A COMMUTATOR IDENTITY FOR BASIC p-GROUPS

PAUL M. WEICHSEL

Abstract

The consequences of the law $(G_{a(1)}, ..., G_{a(n)}) = 1$ in a basic p-group are examined. The principal tools are a combinatorial analysis of the lattice of n-tuples of positive integers and a theorem of the author about higher-commutator subgroups.

1. Introduction

In Weichsel (1973), we investigated the higher commutator structure of basic p-groups and proved, for example, that if G is a basic p-group of sufficiently small class c and $(G_r, G_s) = 1$, then, with suitable restrictions on r and s, to wit $r \leqslant s \leqslant \frac{1}{2}c < \frac{1}{2}(p+2r-1)$, it follows that $(G_r, G_r) = 1$. We also stated a similar result when $(G_2, G_r, G_s) = 1$. The proof that was given of this latter result (Weichsel, 1973, Theorem 4.3) is incomplete, and in attempting to correct it we found it necessary to isolate the purely combinatorial aspects of the theorem in a more general context. As a result we can now prove a far more general result (3.1). The next section contains the combinatorial part and we conclude this introduction by recalling from Weichsel (1973) the main theorem which motivates the combinatorial construction.

Let $(a(1), ..., a(n))$ be an n-tuple of positive integers, and let G be a group. Then if $G_{a(i)}$ is the $a(i)$th term of the lower central series of G, we denote by $(G_{a(1)}, ..., G_{a(n)})$ the verbal subgroup of G generated by all commutators of the form $(g_1, g_2, ..., g_n)$ with $g_i \in G_{a(i)}$. Clearly if $b(i) \geqslant a(i)$ for $i = 1, ..., n$, then $(G_{a(1)}, ..., G_{a(n)}) \supseteq (G_{b(1)}, ..., G_{b(n)})$. Hence there is a natural partial order among the subgroups of the form: $(G_{a(1)}, ..., G_{a(n)})$, for fixed n. Now in Weichsel (1973, Theorem 2.5), we show that under certain conditions, if

$$(G_{a(1)}, ..., G_{a(i-1)}, G_{a(i)+1}, G_{a(i+1)}, ..., G_{a(n)}) = 1$$

for each $i = 1, ..., n$, then $(G_{a(1)}, ..., G_{a(n)}) = 1$. In other words, if each of these subgroups *immediately below* $(G_{a(1)}, ..., G_{a(n)})$ is trivial, then $(G_{a(1)}, ..., G_{a(n)})$ is also trivial. The reader should keep this in mind while reading the next section.

The basic definitions and notation are given in Weichsel (1973).

2. The lattice of n-tuples

Let L be the set of all n-tuples of positive integers, n fixed. If $\alpha = (a(1), ..., a(n))$ and $\beta = (b(1), ..., b(n))$ are elements of L, we say that $\alpha \geqslant \beta$ if $a(i) \geqslant b(i)$ for all $i = 1, ..., n$. Clearly $[L, \leqslant]$ is a lattice with $(1, ..., 1)$ as minimal element. If $\alpha = (a(1), ..., a(n)) \in L$, then we say that the set B is the k-*cover* of α if B is the set of all elements of the form $\beta = (a(1) + t(1), ..., a(n) + t(n))$ with each $t(i)$ a non-negative integer and $\sum_{i=1}^{n} t(i) = k$.

DEFINITION. A subset S of L is said to be *closed* if: (i) whenever $\alpha \in S$ and $\beta \geqslant \alpha$, then $\beta \in S$ and (ii) whenever the 1-cover of an element α is a subset of S, then $\alpha \in S$.

REMARK. The ordering that we have chosen for L is the natural one. When we come to apply these ideas to groups we will have to turn L on its head.

The next result is an immediate consequence of the definition.

2.1 LEMMA. *The intersection of an arbitrary collection of closed subsets of L is closed.*

DEFINITION. Let T be a subset of L. The *closure* of T, denoted by T^*, is the intersection of all closed sets of L which contains T.

We now need to derive a constructive description of the closure operation.

