SMALL POSITIVE VALUES OF INDEFINITE QUADRATIC FORMS

J. H. H. Chalk

(received December 15, 1962)

1. Introduction. Let Λ denote the lattice of points $X = (x_1, \ldots, x_n)$ with integral coordinates. A basis of Λ is a set of n points $X_1, \ldots, X_n$ of Λ such that every point of Λ is expressible in the form $\sum_{i=1}^{n} u_i X_i$, where u_i are integers. It is easy to see that points $X_1, \ldots, X_n$ of Λ form a basis if, and only if

$$\det(X_1, \ldots, X_n) = |x_s^{(r)}| = \pm 1 \quad (r, s = 1, 2, \ldots, n),$$

where $X_r = (x_1^{(r)}, \ldots, x_n^{(r)})$. Let $Q(X) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ be any indefinite quadratic form in the integer variables $x_1, \ldots, x_n$ with real coefficients a_{ij} of determinant $d = d(Q) = |a_{ij}| \neq 0$ $(i, j = 1, 2, \ldots, n)$. It is known that there is a constant $k_n > 0$, depending only on n, such that to each $Q(X)$ there corresponds a basis satisfying $|Q(X_r)| \leqq k_n |d|^{1/n}$, $(r = 1, 2, \ldots, n)$; see G. L. Watson [4]. Recently*, I showed that for a suitably large constant $k_n' > 0$, there is a basis satisfying $0 < Q(X_r) \leqq k_n' |d|^{1/n}$ $(r = 1, 2, \ldots, n)$.

* See [1], Lemma 1 for a proof. An equivalent formulation is stated in Lemma 2 (§ 2).

Canad. Math. Bull. vol. 6, no. 3, September 1963

https://doi.org/10.4153/CMB-1963-027-7 Published online by Cambridge University Press

Consider now the case when the form $Q(X)$ represents arbitrarily small non-zero values for integral $X \neq 0$. It has been conjectured that every indefinite form $Q(X)$ in $n \geqq 5$ variables with incommensurable coefficients a_{ij} satisfies this; so far [2], we know it to be true, provided that $n \geqq 21$. In any event, for forms $Q(X)$ in at least 3 variables which represent arbitrarily small non-zero values, it is easy to deduce from the existence of k'_n that, to every $\varepsilon > 0$, there corresponds a basis $X_1, \ldots, X_n$ satisfying

$$0 \neq |Q(X_r)| < \varepsilon \qquad (r = 1, 2, \ldots, n).$$

The proof** would, in addition, give $Q(X_r) > 0$ except in the one case when the signature $s(Q) = -(n-2)$. The purpose of this note is to present a modification of the argument to secure $0 < Q(X_r) < \varepsilon$ $(r = 1, 2, \ldots, n)$ in all cases. To avoid a succession of constants in our inequalities it is convenient to use the Vinogradov symbol $\ll$, to indicate some implied constant, depending only on n.

I acknowledge gratefully the useful comments of the Referee.

2. Two Lemmas.

LEMMA 1. *For any real* α *and* $a > 0$, $b > 0$, *there is an integer* x *such that*

$$0 < a(x+\alpha)^2 - b \leqq 2(ab)^{1/2} + a \tag{1}$$

Proof. Take x to be the integer for which

$$(b/a)^{1/2} < x + \alpha \leqq (b/a)^{1/2} + 1.$$

** See [1], Theorem 1.

LEMMA 2. *For* $n \geqq 2$, *let* $Q(x_1, \ldots, x_n)$ *be an indefinite quadratic form of determinant* $d \neq 0$. *Then* Q *is equivalent, by an integral unimodular substitution on the variables* $x_1, \ldots, x_n$, *to a form whose coefficients* a_{ij} *satisfy*

$$a_{11} > 0, \ldots, a_{nn} > 0 \tag{2}$$

$$a_{ii} \ll |d|^{1/n} \quad (i=1, 2, \ldots, n). \tag{3}$$

Proof. See [1], Lemma 1.

3. THEOREM. $(n \geqq 3)$ *Let* X_1 *be any primitive point of* Λ *with* $Q(X_1) > 0$ *and put* $\theta = \theta(X_1) = Q(X_1)|d|^{-1/n}$. *Then there is a basis* $X_1, \ldots, X_n$ *of* Λ *satisfying*

$$0 < Q(X_i) \ll \begin{cases} \theta^{\nu_n} |d|^{1/n}, & \text{if } \theta < 1 \\ \theta |d|^{1/n}, & \text{if } \theta \geqq 1, \end{cases} \tag{4}$$

where $\nu_n = (1 - n.2^{-n+1})(n-1)^{-2} > 0$.

