Appendix D

Rank-Level Duality (A Brief Survey) (by Swarnava Mukhopadhyay)

Introduction. Representation theory of $\mathrm{GL}(r)$ and the intersection theory of Grassmannians $\mathrm{Gr}(r, N)$ are deeply connected. In particular the structure constants of the Grothendieck ring of representations of $\mathrm{GL}(r)$ can be read off the structure constants of the cohomology ring of the Grassmannians. Let $\lambda = (\lambda^1 \geq \cdots \geq \lambda^r \geq 0) \in \mathbb{Z}^r$ parameterize rows of a Young diagram and the corresponding representation of $\mathrm{GL}(r)$ will be denoted by V_λ. Let $\mathcal{Y}_{r,s}$ denote the set of Young diagram, with at most r rows and s columns. For any $\lambda \in \mathcal{Y}_{r,s}$, we obtain a new Young diagram $\lambda^t \in \mathcal{Y}_{s,r}$ by interchanging the rows and the columns of λ. We consider $\lambda, \mu, \nu \in \mathcal{Y}_{r,s}$ such that the total number of boxes $|\lambda| + |\mu| + |\nu| = rs$. Then, using the natural duality $\mathrm{Gr}(r, r+s) \simeq \mathrm{Gr}(s, r+s)$, it follows that

$$\dim_{\mathbb{C}}(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{SL}(r)} = \dim_{\mathbb{C}}(V_{\lambda^t} \otimes V_{\mu^t} \otimes V_{\nu^t})^{\mathrm{SL}(s)}.$$

The above ‘strange’ dimension equality is not only numerical but it turns out that the vector spaces are canonically dual to each other (see (Belkale, 2004b) and (Belkale, Gibney and Mukhopadhyay, 2015). It is natural to ask for similar results for groups of other types. However, easy computations with Littlewood–Richardson coefficients show that such equalities do not hold in general. Since conformal blocks are refinements of the spaces of invariants of tensor product representations of semisimple Lie algebras, it is natural to consider conformal blocks as the right objects to study such dualities. They were motivated by direct connections in Goodman and Wenzl (1990); Kuniba and Nakanishi (1991) and Naculich and Schnitzer (1990) between the fusion rules of the Wess–Zumino–Witten models of conformal blocks associated to $\mathfrak{sl}(r)$ at level s and $\mathfrak{sl}(s)$ at level r. In this section, we sketch a general approach to formulate rank-level duality questions and recall known rank-level duality results without proof. We mostly focus on the genus zero case due to its direct connection with conformal blocks dealt in the book and only briefly comment on the

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geometric counterpart of rank-level duality which is known as strange duality. For strange duality questions on surfaces, we refer the reader to Abe (2010, 2015); Marian and Oprea (2009, 2013, 2014) and the references cited there.

D.1 Conformal Embeddings

We will use the notion of conformal embeddings of Lie algebras to formulate rank-level duality questions as a natural map between conformal blocks associated to embedding of Lie algebras and their associated affine branching rules. We refer the reader to Section A.1 of this book for a definition of the Dynkin index of an embedding of simple Lie algebras.

Definition D.1.1 If $(\varphi_1, \varphi_2)\colon \mathfrak{s}_1 \oplus \mathfrak{s}_2 \to \mathfrak{s}$ is an embedding of a semisimple Lie algebra into a simple Lie algebra, we define the Dynkin multi-index to be $(d_{\varphi_1}, d_{\varphi_2})$, where d_{φ_i} is the Dynkin index of the embedding $\mathfrak{s}_i \to \mathfrak{s}$.

Example D.1.2 *Consider the natural embedding of $\varphi\colon \mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \to \mathfrak{sl}(rs)$ given by the tensor product of vector spaces and linear operators on them. The normalized Cartan Killing form on $\mathfrak{sl}(r)$ is given by*

$$(X, Y)_{\mathfrak{sl}(r)} = \mathrm{Trace}(X.Y),$$

where X, Y are $(r \times r)$-matrices with zero trace. The image of X under φ is $rs \times rs$ matrix given by s-diagonal copies of the matrix X. Hence it follows that the Dynkin multi-index of the embedding is (s, r).

Conformal Embeddings and their Classifications

Consider an embedding of Lie algebras $\varphi\colon \mathfrak{s}_1 \oplus \mathfrak{s}_2 \to \mathfrak{s}$ as before and extend it to a map $\widehat{\varphi}\colon \widehat{\mathfrak{s}}_1 \oplus \widehat{\mathfrak{s}}_2 \to \widehat{\mathfrak{s}}$ of affine Lie algebras as follows:

$$\widehat{\varphi}(X \otimes f) = \varphi(X) \otimes f,$$
$$\widehat{\varphi}(c_1) := d_{\varphi_1}.c \ \text{ and } \ \widehat{\varphi}(c_2) = d_{\varphi_2}.c.$$