2.2 THEOREM. *Let T be a subset of L and define the set $\bar{T}$ by: $\bar{T} = A \cup B$ with $A = \{\alpha \in L \mid \alpha \geqslant \beta \text{ for some } \beta \in T\}$ while an element of L is in B if and only if its k-cover is a subset of A for some k. Then $\bar{T} = T^*$.*

PROOF. We will first show that $\bar{T}$ is closed. Let $\alpha \in \bar{T}$ and let $\beta \geqslant \alpha$. If $\alpha \in A$, then $\alpha \geqslant \gamma$ for some $\gamma \in T$ and so $\beta \geqslant \alpha \geqslant \gamma \in T$, which implies that $\beta \in A \subseteq \bar{T}$. If $\alpha \notin A$, then $\alpha \in B$. Thus there is a positive integer k such that the k-cover of α is in A. It follows that the k-cover of β is also in A and hence that $\beta \in B \subseteq \bar{T}$. For let $\alpha = (a(1), ..., a(n))$ and $\beta = (b(1), ..., b(n))$. Then a typical element of the k-cover of β is

$$(b(1) + t(1), ..., b(n) + t(n))$$

with $\sum_{i=1}^{n} t(i) = k$. Hence since $\beta \geqslant \alpha$,

$$(b(1) + t(1), ..., b(n) + t(n)) \geqslant (a(1) + t(1), ..., a(n) + t(n)) \in A$$

and so the k-cover of β is in A. We therefore conclude that $\bar{T}$ satisfies requirement (i) in the definition of a closed set.

Now let α be an element of L whose 1-cover is a subset of $\bar{T}$. If all of the elements of the 1-cover are in A, then clearly, $\alpha \in B \subseteq \bar{T}$. Now suppose that at least one element of the 1-cover of α, say γ, is not in A and is therefore in B. Then there exists a smallest positive integer k' such that the k'-cover of γ is in A. Now let k be the largest of these k''s. That is, for each element δ, of the 1-cover of α which is in B, the k-cover of δ is in A. We now claim that the $(k+1)$-cover of α is in A.

For if $\alpha = (a(1), ..., a(n))$ and $\theta = (a(1)+t(1), ..., a(n)+t(n))$ is an element of the $(k+1)$-cover of α, then for some i, $t(i) \geq 1$. Hence

$$\theta = (a(1)+t(1), ..., a(i)+1+(t(i)-1), ..., a(n)+t(n))$$

with $t(i)-1 \geq 0$. Now consider the element $\alpha_i = (a(1), ..., a(i)+1, ..., a(n))$ of the 1-cover of α. If $\alpha_i \in A$, then $\theta \in A$. If $\alpha_i \in B$, then by the choice of k, the k-cover of α_i is in A. But θ is the k-cover of α_i and again $\theta \in A$. Hence the $(k+1)$-cover of α is in A and so $\alpha \in B \subseteq \bar{T}$. Thus $\bar{T}$ is closed.

The theorem will follow if we can show that $\bar{T}$ is a subset of every closed set containing T. Thus let R be a closed set which contains T and let $\alpha \in \bar{T}$. If $\alpha \in A$, then $\alpha \geq \beta \in T \subseteq R$ for some β and since R is closed, $\alpha \in R$. If $\alpha \in B$, then for some k, the k-cover of α is in A which is a subset of R. Now every element of the $(k-1)$-cover of α is in R since the 1-cover of such an element is a subset of the k-cover of α. Hence by induction, the 1-cover of α is in R and so since R is closed, $\alpha \in R$. Thus $\bar{T} \subseteq R$ and therefore $\bar{T} = T^*$.

Now for purposes of application we need to consider special subsets of L which are invariant under permutation of the n-tuples.

DEFINITION. A subset S of L is called *symmetric* if for each $\alpha = (a(1), ..., a(n)) \in S$, $\alpha\pi = (a(\pi 1), ..., a(\pi n)) \in S$ for all $\pi \in \Sigma_n$, the symmetric group of degree n.

2.3 LEMMA. *If T is symmetric, then T^* is symmetric.*

PROOF. Recall that $T^* = A \cup B$ as in 2.2. If $\alpha \in A$, $\alpha = (a(1), ..., a(n))$, then $\alpha \geq \beta$ for some $\beta \in T$. If $\beta = (b(1), ..., b(n))$, then $a(i) \geq b(i)$ for $i = 1, ..., n$. Hence $(a(\pi 1), ..., a(\pi n)) \geq (b(\pi 1), ..., b(\pi n)) \in T$ since T is symmetric. Thus A is symmetric. Now if $\gamma \in B$, then for some k, the k-cover of γ is in A which is symmetric. If $\{\mu_\phi\}$ is the k-cover of γ, then $\{\mu_\phi \pi\}$ is the k-cover of $\gamma\pi$ and since A is symmetric, $\gamma\pi$ is in B. Thus B is symmetric and hence $\bar{T}$ is.

