

In (**8**) it was shown that functions in V_k are close-to-convex in U if $2 \leqq k \leqq 4$. Here we show that V_k is a subclass of the class $K(\alpha)$ of close-to-convex functions of order α (**10**) for $\alpha = \frac{1}{2}k - 1$, and we give an upper bound for the valency of functions in V_k for $k > 4$.

In the third section we derive an upper bound for the integral means of $f'(z)$, and consequently for the coefficients of functions $f(z)$ in V_k; this improves a result in (3). We conclude with various estimates for the Maclaurin coefficients of functions in V_k when $f(U)$ is bounded or of finite area.

2. Valency

Theorem 2.1. *Suppose that $f(z)$ belongs to V_k, and assumes some value in $f(U)$ p times. Then $p = 1$ if $k = 2$, and $p < \frac{1}{2}k$ if $k > 2$.*

Proof. If $k = 2$, $f(z)$ is convex in U, and so is univalent. Hence we need only consider $k > 2$.

Suppose that $w = f(z)$ assumes some value v p times in U, at the distinct points $z_1, z_2, \ldots, z_p$. Then there is an r_0 such that $|z_k| < r_0 < 1$ for $1 \leqq k \leqq p$, and $f(z) \neq v$ on $|z| = r_0$. Let $C(r_0) = f(|z| = r_0)$. Then the winding number of $C(r_0)$ round v is

$$\frac{1}{2\pi i}\int_{C(r_0)} \frac{dw}{w-v} = \frac{1}{2\pi i}\int_{|z|=r_0} \frac{f'(z)dz}{f(z)-v} = p,$$

since $f'(z) \neq 0$ in U. Consequently the tangent rotation round $C(r_0)$ is at least $2p\pi$, and so

$$2p\pi \leqq \int_{\substack{0 \\ |z|=r_0}}^{2\pi} |\operatorname{Re}(1 + zf''/f')|\, d\theta \equiv I(r_0) < \limsup_{r_0 \to 1} \int_0^{2\pi} |\operatorname{Re}(1 + zf''/f')|\, d\theta \leqq k\pi,$$

since, for some t between 0 and 1, $I(r_0)$ is strictly increasing on $(t, 1)$. Thus $p<\frac{1}{2}k$, as required.

Note. The function

$$f(z)=\frac{1}{k}\left\{\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}k}-1\right\} \tag{2.1}$$

belongs to V_k (**6**), its valency is 1 if $k=2$, $[\frac{1}{2}k]$ if $k>2$ and k is not an even integer, and $\frac{1}{2}k-1$ if k is an even integer. This shows that the bounds of the theorem cannot be improved in general.

Theorem 2.2. *Suppose that* $f(z)$ *belongs to* V_k, *where* $2 \leqq k \leqq 4$. *Then* $f(z)$ *belongs to* $K(\frac{1}{2}k-1)$.

Proof. Choose any r, $0<r<1$, and let $C(r)=f(|z|=r)$. It is clear, geometrically, that, since the tangent to $C(r)$ cannot turn through more than $k\pi$ radians, the tangent cannot bend back on itself more than $(\frac{1}{2}k-1)\pi$ radians. Since r is arbitrary, the result follows at once.

If we are given a bound for the rate of growth of the derivative of a function in V_k, integration gives a bound for the rate of growth of the function itself. However we now establish a result in the opposite direction, using Theorem 2.1 and the theory of multivalent functions.

Theorem 2.3. *Suppose that* $f(z)$ *belongs to* V_k, *and* $M(r)=\max_{|z|=r}|f(z)|$. *Then*

$$|f'(z)| \leqq 2k(1-r^2)^{-1}\{1+M(r)\} \quad (|z|=r). \tag{2.2}$$

Proof. Since $f'(z) \neq 0$ in U, it follows from (**4**, Theorem 217) that, unless $f(z) \equiv z$, there is a number w_0, $|w_0|<1$, such that $f(z)-w_0$ does not vanish in U.

However $f(z)-w_0$ is also at most $\frac{1}{2}k$ valent in U. Consequently, by (**2**, Theorem 5.1), we have

$$|f'(z)| \leqq 2k(1-r^2)^{-1}|f(z)-w_0|, \tag{2.3}$$

from which (2.2) follows at once.