Let ℓ be a non-negative integer and given a weight $\lambda \in D_\ell(\mathfrak{s})$, consider the highest-weight, integrable irreducible $\widehat{\mathfrak{s}}$-module $\mathscr{H}_\lambda(\mathfrak{s}, \ell)$ of highest weight λ. The module $\mathscr{H}_\lambda(\mathfrak{s}, \ell)$ gets $\widehat{\mathfrak{s}}_1 \oplus \widehat{\mathfrak{s}}_2$-module structure via the map $\widehat{\varphi}$. Since $\mathscr{H}_\lambda(\mathfrak{s}, \ell)$ is integrable as a $\widehat{\mathfrak{s}}$-module, it follows that $\mathscr{H}_\lambda(\mathfrak{s}, \ell)$ is also integrable as a $\widehat{\mathfrak{s}}_1 \oplus \widehat{\mathfrak{s}}_2$-module at level $(d_{\varphi_1}.\ell, d_{\varphi_2}.\ell)$. By complete reducibility of integrable $\widehat{\mathfrak{s}}_1 \oplus \widehat{\mathfrak{s}}_2$-modules, we get that $\mathscr{H}_\lambda(\mathfrak{s}, \ell)$ will decompose as a direct sum of integrable $\widehat{\mathfrak{s}}_1 \oplus \widehat{\mathfrak{s}}_2$-modules at level $(d_{\varphi_1}.\ell, d_{\varphi_2}.\ell)$. However since $\mathscr{H}_\lambda(\mathfrak{s}, \ell)$

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are infinite dimensional, the number of components in the decomposition may not be finite in general. This motivates us to consider only a special class of embeddings known as *conformal embeddings*.

Remark D.1.3 The notation $\mathscr{H}(\lambda_\ell)$ was used in Chapter 1 to denote the irreducible integrable representation of highest weight λ at level ℓ. However to stress the dependence on the Lie algebra and the level, we use the notation $\mathscr{H}_\lambda(\mathfrak{s}, \ell)$ in this section.

Definition D.1.4 An embedding $\varphi\colon \mathfrak{s}_1 \oplus \mathfrak{s}_2 \to \mathfrak{s}$ is conformal at level k if the following holds:

$$\frac{d_{\varphi_1}k.\dim \mathfrak{s}_1}{d_{\varphi_1}k + h^\vee(\mathfrak{s}_1)} + \frac{d_{\varphi_2}k.\dim \mathfrak{s}_2}{d_{\varphi_2}k + h^\vee(\mathfrak{s}_2)} = \frac{k.\dim \mathfrak{s}}{k + h^\vee(\mathfrak{s})}, \tag{1}$$

where $h^\vee(\mathfrak{g})$ is the dual Coxeter number of a simple Lie algebra $\mathfrak{g}$ and d_{φ_i} is the Dynkin-index of the embedding $\mathfrak{s}_i \to \mathfrak{s}$.

It was pointed out in Kac (1990) that (1) is satisfied only when $k = 1$. Conformal embeddings have been classified independently by Bais and Bouwknegt (1987) and Schellekens and Warner (1986). We give some examples below:

- $\mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \to \mathfrak{sl}(rs)$ with Dynkin multi-index (s, r).
- $\mathfrak{sp}(2r) \oplus \mathfrak{sp}(2s) \to \mathfrak{so}(4rs)$ with Dynkin multi-index (s, r).
- $\mathfrak{so}(r) \oplus \mathfrak{so}(s) \to \mathfrak{so}(rs)$ with Dynkin multi-index (s, r), with $r, s \geq 5$.
- $\mathfrak{g}_2 \oplus \mathfrak{f}_4 \to \mathfrak{e}_8$ with Dynkin multi-index $(1, 1)$.
- $\mathfrak{so}(r) \to \mathfrak{sl}(r)$ with Dynkin index 2 for $r \geq 4$.

We now list two important properties that make conformal embeddings special. We refer the reader to Kac (1990) for more details:

(i) An embedding $\varphi\colon \mathfrak{s}_1 \oplus \mathfrak{s}_2 \to \mathfrak{s}$ is conformal if and only if any irreducible integrable $\widehat{\mathfrak{s}}$-module $\mathscr{H}_\Lambda(\mathfrak{s}, 1)$ of level one decomposes into a finite direct sum of $\mathfrak{s}_1 \oplus \mathfrak{s}_2$ modules of level $(d_{\varphi_1}, d_{\varphi_2})$.

(ii) If $\varphi\colon \mathfrak{s}_1 \oplus \mathfrak{s}_2 \to \mathfrak{s}$ is a conformal embedding, then the action of the Virasoro operators are the same, i.e. for any integer N, the following equality holds as operators on $\mathscr{H}_\Lambda(\mathfrak{s}, 1)$:

$$L_N^{\mathfrak{s}_1} + L_N^{\mathfrak{s}_2} = L_N^{\mathfrak{s}} \in \mathrm{End}(\mathscr{H}_\Lambda(\mathfrak{s}, 1)),$$

where we refer the reader to Section 3.2 for a definition of Virasoro operators.

