STAR DIAGRAMS AND THE SYMMETRIC GROUP

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Introduction. The irreducible representations of the symmetric group S_n, were shown by A. Young to be in one-to-one correspondence with certain arrays of n nodes. E.g. for $n = 12$ and the partition $\lambda = [4, 4, 3, 1]$ we have the array

$$\lambda: \begin{array}{llll} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & & & \end{array}$$

which we call a "Young diagram." The question arises as to the manner in which various properties of the representations are reflected in their corresponding Young diagrams.

The study of modular representations [1] has shown that, relative to a given prime, p, the ordinary (non-modular) irreducible representations of a group gather into "p-blocks". Two irreducible representations of S_n belong to the same p-block if and only if their corresponding diagrams have the same "p-core" (see 1.8). This was conjectured by T. Nakayama in 1940 [3], and proven by R. Brauer and G. de B. Robinson in 1947 [2]. The proof involved an auxiliary diagram—the "star diagram" of the Young diagram concerned. It is the purpose of the present paper to discuss the construction of the star diagram in greater detail, and to place greater emphasis upon it than has hitherto been done.

The distribution into p-blocks has to do with the power, $e(z)$, to which p divides the degree, z, of the representation concerned. Nakayama [4] obtained a formula for $e(z)$ in terms of the diagram's "p-series" (see 1.7), namely

$$e(z) = e(n!) - \sum_t e(p^{e_t}!).$$

This formula, however, was not suitable for the proof of his conjecture. Nakayama was led to the p-series of λ by the study of the "p-hook" (see 1.4) structure of λ. Robinson [5] showed (see below) that this p-hook structure could be represented by an associated diagram, λ^*—the "star diagram" of λ (usually denoted $\lambda_p{}^*$: we omit the subscript, reserving the space for another use later on). The diagram λ^* is in general skew (see 1.2) and to such a diagram corresponds a reducible representation [6] of S_m, where m is the number of nodes of the diagram. The following formula [2] was used in the proof of the conjecture:

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A: $$e(z_\lambda) = e(n!) - e((n - A)!) + e(z_{\lambda^*}).$$

(A denotes the number of nodes of the p-core of λ, and z_{λ^*} is the degree of the reducible representation corresponding to λ^*.)

The proof of formula **A**, however, was based on Nakayama's formula, which involved the p-series, —an entity not appearing in A. Accordingly it was felt that full use was not being made of the star diagram, λ^*, and it was hoped that a proof could be developed in terms of it alone.

The present paper begins with a proof of the following existence theorem for λ^*:

B: *Given a right diagram, λ, and a positive integer, q, there exists a diagram, λ^*, such that there is a one-to-one correspondence between kq-hooks of λ and k-hooks of λ^*.*

An auxiliary theorem, **B′**, shows that λ^* represents the actual q-hook *structure* of λ, with regard to removal of q-hooks from λ. Simple considerations of congruence provide a new proof of the fact that λ^* has at most q disjoint constituents.

The following theorem exhibits the connection between λ and λ^* in a form which leads to a new proof of **A**:

C: *Gather the δ's of λ (see 1.9) into classes of δ's which are congruent (mod q). For each such class form the diagram whose δ's are those of this class. The star diagrams of the diagrams thus formed are the constituents of λ^*.*

A proof of **A** is then given which depends only on λ^*, and pairs the factors p of z_λ and z_{λ^*} in an explicit manner. (In **A** we take q to be a prime, p.)

The diagram λ^* and the p-series of λ are shown to be related in the following way:

D: *Given λ, form λ^*, $(\lambda^*)^* = \lambda^{2*}, \ldots, \lambda^{r*}$, where λ^{r*} is a p-core. Suppose the p-core of λ^{i*} has A_i nodes. Then the p-series of λ is*

$$p^r, \ldots (A_r \text{ times});\ p^{r-1}, \ldots (A_{r-1} \text{ times}); \ldots \quad ;\ p, \ldots (A_1 \text{ times}).$$

This leads to a new proof of Nakayama's p-series formula for $e(z)$, based on Theorem **A**. All told, emphasis on the star diagram, rather than the p-series, is seen to recast the theory in a more orderly and understandable manner.

1. Notation, definitions. We shall find it convenient to collect the numerous definitions which are required into a preliminary section.

1.1 A *right* (Young) *diagram* is an array of n nodes with straight top and left sides, and whose rows are in non-increasing order of length. A node which has no node one row below and one column to the right of it is said to be a node of the *rim* of the diagram.

1.2 A *skew diagram* is what is obtained by removing from the top left corner of a right diagram another right diagram which is contained in it. E.g.

1.21
$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \circ & \circ \\ \cdot & \cdot & \cdot & \circ & \\ \cdot & \circ & \circ & \circ & \\ \circ & \circ & \circ & & \\ \circ & & & & \end{array}$$

The circled nodes form a skew diagram.