3. The coefficient problem for V_k

One of our principal tools here will be

Theorem 3.1. *The function* $f(z)$, *of the form* (1.1), *belongs to* V_k *if and only if there are two functions* $s_1(z)$ *and* $s_2(z)$, *normalized and starlike in* U, *such that*

$$f'(z)=\frac{(s_1/z)^{\frac{1}{2}k+\frac{1}{2}}}{(s_2/z)^{\frac{1}{2}k-\frac{1}{2}}}. \tag{3.1}$$

Proof. This follows at once from Paatero's integral representation for functions in V_k (**7**).

From this we obtain

Theorem 3.2. *Suppose that f(z) belongs to V_k, and*

$$I_\lambda(r) = \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^\lambda d\theta, \tag{3.2}$$

where $0<r<1$, *and* $(\frac{1}{2}k+1)\lambda>1$. *Then*

$$\limsup_{r\to 1}\,(1-r)^{(\frac{1}{2}k-1)\lambda-1} I_\lambda(r) \leqq A(k,\lambda), \tag{3.3}$$

where

$$A(k,\lambda) = \frac{2^{(\frac{1}{2}k-1)\lambda}\Gamma(\frac{1}{4}k\lambda+\frac{1}{2}\lambda+\frac{1}{2})}{\pi^{\frac{1}{2}}(\frac{1}{2}k\lambda+\frac{1}{2}\lambda-1)\Gamma(\frac{1}{4}k\lambda+\frac{1}{2}\lambda)}. \tag{3.4}$$

Furthermore the constant $A(k, \lambda)$ cannot be improved over the whole class V_k.

Proof. By Theorem 3.1, we may suppose that $f'(z)$ is given by (3.1). Then $|s_2(z)/z| \geqq (1+|z|)^2$ by the Koebe distortion theorem, and $s_1(z)/z$ is subordinate to $(1-z)^{-2}$ in U (**5**). Consequently, on integrating (3.1), we have

$$\begin{aligned} I_\lambda(r) &\leqq \frac{1}{2\pi}(1+r)^{(\frac{1}{2}k-1)\lambda}\int_0^{2\pi} |1+re^{i\theta}|^{-(\frac{1}{2}k+1)\lambda}d\theta \\ &\equiv (1+r)^{(\frac{1}{2}k-1)\lambda} J_{(\frac{1}{2}k+1)\lambda}(r), \text{ say.} \end{aligned} \tag{3.5}$$

In fact, Pommerenke (**9**) has shown that

$$\begin{aligned} J_m(r) &\sim \frac{\Gamma(m-1)}{2^{m-1}\Gamma^2(\frac{1}{2}m)}\cdot\frac{1}{(1-r)^{m-1}} \quad (m>1,\ r\to 1) \\ &= \frac{\Gamma(\frac{1}{2}m+\frac{1}{2})}{\pi^{\frac{1}{2}}(m-1)\Gamma(\frac{1}{2}m)}\cdot\frac{1}{(1-r)^{m-1}}, \end{aligned} \tag{3.6}$$

using the recurrence and duplication formulae for the Gamma function. Substituting (3.6) into (3.5) with $m = (\frac{1}{2}k+1)\lambda$, we get (3.3) and (3.4).

The constant $A(k, \lambda)$. Choosing $s_1(z) = z(1-z)^{-2}$, $s_2(z) = z(1+z)^{-2}$ (so that $f(z)$ is given by (2.1)), and any constant $B(k, \lambda)<A(k, \lambda)$, it is easy to show that

$$I_\lambda(r) > B(k,\lambda)(1-r)^{1-(\frac{1}{2}k+1)\lambda}$$

for r sufficiently near to 1 (intuitively because s_1 is large only near $z = 1$, where s_2 is near $\frac{1}{4}$).

Using the standard inequality (**2**, p. 11)

$$|a_n| < \frac{e}{n} I_1\left(1-\frac{1}{n}\right), \tag{3.7}$$

we deduce

Corollary 3.3. *Suppose that f(z) is of the form* (1.1), *and belongs to V_k. Then*

$$\limsup_{n\to\infty}\,(n^{1-\frac{1}{2}k}|a_n|) \leqq \frac{e2^{\frac{1}{2}k}\Gamma(k/4+1)}{\pi^{\frac{1}{2}}(k-1)\Gamma(k/4+\frac{1}{2})}. \tag{3.8}$$