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Branching Rules of Conformal Embeddings

The theory of conformal embeddings has found very interesting applications in theoretical physics. Given a level-one highest-weight irreducible, integrable $\widehat{\mathfrak{s}}$-module $\mathscr{H}_\Lambda(\mathfrak{s},1)$, it is interesting to find the finite list of representations of $\widehat{\mathfrak{s}}_1 \oplus \widehat{\mathfrak{s}}_2$-representations that appear in the decomposition of $\mathscr{H}_\Lambda(\mathfrak{s},1)$ along with their multiplicities. In Kac and Peterson (1981), 'string functions' were introduced to study branching rules of conformal embeddings. Branching rules for conformal embeddings were derived by studying asymptotics of these string functions. We recall the following results on the branching rules for some conformal embeddings and we refer the reader to Altschuler, Bauer and Itzykson (1990); Cellini et al. (2006); Hasegawa (1989); Kac and Sanielevici (1988); Kac and Wakimoto (1988); Levstein and Liberati (1995) for more details.

We consider the conformal embedding $\mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \rightarrow \mathfrak{sl}(rs)$. The level-one weights of $D_1(\mathfrak{sl}(rs)) = \{\omega_0, \ldots, \omega_{rs-1}\}$. The level-$s$ weights of $\mathfrak{sl}(r)$ are parameterized by the set $\mathscr{Y}_{r-1,s}$. If $\lambda \in \mathscr{Y}_{r-1,s}$, we define the reduced transpose λ^T to be a Young diagram in $\mathscr{Y}_{s-1,r}$ obtained by taking the usual transpose and deleting any column of length s.

Theorem D.1.5 *The module $\mathscr{H}_\lambda(\mathfrak{sl}(r),s) \otimes \mathscr{H}_{\lambda^T}(\mathfrak{sl}(s),r)$ appears with multiplicity one in the branching of $\mathscr{H}_{\omega_{|\lambda|}}(\mathfrak{sl}(rs),1)$, where $|\lambda|$ denote the number of boxes in the Young diagram of λ. All the of the other components are obtained by the permutations of the weights under the action of the automorphisms of the affine Dynkin diagrams.*

We refer the reader to Altschuler, Bauer and Itzykson (1990); Hasegawa (1989) for complete details of the branching rules for this conformal embedding.

Next we consider the embedding $\mathfrak{sp}(2r) \oplus \mathfrak{sp}(2s) \rightarrow \mathfrak{so}(4rs)$. The level-one weights of $\mathfrak{so}(4rs)$ are $\{\omega_0, \omega_1, \omega_+, \omega_-\}$, where $\omega_\pm$ are the spin representations. The level-s representations of the Lie algebra $\mathfrak{sp}(2r)$ are parameterized by the set $\mathscr{Y}_{r,s}$. The branching rules for this conformal embedding can be found in Hasegawa (1989). If Y is in $\mathscr{Y}_{r,s}$, then Y^* denote the Young diagram in $\mathscr{Y}_{s,r}$ obtained by first exchanging the rows and the columns and then taking the complement in a rectangle of size $(s \times r)$.

Theorem D.1.6 *The modules $\mathscr{H}_Y(\mathfrak{sp}(2r),s) \otimes \mathscr{H}_{Y^*}(\mathfrak{sp}(2s),r)$ appears in the branching rules of $\mathscr{H}_{\omega_+}(\mathfrak{so}(4rs),1)$ (respectively $\mathscr{H}_{\omega_-}(\mathfrak{so}(4rs),1)$) if and only if the number of boxes of Y is even (respectively odd). If (Y, Y^*) appears in the branching of ω_+ or ω_-, then the multiplicity is always one. Moreover this list if complete.*

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Next we consider the case $\mathrm{G}_2 \times \mathrm{F}_4 \to \mathrm{E}_8$. The only level-one representation of $\mathfrak{e}_8$ is ω_0. The level-one representations of $\mathfrak{g}_2$ and $\mathfrak{f}_4$ are $\{\omega_0, \omega_1\}$ and $\{\omega_0, \omega_4\}$, respectively.

Theorem D.1.7 *The module (λ, μ) appears in the branching rule of $\mathscr{H}_{\omega_0}(\mathfrak{e}_8, 1)$ if and only if $(\lambda, \mu) = (\omega_0, \omega_0)$ or (ω_1, ω_4). Moreover if (λ, μ) appears, they always appear with multiplicity one.*

Next we consider the example $\mathfrak{so}(r) \to \mathfrak{sl}(r)$. Let $r \geq 5$, if r is odd and $r \geq 8$, if r is even. With the above assumptions, the following result is due to Kac and Wakimoto (1988, p. 213).

Theorem D.1.8 *The module $\mathscr{H}_{\omega_i}(\mathfrak{sl}(r), 1)$ restricted to $\widehat{\mathfrak{so}}(r)$ decomposes as follows:*

(i) *If $i = 0$, then $\mathscr{H}_{\omega_0}(\mathfrak{sl}(r), 1) \simeq \mathscr{H}_{\omega_0}(\mathfrak{so}(r), 2) \oplus \mathscr{H}_{2\omega_1}(\mathfrak{so}(r), 2)$.*
(ii) *If $1 \leq i \leq \lfloor r/2 \rfloor - 2$, then $\mathscr{H}_{\omega_i}(\mathfrak{sl}(r), 1) \simeq \mathscr{H}_{\omega_i}(\mathfrak{so}(r), 2)$.*
(iii) *If $r = 2m + 1$, then*

i $\mathscr{H}_{\omega_{m-1}}(\mathfrak{sl}(2m+1), 1) \simeq \mathscr{H}_{\omega_{m-1}}(\mathfrak{so}(2m+1), 2)$,
ii $\mathscr{H}_{\omega_m}(\mathfrak{sl}(2m+1), 1) \simeq \mathscr{H}_{2\omega_m}(\mathfrak{so}(2m+1), 2)$.