2.4 THEOREM. *Let $\alpha = (a(1), a(2), ..., a(n))$ be a fixed element of L with $n \geq 2$ and $a(1) \geq a(2) \geq ... \geq a(n)$. Put $\alpha_0 = (a(2), a(2), a(3), ..., a(n)) = (c(1), c(2), ..., c(n))$.*

Let $S = \{\alpha\pi = (a(\pi 1), ..., a(\pi n)) \mid \pi \in \Sigma_n\}^*$, written as $S = A \cup B$ in the notation of 2.2. Then

(i) if $a(1) > a(2)$ the k-cover of α_0 is in A if and only if

$$k \geq k_0 = 2(a(1)-a(2)-1)+(a(1)-a(3)-1)+...+(a(1)-a(n)-1)+1,$$

and

(ii) the set of minimal elements of S is precisely

$$\{\alpha_0 \pi \mid \pi \in \Sigma_n\}.$$

PROOF. (i) Suppose $k \geqslant k_0$ and that $k = k_1 + k_2 + \ldots + k_n$ is a partition of k. Clearly either k_1 or $k_2 > a(1) - a(2) - 1$ or for some $i \geqslant 3$, $k_i > a(1) - a(i) - 1$. Hence at least one entry in $(a(2) + k_1, a(2) + k_2, \ldots, a(n) + k_n)$ is greater than or equal to $a(1)$ and then the others are greater than or equal to $a(2), a(3), \ldots, a(n)$ respectively. Thus every element of the k-cover of α_0 is $\geqslant \alpha\pi$ for some $\pi \in \Sigma_n$ and so is in A. Conversely, suppose that every element of the k-cover of α_0 is in A. Then if we define $k_i = a(1) - a(i) - 1$ for $i \geqslant 2$ and $k_1 = k - (k_2 + \ldots + k_n)$, then we must have $a(2) + k_1 \geqslant a(1)$. Hence $k \geqslant k_0$.

(ii) It follows from (i) that $\alpha_0 \in S$ because once the k_0-cover of α_0 is in S, the $(k_0 - 1)$-cover of α_0 whose 1-cover is the k_0-cover of α_0 is also in S. We proceed by induction.

Now let $\beta \in S$, $\beta = (b(1), \ldots, b(n))$ with $b(1)$ the largest entry. Either $\beta \in A$ or for some l, the l-cover of β is in A. In any event, for some $l \geqslant 0$,

$$(b(1) + l, b(2), \ldots, b(n)) \in A,$$

and so for some $\pi \in \Sigma_n$,

$$(b(1) + l, b(2), \ldots, b(n)) \geqslant (a(\pi 1), a(\pi 2), \ldots, a(\pi n)).$$

This means that $b(1) \geqslant a(2)$ and that $b(2), b(3), \ldots, b(n)$ are greater than $a(2), a(3), \ldots, a(n)$ in some order. That is, $\beta \geqslant \alpha_0 \sigma$ for some $\sigma \in \Sigma_n$. Since S is symmetric this means that for all $\beta \in S$, $\beta \geqslant \alpha_0 \sigma$ for some $\sigma \in \Sigma_n$.

It remains to show that $\alpha_0 \sigma$ is minimal for each σ and this will follow if we can show that α_0 is minimal. Now if $\beta = (b(1), \ldots, b(n)) \in S$ and $\beta < \alpha_0$, then for some $\sigma \in \Sigma_n$, $\alpha_0 \sigma \leqslant \beta < \alpha_0$. However, this would mean then $\sum_{i=1}^{n} c(\sigma i) < \sum_{i=1}^{n} c(i)$, a contradiction.

If $\alpha = (a(1), \ldots, a(n)) \in L$, we say that the *weight of* α is $w(\alpha) = \sum_{i=1}^{n} a(i)$.