Since, for positive x,

$$\log \Gamma(x) = (2\pi)^{\frac{1}{2}} + (x-\tfrac{1}{2}) \log x - x + \theta(x)/12x,$$

where $0<\theta(x)<1$ (**6**, p. 153), we see that

$$\frac{\Gamma(k/4+1)}{(k-1)\Gamma(k/4+\frac{1}{2})} \sim k^{-\frac{1}{2}} \text{ as } k\to\infty.$$

In the opposite direction, we have

Theorem 3.4. *Suppose that $f(z)$ is given by* (2.1). *Then*

$$a_n \sim \frac{2^{\frac{1}{2}k}}{k\Gamma(\frac{1}{2}k)} n^{\frac{1}{2}k-1} \text{ as } n\to\infty. \tag{3.9}$$

This is verified by an argument similar to that of (**4**, p. 93), and improves the estimate in (**3**).

Now let us observe that, with very little technical effort, it is possible to obtain a coefficient estimate for functions in V_k. This is based on

Theorem 3.5. *Suppose that $f(z)$ belongs to V_k, $M(r) = \max_{|z|=r} |f(z)|$, and*

$$L(r) = \int_0^{2\pi} r\,|f'(re^{i\theta})|\,d\theta$$

is the length of $f(|z|=r)$. Then

$$2M(r)<L(r)<2^{\frac{3}{2}}(2k+1)M(r). \tag{3.10}$$

Proof. The left inequality of (3.10) is a consequence of the fact that $f(|z|=r)$ is a closed curve round the origin, and the right inequality is Theorem 3.3 of (**1**) (whose proof was totally elementary).

Corollary 3.6. *A function in V_k is bounded if and only if its derivative belongs to the Hardy class H_1.*

We now have

Theorem 3.7. *Suppose that $f(z)$ is of the form* (1.1), *and belongs to V_k. Then, if $M(r) = \max_{|z|=r} |f(z)|$,*

$$|a_n| < \frac{e}{n\pi} 2^{\frac{1}{2}}(2k+1)M\left(1-\frac{1}{n}\right) \quad (n>1). \tag{3.11}$$

This follows by applying (3.7) to (3.10), using the fact that $L(r) = rI_1(r)$.

It has been conjectured (**7**) that, if $f(z)$ belongs to V_k, the moduli of its coefficients do not exceed the corresponding coefficients of the function (2.1). In this direction we have

Theorem 3.8. *Suppose that $f(z)$ is of the form* (1.1), *belongs to V_k, and is given by* (3.1). *Then*

$$a_n = o(n^{\frac{1}{2}k-1}) \text{ as } n\to\infty, \tag{3.12}$$

unless

$$s_1(z) = z(1-tz)^{-2} \text{ for some } |t| = 1. \tag{3.13}$$

Proof. It follows from (**10**, Theorem 1) that, if $s_1(z)$ is normalized and starlike in U and is not of the form (3.13),

$$\lim_{r\to 1} (1-r)^2 \max_{|z|=r} |s_1(z)| = 0.$$

Then, from (3.1),

$$\max_{|z|=r} |f'(z)| = o\{(1-r)^{-\frac{1}{2}k-1}\}.$$

Integrating (3.14) we get

$$\max_{|z|=r} |f(z)| = o\{(1-r)^{-\frac{1}{2}k}\},$$

so that, by Theorem 3.7,

$$a_n = o(n^{\frac{1}{2}k-1}).$$

Furthermore we observe that, if $1/s_2(z)$ is continuous near the point $z = 1/t$, the coefficient conjecture is certainly true for sufficiently large indices; this is easily verified by applying the techniques of (**4**, p. 93) to Theorem 3.8.

Finally we note the following result, which seems rather interesting in view of the conjecture.

Theorem 3.9. *Suppose that* $g(z)$ *belongs to* V_k, $f(z)$ *is of the form* (1.1), *and*

$$f'(z) = g'(z)\left(\frac{1+z}{1-z}\right)^m$$

for some $m \geqq 0$. *Then* $f(z)$ *belongs to* V_{k+2m}.