1.3 A *disjoint* (*skew*) *diagram* is one which consists of constituents having no rows or columns with nodes in common. (See star diagram of 2.1).

1.4 A *right hook* of a diagram, λ, consists of a node of λ, together with all nodes directly below it, and directly to the right of it. If it has q nodes, it is called a *q-hook*, or a hook of *length q*. We shall call the top-right and bottom-left nodes of a hook its *top* and *bottom* nodes, respectively. E.g.

1.41
$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \circ & \circ & \circ + \\ \cdot & \circ & + & + & + \\ \cdot & \circ + & + & & \end{array}$$

The circle nodes form a 6-hook. (Two of them are marked "+" as well).

1.5 Each right hook has an associated *skew hook*, of the same length, consisting of all the nodes along the rim from the top node of the given hook to its bottom node. (The nodes in 1.41 marked "+" form a skew hook.) A piece of the rim is the skew hook of some right hook if, and only if, its top node has no node to the right of it (in its row) and its bottom node has no node below it (in its column).

1.6 To *remove a hook* from λ, we erase the nodes of its associated skew hook. (The removal of the 6-hook of 1.41 leaves the diagram whose nodes are those not marked "+".)

1.7 The *p-series* of λ: Let the longest hook of λ whose length is a power of p be of length p^{e_1}. Remove this hook from λ. Let the longest hook of the remaining diagram whose length is a power of p, but not greater than p^{e_1}, be of length p^{e_2}. Remove this hook and repeat the process until all such hooks are removed. The resulting sequence,

$$p^{e_1}, p^{e_2}, p^{e_3}, \ldots$$

is the p-series of λ. (The 2-series of 1.81 is 8,4.)

1.8 The *p-core* of λ: The result of the successive removal of the hooks of the p-series is a right diagram—the p-core of λ. It is uniquely determined by p

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and λ, and is obtained also when all kp-hooks ($k = 1,2,3,\ldots$) are removed from λ in any order whatever [4]. E.g.

1.81
$$\begin{array}{ccccc} \circ & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & \\ \cdot & & & & \end{array}$$

The 2-core consists of a single node (circle).

1.9 The *δ-numbers* of λ: The lengths of the hooks which begin in the top row of λ are called the "δ-numbers" or "δ's" of λ. They are numbered in the order of their lengths, i.e. $\delta_1 > \delta_2 > \delta_3 > \ldots$. "δ" is also used to refer to the hook whose length is δ. This convenient ambiguity causes no difficulties. (The δ's of 1.21 are 9, 7, 6, 4, 1 and those of 1.41 are 8, 7, 6, 4, 3.)

2. Star diagrams. We prove the following existence theorem.

B. *Given the right diagram λ, and a positive integer q, there exists a diagram λ* (called the "star diagram" of λ) such that there is a one-to-one correspondence between kq-hooks of λ and k-hooks of λ*.*

Example:

2.1 $q = 2$ **λ:**
$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & + & + & \\ \cdot & + & \cdot & \\ \cdot & + & & \end{array}$$
λ*:
$$\begin{array}{ccc} & \cdot & \cdot \\ & + & + \\ \cdot & & \\ \cdot & & \end{array}$$

It is sufficient and simpler to consider skew hooks (see 1.5) and we shall now refer to these simply as "hooks". We shall consider the nodes of a hook to be ordered from top to bottom and right to left. A node which has no node to the right of it (in its row) we shall call an "H", and a node which has no node below it (in its column)we shall call an "F". An F cannot precede (immediately) an H. If a node is not an F, then the node which follows it is an H.

Carrying out Robinson's construction[5], we take the longest (any one of the longest) kq-hook, J_1, in λ, of length k_1q, say, and consider the chain, C_1, of all kq-hooks having the same top node, H_1, as J_1. Suppose the lengths of these kq-hooks are k_1q, k_2q, k_3q, $\ldots$ k_sq; we construct the diagram whose δ's are $k_1, k_2, \ldots, k_s$ ($k_1 > k_2 > k_3 > \ldots > k_s$.) This diagram we take to be the first constituent, λ^*_1, of λ^*. (In 2.1, $k_1 = 3$, $k_2 = 2$.) This gives a one-to-one correspondence between the kq-hooks of C_1 and k-hooks of λ^*_1 which begin in the top row of λ^*_1. More than this follows, however.

Consider *any* k-hook, K, of λ^*_1, consisting of the $(r+1)$th, $\ldots$, $(r+k)$th nodes of the rim of λ^*_1. Its bottom node is the bottom node of an $(r+k)$-hook of λ^*_1, beginning in the top row of λ^*_1: this hook corresponds to an $(r+k)q$-hook of λ, beginning at H_1, and whose bottom node, an F, is the $(r+k)q$th node of J_1.