(iv) *If $r = 2m$, then*

i $\mathscr{H}_{\omega_{m-1}}(\mathfrak{sl}(2m), 1) \simeq \mathscr{H}_{(\omega_{m-1}+\omega_m)}(\mathfrak{so}(2m), 2)$,
ii $\mathscr{H}_{\omega_m}(\mathfrak{sl}(2m), 1) \simeq \mathscr{H}_{2\omega_{m-1}}(\mathfrak{so}(2m), 2) \oplus \mathscr{H}_{2\omega_m}(\mathfrak{so}(2m), 2)$.

D.2 Rank-Level Duality: General Formulation

Let $\varphi\colon \mathfrak{s}_1 \oplus \mathfrak{s}_2 \to \mathfrak{s}$ be a conformal embedding with Dynkin index (ℓ_1, ℓ_2). For any level-one weight Λ of $\widehat{\mathfrak{s}}$, we denote by I_Λ the set of highest weights of $\widehat{\mathfrak{s}}_1 \oplus \widehat{\mathfrak{s}}_2$ that appear in the decomposition of $\mathscr{H}_\Lambda(\mathfrak{s}, 1)$ as $\widehat{\mathfrak{s}}_1 \oplus \widehat{\mathfrak{s}}_2$-modules via φ. Given an n tuple $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ of level-one weights of $\widehat{\mathfrak{s}}$, we consider two n-tuples $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ of level ℓ_1 (respectively ℓ_2) of $\widehat{\mathfrak{s}}_1$ (respectively $\widehat{\mathfrak{s}}_2$) such that the following holds:

- For each $1 \leq i \leq n$, we have $(\lambda_i, \mu_i) \in I_{\Lambda_i}$.
- The multiplicity of $\mathscr{H}_{\lambda_i}(\mathfrak{s}_1, \ell_1) \otimes \mathscr{H}_{\mu_i}(\mathfrak{s}_2, \ell_2)$ is one.

Taking tensor product over n chosen factors, we get a map

$$\widetilde{\varphi}\colon \bigotimes_{i=1}^{n} \Big(\mathscr{H}_{\lambda_i}(\mathfrak{s}_1, \ell_1) \otimes \mathscr{H}_{\mu_i}(\mathfrak{s}_2, \ell_2) \Big) \to \bigotimes_{i=1}^{n} \mathscr{H}_{\Lambda_i}(\mathfrak{s}, 1). \tag{1}$$

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Let $(\Sigma, \vec{p}, \vec{z})$ be a point of the Deligne–Grothendieck–Knudsen–Mumford moduli stack $\widehat{\overline{\mathcal{M}}}_{g,n}$ of pointed stable curves with n marked points and a choice of formal coordinates at the marked points. The map $\widetilde{\varphi}$ is equivariant with respect to the action of $(\mathfrak{s}_1 \oplus \mathfrak{s}_2) \otimes H^0(\Sigma, \mathcal{O}_\Sigma(*\vec{p}))$ on the left and $\mathfrak{s} \otimes H^0(\Sigma, \mathcal{O}_\Sigma(*\vec{p}))$ action on the right. Taking coinvariants with respect to these, we get a map of the conformal blocks:

$$\widetilde{\varphi}\colon \mathcal{V}_\Sigma(\mathfrak{s}_1, \vec{\lambda}, \ell_1) \otimes \mathcal{V}_\Sigma(\mathfrak{s}_2, \vec{\mu}, \ell_2) \to \mathcal{V}_\Sigma(\mathfrak{s}, \vec{\Lambda}, 1). \tag{2}$$

The above map can be defined as a map of locally free sheaves of covacuas on $\widehat{\overline{\mathcal{M}}}_{g,n}$.

Question D.2.1 One can ask the following natural questions, all of which are broadly known as *rank-level duality questions*:

(i) If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are both nontrivial and simple and $\dim_{\mathbb{C}} \mathcal{V}_\Sigma(\mathfrak{s}, \vec{\Lambda}, 1)$, then is $\widetilde{\varphi}$ a perfect pairing?
(ii) Is there a natural section in $\mathcal{V}_\Sigma(\mathfrak{s}, \vec{\Lambda}, 1)$ that induces a duality between $\mathcal{V}_\Sigma(, \mathfrak{s}_1, \vec{\lambda}, \ell_1)$ and $\mathcal{V}_\Sigma^\dagger(\mathfrak{s}_2, \vec{\mu}, \ell_2)$?
(iii) If $\mathfrak{s}_2$ is trivial, is $\widetilde{\varphi}$ surjective?