2.5 COROLLARY. *Let S be the set defined in 2.4. Then the minimal elements of S (in the ordering of L) are precisely the elements of minimal weight in S.*

PROOF. It follows from 2.4 that the minimal elements of S all have weight $w(\alpha_0)$. Hence any element, say γ, of S of smaller weight is not minimal in S. But since all descending chains of L are finite, γ must be bounded from below by an element δ of S, minimal in S. Thus $w(\delta) \leqslant w(\gamma) < w(\alpha_0)$, a contradiction.

3. The commutator-subgroups $(G_{a(1)}, \ldots, G_{a(n)})$

We now recall the interpretation of the lattice L in group-theoretical terms. The n-tuple $\alpha = (a(1), \ldots, a(n))$ corresponds to the commutator-subgroup $(G_{a(1)}, \ldots, G_{a(n)})$ of the group G. The ordering in L corresponds inversely to the ordering of subgroups by set inclusion. (We are not suggesting that the two lattices are isomorphic. For it may happen that whereas $(2, 3) < (2, 4)$, (G_2, G_3) and (G_2, G_4)

may be equal. We only claim that when $(a(1), ..., a(n)) \leqslant (b(1), ..., b(n))$ in L, then $(G_{a(1)}, ..., G_{a(n)}) \supseteq (G_{b(1)}, ..., G_{b(n)})$ in G.) We now state Theorem 2.5 of Weichsel (1973) in the terms of the present paper.

3.1 THEOREM. *Let G be a basic p-group of class c. Suppose that the commutator subgroups of G associated with $R \subseteq L$ are all trivial. If the 1-cover of $\alpha = (a(1), ..., a(n))$ is R, and if $\sum_{i=1}^{n} a(i) \not\equiv c \pmod{p-1}$, then $(G_{a(1)}, ..., G_{a(n)}) = 1$.*

The principal result of this section applies 2.4, whose proof depends on the description of closure via k-covers given in 2.2. Thus in the light of 3.1 above it suffices to impose a condition on the weights of the n-tuples involved. Hence let $\alpha = (a(1), ..., a(n))$ and we consider the set $\{\alpha\pi = (a(\pi 1), ..., a(\pi n)) \mid \pi \in \Sigma_n\}$. We may as well assume that $a(1) \geqslant a(2) \geqslant a(3) \geqslant ... \geqslant a(n)$. Since the argument by k-covers is just an iteration of 1-covers, we need to impose a restriction that guarantees the congruence condition of 3.1 above for any n-tuple between $(a(2), a(2), a(3), ..., a(n))$ and $(a(2)+t(1), a(2)+t(2), a(3)+t(3), ..., a(n)+t(n))$ with

$$\sum_{i=1}^{n} t(i) = k = 2(a(1)-a(2)-1)+(a(1)-a(3)-1)+...+(a(1)-a(n)-1)+1.$$

This is equivalent to the requirement that if d is an integer satisfying

$$2a(2) + \sum_{i=3}^{n} a(i) \leqslant d < 2a(2) + \sum_{i=3}^{n} a(i) + k,$$

then $d \not\equiv c \pmod{p-1}$. Now

$$2a(2) + \sum_{i=3}^{n} a(i) + k = n(a(1)-1)+1$$

and therefore if we impose:

$$n(a(1)-1)+1 \leqslant c$$

and

$$c-(p-1) < 2a(2) + \sum_{i=3}^{n} a(i)$$

then $d \not\equiv c \pmod{p-1}$. We note that $2a(2) + \sum_{i=3}^{n} a(i) \geqslant na(n)$ and we have proved the following result.

3.2 THEOREM. *Let G be a basic p-group of class c. Let $\alpha = (a(1), ..., a(n))$ be an n-tuple of positive integers satisfying:*

$$1 \leqslant a(n) \leqslant ... \leqslant a(2) < a(1) \leqslant \frac{c+n-1}{n} < \frac{p+na(n)+n-2}{n}.$$

If $(G_{a(\pi 1)}, ..., G_{a(\pi n)}) = 1$ for all $\pi \in \Sigma_n$, then $(G_{a(\pi 2)}, G_{a(\pi 2)}, G_{a(\pi 3)}, ..., G_{a(\pi n)}) = 1$ for all $\pi \in \Sigma_n$.

Reference

Paul M. Weichsel (1973), "Basic *p*-groups: higher commutator structure", *J. Austral. Math. Soc.* **16**, 200–206.

University of Illinois
Urbana, Illinois