Suppose the rqth node of J_1 were to the right of the $(rq + 1)$th node. Then it would be an F, and the first rq nodes of J_1 would form an rq-hook corresponding to an r-hook consisting of the first r nodes of the rim of λ^*_1. But the $(r + 1)$th node of the rim of λ^*_1 is the top node of the hook K, and such a node cannot follow the bottom node of a hook. Hence the $(rq + 1)$th node of J_1 is an H.

Therefore, corresponding to K there is a kq-hook of λ, beginning at the $(rq + 1)$th node of J_1.

Next, let I be a kq-hook of λ whose bottom node is the tqth node of J_1, which is the bottom node of a hook of C_1. The tth node of the rim of λ^*_1 must then be an F; k cannot exceed t, for otherwise J_1 would not be the longest hook of λ whose length is a multiple of q. The top node of I is an H—hence the preceding node is not an F, and the first $(t - k)q$ nodes of J_1 do not form a hook. Hence the first $(t - k)$ nodes of the rim of λ^*_1 do not form a hook, and the $(t - k + 1)$th node is an H.

Hence the k nodes of the rim of λ^*_1 which follow the $(t - k)$th node form a k-hook corresponding to I.

So far we have a one-to-one correspondence between k-hooks of λ^*_1 *and kq-hooks of* λ *whose bottom nodes are bottom nodes of hooks of* C_1. For each k-hook of λ^*_1 there is a δ-hook of λ^*_1 having the same bottom node: the corresponding kq-hook of λ has the same bottom node as the hook of λ corresponding to this δ-hook.

Continuing Robinson's construction, we take the longest kq-hook, J_2, of λ *which is not already represented* in λ^*_1 and repeat the previous construction, obtaining λ^*_2. We continue in this way until all the kq-hooks of λ are used up, obtaining diagrams $\lambda^*_1, \lambda^*_2, \ldots \lambda^*_w$, which we arrange disjointly to form λ^*.

It remains to show that a given kq-hook, M, of λ is represented in only one constituent of λ^*. Let λ^*_a, λ^*_b, $a \neq b$, correspond to the chains C_a, C_b of kq-hooks. It is sufficient to show that the bottom node of M cannot be the bottom node of a hook of C_a and a hook of C_b.

Suppose it were, and suppose the top node of J_a were m nodes above the top node of J_b. Then m would be divisible by q, and the hook running from the top of J_a to the bottom of J_b would belong to C_a. But then J_b would be represented in λ^*_a, and also $a > b$, since J_a would be longer than J_b: thus J_b would already have been represented, contrary to hypothesis.

Hence the correspondence is one-to-one, and the theorem is proven.

The following theorem shows that if corresponding hooks are removed from λ and λ^*, the relationship between them is unaltered.

B′. *If a k-hook is removed from* λ^*, *leaving* $\bar{\lambda}^*$, *and the corresponding kq-hook is removed from* λ, *leaving* $\bar{\lambda}$, *then* $(\bar{\lambda})^* = \bar{\lambda}^*$.

Nakayama showed that the removal of a kq-hook can be accomplished by the successive removal of k q-hooks. Hence it is sufficient to consider the removal of a single node from λ^*. This will affect only the constituent, λ^*_i,

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in which it appears. If it is the top node of λ^*_i, it will (when removed) reduce all the δ's of that constituent by 1: if it is not the top node, it will reduce exactly one δ by 1—namely the δ of which it is the bottom node. Let B be the q-hook of λ corresponding to the node removed from λ^*, and let h,f be its top and bottom nodes respectively. Consider the effect of removing B from λ.

When B is removed, the node (if there is one) preceding h (on the rim of λ) becomes an F. (It was not previously an F.) Any other nodes preceding B remain unaffected. The node following f (if there is one) becomes an H. (It was not previously an H.) All other nodes following B remain unaffected. If there are nodes preceding B and nodes following B, then when B is removed, q new nodes become members of the rim in its place. These are the nodes situated one space diagonally up and to the left of the nodes of B: they form a piece of rim identical in shape to B. The node corresponding to h is not an H (of $\bar{\lambda}$), however, and the node corresponding to f is not an F. Otherwise these q nodes are H's or F's, or both, according as the corresponding nodes of B are H's or F's or both. If there are no nodes preceding B, then there may be less than q new nodes becoming part of the rim of the diagram: the same thing may happen if there are no nodes following B. In any case, however, the above remarks apply to as many new members of the rim as there may be.

Consider first the case where the node removed from λ^*_i is the top node of λ^*_i. When it is removed, all the δ's of λ^*_i are reduced by 1. We must check and see that the hooks of C_i are all reduced by q, and that the other chains remain unaffected as to the lengths of their hooks. B in this case will be the smallest hook of C_i, and h will be the top node of the hooks of C_i. The node corresponding to h (see above) if there is one at all, is not an H, and the node following f, if there is one, becomes an H. Thus we have a chain of $\bar{\lambda}$ whose node is q nodes below h, and the bottom nodes of the hooks remain unaltered. This reduces the lengths of the hooks of C by 1, as required.