Proof. Since X_1 is primitive, we may, after a suitable integral unimodular substitution applied to $x_1, \ldots, x_n$, suppose that $X_1 = (1, 0, \ldots, 0)$, whence

$$0 < a_{11} = Q(X_1). \tag{5}$$

Let $Q^*(Y) = \sum_{i,j=1}^{n} A_{ij} y_i y_j$ denote the form, of determinant d^{n-1}, adjoint to $Q(X)$, and consider its section $Q^*(0, y_2, \ldots, y_n)$. This is a quadratic form in $n-1$ variables and has determinant $a_{11} d^{n-2} \neq 0$. Now *any* real non-singular quadratic form f in s variables represents a non-zero value $\ll |d(f)|^{1/s}$, by

https://doi.org/10.4153/CMB-1963-027-7 Published online by Cambridge University Press

classical inequalities (see e.g. H. Blaney, J. London Math. Soc., 23 (1948), 153-160 for the case of indefinite forms, and J. F. Koksma, Diophantische Approx., Kap II, § 6 for the definite case). Thus, in particular, there are relatively prime integers $y_2', \ldots, y_n'$ such that

$$0 \neq |Q^*(0, y_2', \ldots, y_n')| \ll |a_{11} d^{n-2}|^{1/(n-1)} . \tag{6}$$

Applying an appropriate integral unimodular substitution to the variables $y_2, \ldots, y_n$, we can suppose, without loss of generality, that $(y_2', \ldots, y_n') = (0, \ldots, 0, 1)$; whence

$$0 \neq |A_{nn}| \ll |a_{11} d^{n-2}|^{1/(n-1)} . \tag{7}$$

In order to preserve the reciprocal relation between Q and Q^*, we also apply the contravariant substitution to $x_2, \ldots, x_n$, which is integral and unimodular, and, moreover, leaves the coefficient of x_1^2 in $Q(X)$ invariant. Thus we preserve the relation (5). By completing the square on x_1 in $Q(X)$, we may write

$$Q(X) = a_{11} (x_1 + \ell_1)^2 + q(x_2, \ldots, x_n) , \tag{8}$$

where ℓ_1 is a linear form in $x_i (i \geqq 2)$ and $q(x_2, \ldots, x_n)$ is a quadratic form of determinant $d/a_{11} \neq 0$. We now consider two cases according as $q(x_2, \ldots, x_n)$ is indefinite or otherwise. (In the latter case, it will be observed that q, being non-singular, is negative definite, since $Q(X)$ is indefinite, by hypothesis.) We proceed by induction on n, assuming the theorem to hold for indefinite forms in $n-1$ variables if $n \geqq 4$.

Case 1. Suppose that $q(x_2, \ldots, x_n)$ is indefinite. Then Lemma 2 may be applied directly to $-q(x_2, \ldots, x_n)$. Hence

https://doi.org/10.4153/CMB-1963-027-7 Published online by Cambridge University Press

there are integer sets $(x_2^{(r)}, \ldots, x_n^{(r)})$, $r = 2, \ldots, n$ with $|x_s^{(r)}| = \pm 1$ $(r, s = 2, \ldots, n)$, satisfying

$$0 < -q(x_2^{(r)}, \ldots, x_n^{(r)}) = b_r \ll |d/a_{11}|^{1/(n-1)}. \qquad (9)$$

For these values,

$$Q(x_1, x_2^{(r)}, \ldots, x_n^{(r)}) = a_{11}(x_1 + \ell_1^{(r)})^2 - b_r, \text{ say,} \qquad (10)$$

where
$$0 < a_{11} b_r \ll a_{11} |d/a_{11}|^{1/(n-1)}$$

$$\ll a_{11}^{\frac{n-2}{n-1}} |d|^{1/(n-1)}. \qquad (11)$$

For each such r, we use Lemma 1 to select $x_1 = x_1^{(r)}$ say, giving

$$0 < Q(x_1^{(r)}, \ldots, x_n^{(r)}) \ll a_{11} + (a_{11} b_r)^{1/2}$$

$$\ll a_{11} + a_{11}^{\frac{1}{2}(\frac{n-2}{n-1})} |d|^{1/2(n-1)};$$

thus $X_1 = (1, 0, \ldots, 0)$, $X_r = (x_1^{(r)}, \ldots, x_n^{(r)})$, $r = 2, \ldots, n$ form a basis of Λ satisfying