The *propagation of vacua* (Section 2.2) identifies $\mathcal{V}_\Sigma(\mathfrak{g}, \vec{\lambda}, \ell)$ associated to any simple Lie algebra $\mathfrak{g}$ with weights $\vec{\lambda}$ at level ℓ with the sheaf of covacua $\mathcal{V}_\Sigma(\mathfrak{g}, \vec{\lambda}, \omega_0, \ell)$. There is a natural commutative diagram

$$\begin{tikzcd}
\mathcal{V}_\Sigma(\mathfrak{s}_1, \vec{\lambda}, \ell_1) \otimes \mathcal{V}_\Sigma(\mathfrak{s}_2, \vec{\mu}, \ell_2) \arrow[r] \arrow[d] & \mathcal{V}_\Sigma(\mathfrak{s}, \vec{\Lambda}, 1) \arrow[d] \\
\mathcal{V}_\Sigma(\mathfrak{s}_1, (\vec{\lambda}, \omega_0), \ell_1) \otimes \mathcal{V}_\Sigma(\mathfrak{s}_2, (\vec{\mu}, \omega_0), \ell_2) \arrow[r] & \mathcal{V}_\Sigma(\mathfrak{s}, (\vec{\Lambda}, \omega_0), 1).
\end{tikzcd} \tag{1}$$

Here the horizontal maps are given by branching rules of conformal embeddings and the vertical maps are isomorphisms given by *propagation of vacua*. Using Diagram (1), new rank results on rank-level duality results are obtained.

Remark D.2.2 The rank-level duality map of sheaves of covacua attached to a family of pointed curves with formal coordinates does not descend to a map of locally free sheaves of covacua on $\overline{\mathcal{M}}_{g,n}$. However, using the fact that the embedding is conformal, it was shown in Mukhopadhyay (2016c) that up to some correction factors involving the difference of trace anomaly and Psi-classes, the map $\widetilde{\varphi}$ descends to a map of locally free sheaves on $\overline{\mathcal{M}}_{g,n}$.

Remark D.2.3 In all the known examples, the rank-level duality map fails to be an isomorphism/perfect pairing if Σ is nodal. The behavior of the rank-level duality along the boundary of $\overline{\mathcal{M}}_{g,n}$ has important implications regarding

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the first Chern class of conformal blocks bundles and their positivity on $\overline{\mathcal{M}}_{0,n}$. We refer the reader to Mukhopadhyay (2016c) for further details.

Applications of Verlinde Formula in Rank-Level Duality

The Verlinde formula has many applications to questions on rank-level duality. The dimensions of level-one conformal blocks have been computed in the works of Fakhruddin (2012); Mukhopadhyay (2016b); Nakanishi and Tsuchiya (1992) using the Verlinde formula formalisms for three points and factorization rules.

Theorem D.2.4 *The following dimensions formula for level-one conformal blocks hold:*

(i) *If $\mathfrak{s} = \mathfrak{sl}(r)$, let $\vec{\Lambda} = (\omega_{i_1}, \dots, \omega_{i_n})$ and Σ be a stable curve of genus g, then the dimension of $\mathcal{V}_\Sigma(\mathfrak{sl}(r), \vec{\Lambda}, 1)$ is r^g if r divides $i_1 + \cdots + i_n$ and is zero otherwise.*

(ii) *If $\mathfrak{s} = \mathfrak{so}(2r+1)$, the level-one weights are ω_0, ω_1 and ω_r. Let $\vec{\Lambda} = (\omega_1, \dots, \omega_1)$, then the dimension of $\mathcal{V}_{\mathbb{P}^1}(\mathfrak{so}(2r+1), \vec{\Lambda}, 1)$ is 1-dimensional if n is even and is zero otherwise. Let $n = m_1 + m_2$ and $\vec{\Lambda}$ consists of m_1-copies of ω_1 and m_2-copies of ω_r. Then the dimension of the space of covacua $\mathcal{V}_{\mathbb{P}^1}(\mathfrak{sl}(2m+1), \vec{\Lambda}, 1)$ is $2^{\frac{m_2}{2}-1}$ if m_2 is even and is zero otherwise.*

(iii) *If $\mathfrak{s} = \mathfrak{so}(2r)$, let $\vec{\Lambda} = (\omega_{i_1}, \dots, \omega_{i_n})$ and $Sigma$ be a stable curve of genus g, then the dimension of $\mathcal{V}_{\mathbb{P}^1}(\mathfrak{so}(2r), \vec{\Lambda}, 1)$ is one if $\omega_{i_1} + \cdots + \omega_{i_n}$ is in the root lattice and is zero otherwise. If $\vec{\Lambda} = (\omega_0, \dots, \omega_0)$, then the dimension of the space of covacua $\mathcal{V}_\Sigma(\mathfrak{so}(2r), \vec{\Lambda}, 1)$ is 4^g, where g is the genus of Σ.*

(iv) *Let $\mathfrak{s}$ be of type G_2 or F_4, and $\vec{\Lambda} = (\omega_1, \dots, \omega_1)$ for $\mathfrak{s} = \mathfrak{g}_2$ or $\vec{\Lambda} = (\omega_4, \dots, \omega_4)$ for $\mathfrak{s} = \mathfrak{f}_4$. Then the dimension of $\mathcal{V}_\Sigma(\mathfrak{g}, \vec{\Lambda}, 1)$ is*

$$\left(\frac{1+\sqrt{5}}{2}\right)^n \left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{1-\sqrt{5}}{2}\right)^n \left(\frac{5-\sqrt{5}}{2}\right)^{g-1}. \quad (1)$$

Here Σ is any n-pointed stable curve of genus g.