Proof. For any $z = re^{i\theta}$ and $0 \leqq r < 1$, we have

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{2z}{1-z^2} \right| d\theta \leqq \tfrac{1}{2}\int_0^{2\pi} \operatorname{Re} \frac{1+z}{1-z}\, d\theta + \tfrac{1}{2}\int_0^{2\pi} \operatorname{Re}\frac{1-z}{1+z}\, d\theta = 2\pi.$$

Consequently

$$\int_0^{2\pi} |\operatorname{Re}(1+zf''/f')|\, d\theta \leqq \int_0^{2\pi} |\operatorname{Re}(1+zg''/g')|\, d\theta + m\int_0^{2\pi} \left|\operatorname{Re}\frac{2z}{1-z^2}\right| d\theta$$
$$\leqq k\pi + 2m\pi;$$

thus $f(z)$ belongs to V_{k+2m} as required.

Note. This theorem can also be proved using Theorem 3.1.

4. More coefficient results

We now consider the connection between the coefficients of functions $f(z)$ in V_k, the area $A(r)$ of $f(|z|<r)$ (taking account of multiplicity), and the maximum modulus $M(r) = \max_{|z|=r} |f(z)|$, $0<r<1$.

Theorem 4.1. *Suppose that $f(z)$ belongs to V_k, and is of the form* (1.1) *and that $f(U)$ has finite area A (taking account of multiplicity). Then*

$$|a_n| < \frac{k}{n}\left(1+\frac{1}{2n}\right)^{\frac{1}{2}}\left(\frac{A}{\pi}\right)^{\frac{1}{2}}. \tag{4.1}$$

Proof. By the Paatero representation theorem, we have

$$1+zf''/f' = (\tfrac{1}{4}k+\tfrac{1}{2})p_1(z)-(\tfrac{1}{4}k-\tfrac{1}{2})p_2(z), \tag{4.2}$$

where $p_i(0) = 1$ and $\operatorname{Re} p_i(z)>0$ in U, $i = 1, 2$. Then

$$f'+zf'' = (\tfrac{1}{4}k+\tfrac{1}{2})p_1 f'-(\tfrac{1}{4}k-\tfrac{1}{2})p_2 f'; \tag{4.3}$$

hence, if $z = re^{i\theta}$, multiplying both sides of (4.3) by $e^{-i(n-1)\theta}$, integrating from 0 to 2π, and using the triangle inequality, we obtain

$$2\pi n^2 r^{n-1}|a_n| \leqq (\tfrac{1}{4}k+\tfrac{1}{2})\int |p_1 f'|\,d\theta+(\tfrac{1}{4}k-\tfrac{1}{2})\int |p_2 f'|\,d\theta. \tag{4.4}$$

Applying Schwarz's inequality for $i = 1, 2$, we obtain

$$\begin{aligned}\left(\int_0^{2\pi} |p_i f'|\,d\theta\right)^2 &\leqq \int |p_i|^2 d\theta \int |f'|^2 d\theta\\ &\leqq \int_0^{2\pi}\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^2 d\theta \int_0^{2\pi} |f'|^2 d\theta\\ &\leqq 2\pi\,\frac{1+r}{1-r}\int_0^{2\pi}|f'|^2 d\theta,\end{aligned} \tag{4.5}$$

using the fact that each $p_i(z)$ is subordinate in U to $\dfrac{1+z}{1-z}$, and that

$$\int_0^{2\pi}\frac{1-r^2}{|1-re^{i\theta}|^2}\,d\theta = 2\pi$$

(as the integrand is the Poisson kernel). From (4.4) and (4.5) we have

$$\begin{aligned}4\pi^2 n^4 |a_n|^2 r^{2n-2} &\leqq \tfrac{1}{4}k^2 . 2\pi . \frac{1+r}{1-r}\int_0^{2\pi}|f'(re^{i\theta})|^2 d\theta\\ &< k^2\pi(1-r)^{-1}\int_0^{2\pi}|f'|^2 d\theta.\end{aligned} \tag{4.6}$$

Multiplying both sides of (4.6) by $r(1-r)$, and integrating from 0 to 1, we get (4.1).

This leads at once to

Theorem 4.2. *Suppose that $f(z)$ belongs to V_k, and is of the form* (1.1). *Then if $A(r)$ is the area of $f(|z|<r)$, and $M(r) = \max_{|z|=r}|f(z)|$,*

$$|a_n| < \frac{ek}{n}\left(\frac{A\left(1-\frac{1}{n}\right)}{\pi}\right)^{\frac{1}{2}} \tag{4.7}$$

and

$$|a_n| < \frac{ek}{2^{\frac{1}{2}}n} M\left(1-\frac{1}{n}\right). \tag{4.8}$$

Proof. Applying Theorem 4.1 to the function $\frac{1}{r}f(rz)$, we deduce that

$$r^{n-1}|a_n| < \frac{k}{n}\left(1+\frac{1}{2n}\right)^{\frac{1}{2}}\left(\frac{A(r)}{\pi}\right)^{\frac{1}{2}}.$$

Substituting $r = 1-\frac{1}{n}$, we obtain (4.7).