Let C_j, $j \neq i$, be a chain with top node c. Then c cannot coincide with h. If it lies above h it is unaffected by the removal of B. (It remains an H.) Those bottom nodes of hooks of C_j which lie below B are unaffected. If a bottom node b of a hook of C_j is a node of B, then it lies at least one column to the right of the column of f, and since c is at least one row above h, and hence above b, b is not in the first row of λ. Hence there is a node corresponding to b—one row above and one column to the left, and this node will be an F. Thus the lengths of the hooks of C_j are unchanged. (More precisely, if λ has a chain C_j, then $\bar{\lambda}$ has a chain of hooks of the same lengths.) Exactly similar arguments deal with the other possible locations of c.

Any chain of $\bar{\lambda}$ corresponds to some chain of λ, for a bottom node of one of its hooks is either an F of λ, corresponding to a node of B, or is the node preceding h on the rim of λ. In the last case the chain corresponds to the chain C_i, and in the other cases it corresponds to the chain of which it (or its corresponding node) is a bottom node. Thus the star diagrams $(\bar{\lambda})^*$ and $\bar{\lambda}^*$ are identical.

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Exactly similar arguments deal with the case where the node removed from λ^* is not the top node of λ^*_i, completing the proof of the theorem.

3. Classes of congruent δ's. A diagram is completely determined by its δ's, and these will be our primary concern in the sections to follow. In this section we make some remarks concerning the congruent (mod q) classes of δ's of a right diagram.

Consider the chain, C_i, of kq-hooks corresponding to λ^*_i. Suppose H_i, the top node of these hooks, is the $(m+1)$th node of the rim of λ. The lengths of the hooks are all divisible by q, and their bottom nodes are bottom nodes of δ's of λ. The lengths of these δ's are just m greater than the lengths of the hooks of C_i. Hence they are all congruent (mod q). We shall refer to them as δ^i's.

A δ^i cannot be congruent (mod q) to a δ^j, $i \neq j$, for suppose it were, and suppose $i > j$. Then the bottom node, f, of the δ^i would lie c nodes (along the rim) below the bottom node of the δ^j, where $c \equiv 0$ (mod q), and f would be the bottom node of a hook of C_j. But f is also the bottom node of a hook of C_i, and this cannot be, as was seen in §2. Hence the δ^i's are not congruent to the δ^j's, $i \neq j$.

Since there can be at most q different classes of numbers congruent (mod q), we have immediately

3.1 *The number of constituents of δ^* is at most q.*

This was formerly proven by Robinson, using a theorem of Nakayama's, and by consideration of the removal of hooks from λ.

The following theorem is used in the proof of Theorem **A**.

C. *Gather the δ's of λ into classes of δ's which are congruent (mod q). For each such class of congruent δ's form the diagram whose δ's are the δ's of this class. The star diagrams of the diagrams thus formed will be the constituents of λ^*.*

To illustrate the theorem let $q = 3$ and consider the diagram

3.2
$$\lambda: \quad \begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & & \\ \cdot & & & & \end{array}$$

The δ's are 9, 7, 5, 4, 2, and the classes of congruents δ's are $\{9\}, \{7, 4\}, \{5, 2\}$, which yield the diagrams

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$$\begin{array}{l} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}, \quad \begin{array}{ll} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \cdot & \end{array}, \quad \begin{array}{ll} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \cdot & \end{array} \quad \text{with star diagrams} \quad \begin{array}{l} \cdot \\ \cdot \\ \cdot \end{array}, \quad \cdot\ \cdot \quad \text{null.}$$

Thus λ^* is

$$\begin{array}{lll} & & \cdot \\ & & \cdot \\ & & \cdot \\ \cdot & \cdot & \end{array}$$

To prove **C**, consider a class, K, of congruent δ's, and let μ be the diagram whose δ's are the members of K. We have just seen that the δ's of λ which have bottom nodes in common with hooks of a chain, C_i, are all congruent (mod q). We will show that, if K is the class of all δ's congruent to those associated in this way with some C_i, then $\mu^* = \lambda^*_i$: otherwise μ^* is null.

Suppose that the hooks of C_i, are of lengths k_1q, $k_2q, \ldots, k_sq$, and that their common top node, H_i, is the $(m + 1)$th node of the rim of λ. Then the members of K are $k_1q + m, k_2q + m, \ldots, k_sq + m$ and possibly some of $m - q$, $m - 2q, \ldots$ as well: these will be the δ's of μ. We wish to show that the $(m + 1)$th node of the rim of μ is an H, but that the $(m + 1 - q)$th, $(m + 1 - 2q)$th, ... nodes are not H's. This will yield a chain of hooks of μ of lengths $k_1q, k_2q, \ldots, k_sq$, as required.