(v) *If $\mathfrak{s} = \mathfrak{e}_6$, the level-one weights are ω_0, ω_1 and ω_6. The representation ω_6 is dual to ω_1. Let $\vec{\Lambda} = (\omega_{i_1}, \dots, \omega_{i_n})$, then the dimension of $\mathcal{V}_{\mathbb{P}^1}(\mathfrak{e}_6, \vec{\Lambda}, 1)$ is one if 3 divides $(i_1 + \cdots + i_n)$ and is zero otherwise.*

(vi) *If $\mathfrak{s} = \mathfrak{e}_7$, the level-one weights are ω_0, ω_7. The representation ω_7 is self dual. Hence the only nontrivial three-pointed conformal blocks on $\mathbb{P}^1$ are associated to the weights $(\omega_0, \omega_0, \omega_0)$ and $(\omega_0, \omega_1, \omega_1)$. The dimension is one in all these cases.*

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(viii) *If* $\mathfrak{s} = \mathfrak{e}_8$*, the only level-one weight of* $\mathfrak{e}_8$ *is* ω_0*. The dimension of* $\mathscr{V}_\Sigma(\mathfrak{e}_8, \omega_0, 1)$ *is always one, where* Σ *is any stable curve of genus* g*.*

We can now use the above calculations to check when the condition that the dimension $\dim_{\mathbb{C}} \mathscr{V}_\Sigma(\mathfrak{s}, \vec{\Lambda}, 1) = 1$ related for Question D.2.1 is satisfied. We can also use the Verlinde formula to compare the dimensions of the $\mathscr{V}_\Sigma(\mathfrak{s}_1, \vec{\lambda}, \ell_1)$ and $\mathscr{V}_\Sigma(\mathfrak{s}_2, \vec{\mu}, \ell_2)$ appearing in the rank-level duality questions. Such comparison involves identities involving the determinant of matrices whole entries are certain cyclotomic polynomials evaluated at the root of unity that appear in the Verlinde formula via the Weyl character formula. We refer the reader to the papers of Altschuler, Bauer and Itzykson (1990); Abe (2008); Donagi and Tu (1994); Mlawer et al. (1991); Mukhopadhyay (2016a,b); Mukhopadhyay and Wentworth (2019); Naculich and Schnitzer (1990); Nakanishi and Tsuchiya (1992); Oxbury and Wilson (1996); Zagier (1996) for detailed calculations.

Results on Rank-Level Duality

In this section, we give a brief survey of the known rank-level duality isomorphisms.

The Case $\mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \to \mathfrak{sl}(rs)$

Let $\vec{\Lambda} = (\omega_{i_1}, \ldots, \omega_{i_n})$ be such that rs divides $i_1 + \cdots + i_n$. Then, the following holds (Nakanishi and Tsuchiya, 1992):

Theorem D.2.5 *For any choice of* $\vec{\lambda}$ *and* $\vec{\mu}$ *such that* $(\lambda_i, \mu_i) \in I_{\Lambda_i}$ *(see Section D.2 for notation), the following rank-level duality map is a perfect pairing:*

$$\widetilde{\varphi}\colon \mathscr{V}_{\mathbb{P}^1}(\mathfrak{sl}(r), \vec{\lambda}, s) \otimes \mathscr{V}_{\mathbb{P}^1}(\mathfrak{sl}(s), \vec{\mu}, r) \to \mathscr{V}_{\mathbb{P}^1}(\mathfrak{sl}(rs), \vec{\Lambda}, 1) \simeq \mathbb{C}.$$

Hence, it induces an isomorphism between $\mathscr{V}^{\dagger}_{\mathbb{P}^1}(\mathfrak{sl}(r), \vec{\lambda}, s) \simeq \mathscr{V}_{\mathbb{P}^1}(\mathfrak{sl}(s), \vec{\mu}, r)$*.*

Rank-level duality in this case for curves of positive genus has been studied by Belkale (2008a, 2009); Marian and Oprea (2007); Oudompheng (2011) using methods from enumerative geometry. This question was formulated using the language of non-abelian theta functions and is also known as the strange duality conjecture (Donagi and Tu, 1994). The level-one case was proved by Beauville, Narasimhan and Ramanan (1989) and the strange duality conjecture for generic curves was proved by Belkale (2008a). Later, Belkale used the notion of conformal embeddings, uniformization theorems and showed that the strange duality map is flat with respect to the Hitchin

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connection, hence proving the result for all curves in Belkale (2009). We describe the statement below.