$$0 < Q(X_r) \ll \theta |d|^{1/n} + \theta^{\frac{1}{2}(\frac{n-2}{n-1})} |d|^{1/n},$$

$$\ll \begin{cases} \theta^{\frac{1}{2}(\frac{n-2}{n-1})} |d|^{1/n} & \text{if } \theta < 1, \\ \theta |d|^{1/n} & \text{if } \theta \geqq 1. \end{cases} \qquad (12)$$

https://doi.org/10.4153/CMB-1963-027-7 Published online by Cambridge University Press

Obviously $\nu_n < \frac{1}{2}(\frac{n-2}{n-1})$ for $n \geqq 3$ and so (4) is established in this case.

Case 2. Suppose that $q(x_2, \ldots, x_n)$ is negative definite. Observe that for $n = 3$,

$$Q(x_1, x_2, 0) = a_{11}(x_1 + \frac{a_{12}}{a_{11}} x_2)^2 + \frac{A_{33}}{a_{11}} x_2^2 , \tag{13}$$

where $A_{33} < 0$, by the hypothesis of this case. By Lemma 1, we can select an integer $x_1^{(2)}$ such that

$$0 < Q(x_1^{(2)}, 1, 0) \ll a_{11} + \left(a_{11} \frac{|A_{33}|}{a_{11}} \right)^{1/2}$$

$$\ll a_{11} + (a_{11} |d|)^{1/4} , \tag{14}$$

by (7). Hence if $x_1^{(1)} = 1$, $x_2^{(1)} = 0$, $x_2^{(2)} = 1$, and $x_1^{(2)}$ is as chosen above, we have

$$0 < Q(x_1^{(r)}, x_2^{(r)}, 0) \ll a_{11} + (a_{11} |d|)^{1/4}$$

$$\ll \theta |d|^{1/3} + \theta^{1/4} |d|^{1/3},$$

$$(r = 1, 2) \tag{15}$$

and $|x_s^{(r)}| = 1$. Now, more generally for $n \geqq 4$, we apply our inductive hypothesis to $Q(x_1, \ldots, x_{n-1}, 0)$ which has determinant $A_{nn} \neq 0$ and clearly is indefinite. Since $Q(1, 0, \ldots, 0) = a_{11}$, we may assume that there are $n-1$ integer sets $X_r = (x_1^{(r)}, \ldots, x_{n-1}^{(r)}, 0)$, $(r = 1, 2, \ldots, n-1)$ with $|x_s^{(r)}| = \pm 1$

https://doi.org/10.4153/CMB-1963-027-7 Published online by Cambridge University Press

and $(x_1^{(1)}, \ldots, x_{n-1}^{(1)}, 0) = (1, 0, \ldots, 0)$ satisfying

$$0 < Q(X_r) \ll \begin{cases} \theta' |A_{nn}|^{1/(n-1)} & \text{if } \theta' > 1 \\ \theta'^{\nu_{n-1}} |A_{nn}|^{1/(n-1)} & \text{if } \theta' \leqq 1 , \end{cases} \tag{16}$$

where $\theta' = a_{11} |A_{nn}|^{-1/(n-1)}$. Now

$$\theta' |A_{nn}|^{1/(n-1)} = \theta |d|^{1/n}$$

and

$$\theta'^{\nu_{n-1}} |A_{nn}|^{1/(n-1)} = a_{11}^{\nu_{n-1}} |A_{nn}|^{(1-\nu_{n-1})(n-1)^{-1}}$$