The mth node of the rim μ is not an F, for suppose it were: then the mth node of the rim of λ would be an F also, but this could not be, since the $(m + 1)$th node, H_i, is an H. Hence the $(m + 1)$th node of the rim of μ is an H.

The $(m + 1 - bq)$th node, g, of the rim of λ is not an H, for if it were then there would be a chain of kq-hooks beginning at g, and the hooks of C_i would already have been represented in the constituent of λ^* corresponding to this chain, contrary to assumption. Hence the $(m - bq)$th node is an F, and so is the $(m - bq)$th node of the rim of μ. Hence the $(m - bq + 1)$th node of the rim of μ is not an H.

Therefore μ has a chain of kq-hooks identical with C_i. It can only have one chain, since all the δ's of μ are congruent (mod q). Hence $\mu^* = \lambda^*_i$.

Finally, suppose that the members of K have no bottom nodes in common with hooks of C_i, for any i. Then μ has no kq-hooks, for, suppose the $(p + 1)$th and $(p + kq)$th nodes of its rim are an H and an F respectively. The $(p+kq)$th node of the rim of λ is then an F. The pth node of the rim of μ is not an F, since the $(p + 1)$th is an H; hence the pth node of the rim of λ is not an F, and hence the $(p + 1)$th node is an H. But this results in a kq-hook of λ

which shares its bottom node with a δ of K, contrary to assumption. Hence μ^* is null.

4. The determination of $e(z_\lambda)$. We shall henceforth require q to be a prime, p. The degree of the irreducible representation corresponding to λ is

4.1
$$z_\lambda = n!\,\frac{\prod\limits_{s<t}(\delta_s - \delta_t)}{\prod\limits_i \delta_i!}$$

The degree of the reducible representation corresponding to λ^* is

4.2
$$z_{\lambda^*} = \frac{B!}{B_1!\,\ldots\,B_k!}\cdot z_{\lambda^*_1}\,\ldots\,z_{\lambda^*_k}$$

where B_i denotes the number of nodes in λ^*_i, $B = \sum\limits_i B_i$, k is the number of constituents of λ^*, and $z_{\lambda^*_i}$ is given by 4.1 applied to λ^*_i.

We shall prove

A:
$$e(z_\lambda) = e(n!) - e(\,(n-A)!) + e(z_{\lambda^*})$$

by pairing off factors p from z_λ and z_{λ^*}. First we deal with the particular case where (1) λ^* has exactly one constituent, and (2) the kp-hooks of the corresponding chain all begin at the top node of λ—that is, they are all δ's of λ. The general case is reduced to this special case by (1) Theorem C, which reduces the problem to consideration of a single constituent, and (2) Lemma 4.3 which enables us to remove rows from the top of λ until the row at which the kq-hooks begin is reached.

4.3 LEMMA. *Suppose λ has no κp-hook beginning in the first row, and suppose the first row of λ is removed, leaving $\bar{\lambda}$ of $\bar{n}$ nodes. Then*

$$e(z_\lambda) - e(z_{\bar{\lambda}}) = e(n!) - e(\bar{n}!).$$

Let us assume λ to have the form

$$\begin{array}{l} \cdot\ \ -\ \ -\ \ -\ \ \cdot\ \ \overbrace{\cdot\ \ -\ \ -\ \ \cdot}^{k} \\ \cdot\ \ -\ \ -\ \ -\ \ \cdot \\ \ \ \ -\ \ - \\ \cdot \end{array}$$

so that the first row is k nodes longer than the second row, where $k \geqq 0$. We wish to compare

$$z_\lambda = n!\frac{\prod\limits_{i<j}(\delta_i - \delta_j)\,.\,\prod\limits_{f<g}(\delta_f - \delta_g)\,.\,\prod\limits_{a<b}(\delta_a - \delta_b)}{\prod\limits_s(\bar{\delta}_s + k + 1)!\,k!\,(k-1)!\ldots 3!\,2!\,1}$$

with
$$z_{\bar{\lambda}} = \bar{n}!\,\frac{\prod(\bar{\delta}_s - \bar{\delta}_t)}{\prod\limits_s(\bar{\delta}_s!)},$$

where (a) δ_i, δ_j of λ have their feet (bottom nodes) in $\bar{\lambda}$ (and in λ).

(b) δ_f, δ_g of λ have their feet in the first row of λ. (Hence in the last k nodes of this row.)

(c) δ_a of λ has its foot in $\bar{\lambda}$ (and in λ), δ_b of λ has its foot in the first row of λ.

(d) $\bar{\delta}_s$, $\bar{\delta}_t$ are δ's of $\bar{\lambda}$.

Note that $(\bar{\delta}_s + k + 1)$, as s varies, and k, $(k-1)$, ..., 3, 2, 1 are just the δ's of λ.