Theorem D.2.6 *Let Σ be a smooth curve of genus $g \geq 2$. Consider the moduli stack $\mathcal{SU}_{\Sigma}(r)$ (respectively $\mathcal{U}_{\Sigma}(s, s(g-1))$) of rank r (respectively rank s) vector bundles on curve Σ with trivial determinant (respectively of degree $s(g-1)$). There is a natural duality between $H^0(\mathcal{SU}_{\Sigma}(r), \mathcal{L}^{\otimes s})$ and $H^0(\mathcal{U}_{\Sigma}(s, s(g-1)), \Theta_s^{\otimes r})$, where $\mathcal{L}$ (respectively Θ_s) (see Chapter 8) is the generator of the Picard group of $\mathcal{SU}_{\Sigma}(r)$ (respectively the generalized theta divisor on $\mathcal{U}_{\Sigma}(s, s(g-1))$).*

R. Oudompheng (2011) proved a parabolic version of the above result. Later, C. Pauly (2014) showed that Theorem D.2.5 in Nakanishi and Tsuchiya (1992) combined with the strange duality result in Beauville, Narasimhan and Ramanan (1989) gives the strange duality stated in Theorem D.2.6.

Remark D.2.7 We could not find a complete reference for $\mathfrak{sl}(r)$ to the proof of the second part of Proposition 1 in Nakanishi and Tsuchiya (1992, Section 5, p. 363). We can use the result of Oudompheng (2011) and results in Mukhopadhyay (2013) to derive Theorem D.2.5. We refer the reader to Section 8.2 of Mukhopadhyay (2013) for further details.

The Case $\mathfrak{sp}(2r) \oplus \mathfrak{sp}(2s) \to \mathfrak{so}(4rs)$

Let n be an even integer and we write $n = 2a + 2b$. Let $\vec{\Lambda} = (\vec{\omega}_+, \vec{\omega}_-)$, where $\vec{\omega}_+$ (respectively $\vec{\omega}_-$) be a $2a$ (respectively $2b$) tuple of the weight ω_+ (respectively ω_-) of $\mathfrak{so}(4rs)$. We state the following theorem (Abe, 2008; Belkale, 2012a):

Theorem D.2.8 *Let Σ be any $2m$-pointed smooth curve of genus g, then there is a natural duality between $\mathcal{V}_{\Sigma}(\mathfrak{sp}(2r), \vec{Y}, s)$ and $\mathcal{V}_{\Sigma}(\mathfrak{sp}(2s), \vec{Y}^*, r)$, where $\vec{Y} = (Y_1, \ldots, Y_n)$ is an n-tuple of Young diagrams in $\mathcal{Y}_{r,s}$ and $\sum_{i=1}^{2m} |Y|$ is even.*

In Theorem D.2.8, the natural duality is induced by a canonical projectively flat (Belkale, 2012a) 'Pfaffian' section (Beauville, 2006). We now state a genus-zero version of the rank-level duality result which is due to T. Abe (2008):

Theorem D.2.9 *The rank-level duality map $\widetilde{\varphi}$ induced by the branching rules with $\vec{\Lambda}$ as above is a perfect pairing for any smooth genus-zero curve with n marked points.*

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Combining Theorem D.2.9, along with the factorization theorem and a result on projective flatness of Pfaffian sections in Belkale (2012a), Theorem D.2.8 follows directly. Moreover, new rank-level dualities for genus-zero smooth curves with n marked points can be obtained from Theorem D.2.9 by applying diagram automorphisms. We refer the reader to Mukhopadhyay (2013) for further details.

The Case $\mathfrak{so}(2r+1) \oplus \mathfrak{so}(2s+1) \to \mathfrak{so}((2r+1)(2s+1))$

Let d be such that $2d+1 = (2r+1)(2s+1)$ and consider an n-tuple $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ of level-one weights of $\mathfrak{so}(2r+1)$ such that

(i) Each Λ_i is either ω_0 of ω_1.
(ii) The number of ω_1 is even.

The following theorem can be found in Mukhopadhyay (2016c):

Theorem D.2.10 *Let $\vec{\Lambda}$ be as above, then the rank-level duality map $\widetilde{\varphi}$ given by the branching rules is a perfect pairing for any smooth genus-zero curve with n marked points.*

Remark D.2.11 Let $\vec{\Lambda} = (\omega_1, \omega_d, \omega_d)$, then the level-one conformal block on $\mathbb{P}^1$ with three marked points is 1-dimensional. Unlike the previous cases, it was shown in Mukhopadhyay and Wentworth (2019) that the rank-level duality map is not in general an isomorphism. We refer the reader to Mukhopadhyay and Wentworth (2019) to computations with Verlinde formula that show that the source and the target are of different dimensions. However an injectivity result (Mukhopadhyay and Wentworth, 2019) involving the maps of conformal blocks still holds. This shows that the monodromy representations of Knizhnik–Zamolodchikov connections on the space of covacua associated to $\mathfrak{so}(2r+1)$ with spin weights are not in general irreducible.