$$\ll a_{11}^{\lambda_{n-1}} |d|^{(1-\lambda_{n-1})n^{-1}}$$

$$= \theta^{\lambda_{n-1}} |d|^{1/n} , \tag{17}$$

where $\lambda_{n-1} = (n-1)^{-2} + (1 - (n-1)^{-2})\nu_{n-1} > \nu_n$ for $n \geqq 4$. Combining these inequalities, we see that if $\theta \geqq 1$, then $\theta |d|^{1/n} \geqq \theta^{\lambda_{n-1}} |d|^{1/n}$, since $\lambda_{n-1} < 1$ for $n \geqq 4$, while if $\theta < 1$, we have $\theta |d|^{1/n} < \theta^{\lambda_{n-1}} |d|^{1/n} < \theta^{\nu_n} |d|^{1/n}$. Thus $X_r (r = 1, 2, \ldots, n-1)$ satisfy (4) when $n \geqq 4$; moreover, since $\nu_3 = \frac{1}{16} < \frac{1}{4}$ we see, by (15), that this is true when $n = 3$. Thus, for $n \geqq 3$, in completing our basis, we consider the point $(x_1^{(n)}, \ldots, x_{n-1}^{(n)}, 1)$, where $x_r^{(n)}$ $(r = 1, \ldots, n-1)$ are arbitrary integers at our disposal. By a theorem of Miss Foster [3] on polynomials $Q(x_1, \ldots, x_{n-1}, 1)$ with an

https://doi.org/10.4153/CMB-1963-027-7 Published online by Cambridge University Press

indefinite section $Q(x_1, \ldots, x_{n-1}, 0)$ we can ensure that

$$0 < Q(x_1^{(n)}, \ldots, x_{n-1}^{(n)}, 1) \ll |\Delta_{n-1}|^{1/(n-1)}$$

$$+ |\Delta_n|^{2^{-n+1}} |\Delta_{n-1}|^{(n-1)\nu_n} \tag{18}$$

where

$$\Delta_{n-1} = d(Q(x_1, \ldots, x_{n-1}, 0)) = A_{nn} ,$$

$$\Delta_n = d(Q(x_1, \ldots, x_n)) = d \quad .$$

Applying (7) to the right hand side of (18), we get

$$|A_{nn}|^{1/(n-1)} + |A_{nn}|^{(n-1)\nu_n} |d|^{2^{-n+1}}$$

$$\ll a_{11}^{(n-1)^{-2}} |d|^{(n-2)(n-1)^{-2}} + a_{11}^{\nu_n} |d|^{(n-2)\nu_n + 2^{-n+1}}$$

$$= \theta^{(n-1)^{-2}} |d|^{1/n} + \theta^{\nu_n} |d|^{1/n}$$

$$\ll \begin{cases} \theta^{\nu_n} |d|^{1/n} & \text{if } \theta < 1 \\ \theta\, |d|^{1/n} & \text{if } \theta \geqq 1 . \end{cases}$$

Thus $X_n = (x_1^{(n)}, \ldots, x_{n-1}^{(n)}, 1)$ completes our basis and satisfies (4). The proof is now complete.

In conclusion, it may be noted that the exponent ν_n in (4) could be improved if some better bound on the right of (18) were known. It has been conjectured (see G. L. Watson, Mathematika, 7 (1960), 141-144) that the term

$$|\Delta_n|^{2^{-n+1}} |\Delta_{n-1}|^{(n-1)\nu_n}$$

is superfluous. Indeed, for $n \geqq 3$ and for forms $Q(X)$ which assume arbitrarily small non-zero values for integral $X \neq 0$, he proves that the right of (18) may be replaced by any $\varepsilon > 0$. On the other hand, since $\nu_n > 0$, the result (18) itself is sufficient (for our purpose) to show that there is a basis with $0 < Q(X_i) < \varepsilon$ whenever $n \geqq 3$ and Q represents arbitrarily small non-zero values.

REFERENCES

1. J. H. H. Chalk, Integral Bases for Quadratic Forms, Canadian J. of Mathematics, 15 (1963), 412-421.

2. H. Davenport and D. Ridout, Indefinite Quadratic Forms, Proc. London Math. Soc., (3), 9 (1959), 544-555.

3. D. M. E. Foster, Indefinite Quadratic Polynomials in n variables, Mathematika, 3 (1956), 111-116, Theorem 2.

4. G. L. Watson, Distinct small values of Quadratic Forms, Mathematika, 7 (1960), 36-40, Theorem 1.

University of Toronto

Note added in Proofs.

Since the point X_1 of Λ can be selected to satisfy $0 < Q(X_1) << |d|^{1/n}$, our theorem may be regarded as a stronger form of Lemma 2. Thus, the appeal to Lemma 2 (which occurs only in Case 1) could be avoided for $n \geqq 3$ variables by replacing it by the more powerful inductive hypothesis. Lemma 2, in the case of 2 variables, is classical and several proofs are known. With this modification our proof of the theorem is more self-contained and, incidentally, provides an alternative verification of Lemma 2 for 3 or more variables.