(i) $(\delta_i - \delta_j)$ depends only on the feet of δ_i, δ_j, hence $\prod_{i<j}(\delta_i - \delta_j) = \prod_{s<t}(\bar{\delta}_s - \bar{\delta}_t)$ and $e(\prod_{i<j}(\delta_i - \delta_j)) = e(\prod_{s<t}\bar{\delta}_s - \bar{\delta}_t))$.

(ii) $k < p$, for otherwise there would be a κp-hook in the first row of λ, contrary to assumption. Hence $\delta_f < p$, $\delta_g < p$, and $e(\prod_{f<g}(\delta_f - \delta_g))$, $e((k)!\,(k-1)!\,(k-2)!\ldots 3!2!1)$ are both zero.

(iii) $\dfrac{(\bar{\delta}_s + k + 1)!}{\bar{\delta}_s!} = (\bar{\delta}_s + k + 1)(\bar{\delta}_s + k)\ldots(\bar{\delta}_s + 1)$; $(\bar{\delta}_s + k + 1)$ is not divisible by p, since λ was assumed to have no κp-hook beginning in the first row. Hence $e\left(\dfrac{(\bar{\delta}_s + k + 1)!}{\bar{\delta}_s!}\right) = e((\bar{\delta}_s + 1)\ldots(\bar{\delta}_s + k))$, and we note that $(\bar{\delta}_s + k), \ldots, (\bar{\delta}_s + 1)$ are the terms of $\prod_{a<b}(\delta_a - \delta_b)$ with a given $\delta_a = \bar{\delta}_s + k + 1$. Hence $e\left[\prod_s \dfrac{(\bar{\delta}_s + k + 1)!}{\prod_s \bar{\delta}_s!}\right] = e(\prod_{a<b}(\delta_a - \delta_b))$. Thus all contributions cancel except those of $n!$ and $\bar{n}\,!$. These yield the required result.

4.4 Lemma. *If λ is a p-core, then $e(z_\lambda) = e(n!)$.*

This is proven by removing rows until the last row is reached, and applying 4.3 on the removal of each row.

4.5 Corollary. *If λ is a p-core, then* $e\left[\dfrac{\prod_{s<t}(\delta_s - \delta_t)}{\prod_i \delta_i!}\right] = 0.$

The following two lemmas are given without proof.

4.6 Lemma. $e((pa)!) - e(a!) = a$, *a any integer.*

4.7 Lemma. *If λ has k columns, then* $\sum_{i=1}^{k}\delta_i = n + \frac{1}{2}k(k-1)$. (A well-known result.[4])

We are now in a position to prove an important special case of the main theorem.

4.8 *If λ^* has exactly one constituent, and each δ of λ^* represents a δ of λ then*

$$e(z_\lambda) = e(n!) - e((n-A)!) + e(z_{\lambda^*}),$$

where z_λ is given by 4.1. We observe that $\delta_s - \delta_t$ makes no contribution to $e(z_\lambda)$ unless $\delta_s \equiv \delta_t \pmod p$. Hence we may write

$$z_\lambda = n!\prod_i \left[\frac{\prod_{s<t}(\delta_s{}^i - \delta_t{}^i)}{\prod_j \delta_j{}^i!}\right].K$$

where the δ^i's are a class of congruent δ's and $\prod_i$ is taken over these classes, and K makes no contribution to $e(z_\lambda)$. Since λ^* has only one constituent, just one class of δ's yields a diagram which is not a p-core. (See **C**.) For the other classes, 4.5 tells us that

$$e\left[\frac{\prod_{s<t}(\delta_s{}^i - \delta_t{}^i)}{\prod_j \delta_j{}^i!}\right] = 0.$$

Hence the contributing part of z_λ is just

$$C = n!\,\frac{\prod_{s<t}(\delta^\circ{}_s - \delta^\circ{}_t)}{\prod_i \delta^\circ{}_i!}$$

where the δ°'s are the δ's corresponding to the δ's of λ^*. Let us denote a δ of λ^* by δ^*. Each δ^* represents a δ of λ, by assumption, and $\delta = p\delta^*$. Hence we may write

$$C = n!\,\frac{\prod_{s<t}(p\delta^*{}_s - p\delta^*{}_t)\,.\,(pB)!}{\prod_i (p\delta^*{}_i)!\ \ (pB)!}$$

where the extra unit factor is added for later convenience, and

$$z_{\lambda^*} = B!\,\frac{\prod_{s<t}(\delta^*{}_s - \delta^*{}_t)}{\prod_i \delta_i{}^*!}. \qquad (B \text{ defined as in 4.2.})$$

It remains to show that

$$e(C) = e(n)! - e((n-A)!) + e(z_{\lambda^*}).$$

By Theorem **B**$'$, if a k-hook is removed from λ^*, and the corresponding kq-hook is removed from λ, then the relationship of diagram to star diagram is preserved. This leads directly to the fact that $pB = n - A$, which accounts for the term $e(n!) - e((n-A)!)$. For the remainder, consider:

(1) $$e[\prod_{s<t}(p\delta^*{}_s - p\delta^*{}_t)] - e[\prod_{s<t}(\delta^*{}_s - \delta^*{}_t)].$$

This is seen to be equal to the number of differences $(\delta^*{}_s - \delta^*{}_t)$, which is $\frac{1}{2}\,k(k-1)$, where k is the number of columns of λ^*.