Since the valency of $f(z)$ is at most $\frac{1}{2}k$, by Theorem 2.1, we have

$$A(r) \leqq \pi M^2(r)\,.\,\tfrac{1}{2}k; \tag{4.9}$$

then (4.8) follows from (4.7) and (4.9).

5. Special subclasses of V_k

We now examine the coefficient problem for functions in V_k which are bounded or of finite area.

If $f(U)$ is bounded, then $f'(z)$ belongs to H_1, by Corollary 3.6; thus, if $f(z)$ is of the form (1.1),

$$a_n = o(n^{-1}) \text{ as } n \to \infty \tag{5.1}$$

(see, for example, (**11**, p. 112)). Although we are unable to show that (5.1) is best possible, we can at least show that the exponent of n cannot be reduced.

Theorem 5.1. *Choose any* $\varepsilon > 0$, *and any* $k \geqq 6+4\varepsilon$. *Then there is a bounded function* $f(z)$ *in* V_k, *of the form* (1.1), *such that*

$$a_n \sim 1/n(\log n)^{1+\varepsilon} \text{ as } n \to \infty. \tag{5.2}$$

Note. In our proof, we use Theorem 3.1 and the fact that, if $g(z)$ is normalized and starlike in U, then so is $z(g/z)^t$ for $0<t<1$.

Proof. The function $s_1(z) = z(1-z)^{-4/(k+2)}$ is starlike in U; also, since $\log(1-z)^{-1}$ is starlike in U, so is

$$s_2(z) = z\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{4(1+\varepsilon)/(k-2)},$$

as $4(1+\varepsilon) \leqq k-2$. Thus the function $f(z)$, of the form (1.1), belongs to V_k, where

$$f'(z) = (1-z)^{-1}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-1-\varepsilon},$$

by Theorem 3.1. Hence

$$na_n \sim (\log n)^{-1-\varepsilon} \text{ as } n \to \infty,$$

using the coefficient estimates in (**4**, p. 93).

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In the case that $f(U)$ is not necessarily bounded, but does have finite area, Theorem 4.1 shows that

$$a_n = O(n^{-1}) \text{ as } n \to \infty. \tag{5.3}$$

We cannot show that (5.3) is best possible, but can establish

Theorem 5.2. *Choose any* $\varepsilon > 0$, *and any* $k \geqq 4+2\varepsilon$. *Then there is a function* $f(z)$ *in* V_k, *of the form* (1.1), *such that* $f(U)$ *has finite area, and*

$$na_n \sim (\log n)^{-\frac{1}{2}-\frac{1}{2}\varepsilon} \text{ as } n \to \infty. \tag{5.4}$$

Proof. The functions

$$s_1(z) = z(1-z)^{-4/(k+2)} \text{ and } s_2(z) = z\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{2(1+\varepsilon)/(k-2)}$$

are normalized starlike functions in U so long as $k \geqq 4+2\varepsilon$. Hence, by Theorem 3.1, the function $f(z)$, of the form (1.1), belongs to V_k, where

$$f'(z) = (1-z)^{-1}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-\frac{1}{2}-\frac{1}{2}\varepsilon}.$$

Then the area of $f(U)$ is

$$\begin{aligned} A(1) &= \int_0^1 \int_0^{2\pi} |f'(re^{i\theta})|^2 r\,dr\,d\theta \\ &= \int_0^1 r^{-\varepsilon}dr \int_0^{2\pi} \left|(1-re^{i\theta})^2\left(\log\frac{1}{1-re^{i\theta}}\right)^{-1-\varepsilon}\right|^{-1} d\theta \\ &< A(\varepsilon)\int_0^1 (1-r)^{-1}\left(\frac{1}{r}\log\frac{1}{1-r}\right)^{-1-\varepsilon} dr \\ &< +\infty; \end{aligned}$$

here we have used the results on integral means in (**4**, p. 96). (5.4) then follows from the coefficient estimates in (**4**, p. 93).

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