Remark D.2.12 A conjectural dimensional equality between the source and target of rank-level duality map for conformal blocks involving Lie algebras of type B_r was proposed in Oxbury and Wilson (1996). This conjecture was proved in Mukhopadhyay and Wentworth (2019), however it was shown there that the associated rank-level duality map is not an isomorphism. It remains an open question to find a rank-level duality for conformal blocks of type B_r on curves of positive genus.

The Case $\mathbf{G_2} \times \mathbf{F_4} \to \mathbf{E_8}$

Consider $\vec{\lambda} = (\omega_1, \ldots, \omega_1)$ (respectively $\vec{\mu} = (\omega_4, \ldots, \omega_4)$) to be the n-tuple of the weight ω_1 of $\mathfrak{g}_2$ (respectively the weight ω_4 of $\mathfrak{f}_4$) and $\vec{\Lambda}$ be n-tuple of

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the weight ω_0 of the vacuum representation of $\mathfrak{e}_8$. The following theorem can be found in Mukhopadhyay (2016b):

Theorem D.2.13 *The map of the space of covacuas induced by the branching rules is a perfect pairing for any n-pointed smooth curve Σ of genus g.*

$$\mathcal{V}_\Sigma(\mathfrak{g}_2, \omega_1, \ldots, \omega_1, 1) \otimes \mathcal{V}_\Sigma(\mathfrak{f}_4, \omega_4, \ldots, \omega_4, 1) \to \mathcal{V}_\Sigma(\mathfrak{e}_8, \omega_0, \ldots, \omega_0, 1).$$

The equality of the dimension of the $\mathfrak{g}_2$ and $\mathfrak{f}_4$ conformal blocks in the statement of Theorem D.2.13 follows from (1).

The Case $\mathfrak{so}(r) \to \mathfrak{sl}(r)$

It follows directly from the branching rules of the conformal embedding $\mathfrak{so}(r) \to \mathfrak{sl}(r)$ that for any stable nodal curve Σ of genus g, the following map is surjective:

$$\mathcal{V}_\Sigma(\mathfrak{so}(r), \omega_0, 2) \oplus \mathcal{V}_\Sigma(\mathfrak{so}(r), 2\omega_1, 2) \to \mathcal{V}_\Sigma(\mathfrak{sl}(r), \omega_0, 1).$$

If Σ is smooth, then using the uniformization theorem and the invariance of the rank-level duality map under the action of two torsion points $J_2(\Sigma)$ of the Jacobian, the following was shown in Mukhopadhyay and Zelaci (2020):

Theorem D.2.14 *The natural map between the moduli stack* $\mathrm{Bun}_{\mathrm{SO}(r)}(\Sigma)$ *of* $\mathrm{SO}(r)$*-bundles on a smooth curve* Σ *and the moduli stack* $\mathrm{Bun}_{\mathrm{SL}(r)}(\Sigma)$ *of* $\mathrm{SL}(r)$*-bundles induces an isomorphism between* $SD\colon H^0(\mathrm{Bun}_{\mathrm{SO}(r)}(\Sigma), \mathcal{D}) \simeq H^0(\mathrm{Bun}_{\mathrm{SL}(r)}(\Sigma), \mathcal{D})$ *where* $\mathcal{D}$ *is the determinant of cohomology line bundles. Moreover the isomorphism* SD *is flat with respect to the Hitchin connection.*

Other Results

Strange duality results associated to some conformal subalgebra $\mathfrak{p}$ of $\mathfrak{e}_8$ was considered in Boysal and Pauly (2010). Here $\mathfrak{p}$ is the Lie algebra of a simply-connected group P and the list of such P is the following:

- $\mathrm{Spin}(8) \times \mathrm{Spin}(8)$,
- $\mathrm{Spin}(16)$,
- $\mathrm{SL}(9)$,
- $\mathrm{SL}(5) \times \mathrm{SL}(5)$,
- $\mathrm{SL}(3) \times \mathrm{E}_6$,
- $\mathrm{SL}(2) \times \mathrm{E}_7$.

Their proof is based on flatness of rank-level duality, Verlinde formula and representations of the Heisenberg group associated to the center of the

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simply-connected group P. Rank-level duality for level-one theta functions for G_2, $SL_2 \times SL_2$ theta functions at level 2 and SL_3 theta functions at level 3 was considered in Grégoire and Pauly (2013). We also refer to the results in Beauville (2006); Pauly and Ramanan (2001); Mukhopadhyay and Zelaci (2020) for level-one rank-level duality results associated to the group $SO(r)$ and theta functions on a Prym variety associated to an étale double cover of a curve.

Remark D.2.15 There are several examples of conformal embeddings for which rank-level duality questions have not been investigated in genus zero. On curves of positive genus, Question D.2.1 usually has a negative answer due to the action of torsion points of the Jacobians of the curves arising from the center of the simply-connected group G. It is not clear how to modify Question D.2.1 for curves of higher genus to accommodate the action of torsion points. Part (1) of Question D.2.1 fails to be a perfect pairing for all curves in $\overline{\mathcal{M}}_{g,n}$; it is interesting to study the stratum of stable nodal curves on which Question D.2.1 holds.

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