(2) $$e((pB)!) - e(B!) = B, \qquad \text{by 4.6}$$

(3) $$e[\prod_i((p\delta^*{}_i)!)] - e[\prod_i(\delta^*{}_i!)] = \sum_i \delta^*{}_i \qquad \text{by 4.6}$$
$$= B + \tfrac{1}{2}k(k-1) \qquad \text{by 4.7.}$$

Then (1), (2) and (3) yield the required result.

Next we extend 4.8 by means of 4.3.

4.9 *If λ^* has exactly one constituent, then the result of 4.8 holds.*

Let H be the top node of the kp-hooks of λ which correspond to δ's of λ^*. Remove rows until the row of H is reached. Denote the succession of diagrams obtained by $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r$, and the number of nodes in λ_i by n_i.

λ_r satisfies the conditions of 4.8, and $\lambda^*_r = \lambda^*$. Suppose the p-core of λ_r has A_r nodes.

$$\begin{aligned} e(z_\lambda) - e(z_{\lambda_r}) &= e(z_\lambda) - e(z_{\lambda_1}) + e(z_{\lambda_1}) - \ldots + e(z_{\lambda_{r-1}}) - e(z_{\lambda_r}) \\ &= e(n!) - e(n_1!) + e(n_1!) - \ldots + e(n_{r-1}!) - e(n_r!), \quad \text{by 4.3} \\ &= e(n!) - e(n_r!). \end{aligned}$$

Since $\lambda^*_r = \lambda^*$, 4.8 yields

$$e(z_{\lambda_r}) = e(n_r!) - e(\,(n_r - A_r)!) + e(z_{\lambda^*}).$$

But $n - A = pB = n_r - A_r$, since $\lambda^*_r = \lambda^*$, hence

$$e(z_\lambda) = e(n!) - e(n_r!) + e(z_{\lambda_r}) = e(n!) - e(\,(n - A)!) + e(z_{\lambda^*}).$$

We can now prove the main theorem.

A: $$e(z_\lambda) = e(n!) - e(\,(n - A)!) + e(z_{\lambda^*}).$$

We have $$z_\lambda = n! \prod_i \left(\frac{\prod_{s<t} (\delta_s{}^i - \delta_j{}^i)}{\prod_j (\delta_j{}^i)!} \right) . K$$

where Π_i is taken over classes of congruent δ's and $K \not\equiv 0$. Each class $\{\delta^i\}$ is the class of δ's of a diagram, λ_i, with n_i nodes and a p-core of A_i nodes, say.

If λ_i is a p-core, then, by 4.5

$$e\left(\frac{\prod_{s<t} (\delta_s{}^i - \delta_t{}^i)}{\prod_j (\delta_j{}^i)!} \right) = 0.$$

Hence we need only consider $\prod_i$ to range over classes $\{\delta^i\}$ for which λ_i is not a p-core. We now apply Theorem **C**, proven in **3**. This tells us that the (λ^*_i)'s are just the constituents of λ^*, and we may suppose the λ_i's to be numbered correspondingly. We have

$$z_\lambda = n! \prod_i \left(\frac{z_{\lambda i}}{n_i!} \right) . K,$$

$$\begin{aligned} e\left(\frac{z_{\lambda_i}}{n_i!} \right) &= - e(\,(n_i - A_i)!) + e\,(z_{\lambda^*_i}) \qquad \text{by 4.9} \\ &= - e(\,(pB_i)!) + e(z_{\lambda^*_i}). \end{aligned}$$

Also, (4.2), $$z_{\lambda^*} = \frac{B!}{\prod_i B_i!} \prod_i z_{\lambda^*_i}.$$

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Hence
$$e(z_\lambda) = e(n!) - \sum_i e((pB_i)!) + \sum_i e(z_{\lambda *_i})$$
$$e(z_{\lambda *}) = e(B!) - \sum_i e(B_i!) + \sum_i (z_{\lambda *_i}).$$

Hence
$$\begin{aligned} e(z_\lambda) - e(z_{\lambda *}) &= e(n!) - e(B!) + \sum_i e(B_i!) - e((pB_i)!) \\ &= e(n!) - e(B!) - \sum_i B_i \qquad \text{by 4.6} \\ &= e(n!) - e((pB)!) + e((pB)!) - e(B!) - B \\ &= e(n!) - e((n - A)!) \qquad \text{by 4.6.} \end{aligned}$$

5. Star diagrams and the p-series. We have derived Robinson's formula for $e(z_\lambda)$ without making use of the p-series of λ. To make the story complete, however, we should reverse the original procedure carried out by Robinson, and derive the p-series formula for $e(z_\lambda)$ from that which we have just proven. The main task is to find a connection between the star diagram and the p-series.

Let us repeat the operation of forming star diagrams. That is, we form λ^*, then we form the star diagrams of the constituents of λ^*, arrange these in order, forming $(\lambda^*)^*$, and so on. Denote the sequence of diagrams thus formed by $\lambda, \lambda^*, \lambda^{2*}, \ldots, \lambda^{r*}$ and the number of nodes in their p-cores by $A, A_1, A_2, \ldots, A_r$, where λ^{r*} is a p-core. A node removable from λ^{i*} represents a p-hook removable from $\lambda^{(i-1)*}$: continuing back to λ we see that it represents a p^i-hook removable from λ.

Consider the last diagram, λ^{r*}. Each node removable from it represents a p^r-hook of λ, and this must be the longest p^i-hook of λ, for otherwise λ^{r*} would have a p-hook and would not be the last diagram of the sequence. Let us remove a node from λ^{r*}, and the corresponding hooks from $\lambda^{(r-1)*}, \ldots, \lambda^*, \lambda$. The result is the removal of a p^r-hook from λ. When all A_r nodes of λ^{r*} have been removed, we will have removed A_r p^r-hooks from λ, and no more p^r-hooks remain on λ. We will also have removed all nodes from $\lambda^{(r-1)*}$ except its p-core. Removing these one at a time, we remove A_{r-1} p^{r-1}-hooks from what is left of λ. And so on, until all nodes have been removed from λ^*, and only the p-core of λ remains. What we have done is to remove the hooks of the p-series from λ. Hence the p-series of λ is

$$p^r, \ldots (A_r \text{ times}); p^{r-1}, \ldots (A_{r-1} \text{ times}); \ldots \quad ; p, \ldots (A_1 \text{ times})$$

which proves Theorem **D**.

Next we prove Nakayama's formula for $e(z_\lambda)$, namely

5.2
$$e(z_\lambda) = e(n!) - \sum A_i e(p^i!).$$

Let the number of nodes in λ^{i*} be n_i. By removing all the nodes from $\lambda^{(i+1)*}$, and the corresponding p-hooks from λ^{i*} we see that $n_i - A_i = pn_{i+1}$.

5.21
$$n = A_1 + pA_2 + p^2A_3 + \ldots + p^{r-1}A_r.$$

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Proof:

$$\begin{aligned} n_1 - A_1 &= pn_2 \\ &= p(A_2 + pn_3) \\ &= pA_2 + p^2(A_3 + pn_4) \\ &= pA_2 + p^2A_3 + p^3A_4 + \ldots + p^{r-1}A_r \end{aligned}$$

by a simple induction.

The proof of 5.2 is based on an induction over n. Since each B_i is less than n, we may assume the theorem for each of the constituents of λ^*. The p-series for λ^* is just the sum of the p-series of its constituents, and hence we may assume the theorem true for λ^*. The p-series of λ^* is

$$p, \ldots (A_2 \text{ times}); \; p^2, \ldots (A_3 \text{ times}); \ldots ; \; p^r \ldots (A_{r+1} \text{ times}).$$

Hence induction over n implies that

$$e(z_{\lambda^*}) = e(n_1!) - [A_2e(p!) + A_3e(p^2!) + \ldots + A_re(p^{r-1}!)].$$

By Theorem **A**, $e(z_\lambda) = e(n!) - e\,((n - A)!) + e(z_{\lambda^*})$

$$= e(n!) - e\,((pn_1)!) + e(z_{\lambda^*}).$$

Using the above expression for $e(z_{\lambda^*})$, we have

$$\begin{aligned} e(z_\lambda) &= e(n!) - e((pn_1)!) + e(n_1!) - [A_2e(p!) + A_3e(p^2!) + \ldots + A_re(p^{r-1}!)] \\ &= e(n!) - n_1 - [A_2e(p!) + \ldots + A_re(p^{r-1}!)] \\ &= e(n!) - [A_1 + pA_2 + \ldots + p^{r-1}A_r] \qquad \text{by 5.21} \\ &\qquad - [A_2e(p!) + \ldots + A_re(p^{r-1}!)] \\ &= e(n!) - [A_1 + A_2(e(p^2!) - e(p!) + \ldots] \\ &\qquad - [A_2e(p!) + \ldots + A_r\,e(p^{r-1}!)\,] \\ &= e(n!) - [A_1e(p!) + A_2e(p^2!) + \ldots + A_re(p^r!)]. \end{aligned}$$

The theorem is easily established for $n = 1$, or other small values, which is sufficient to start the induction.

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