



Quaternionic Dolbeault complex and vanishing theorems on hyperkähler manifolds

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ABSTRACT

Let (M, I, J, K) be a compact hyperkähler manifold, $\dim_{\mathbb{H}} M = n$, and L a non-trivial holomorphic line bundle on (M, I) . Using the quaternionic Dolbeault complex, we prove the following vanishing theorem for holomorphic cohomology of L . If $c_1(L)$ lies in the closure \hat{K} of the dual Kähler cone, then $H^i(L) = 0$ for $i > n$. If $c_1(L)$ lies in the opposite cone $-\hat{K}$, then $H^i(L) = 0$ for $i < n$. Finally, if $c_1(L)$ is neither in \hat{K} nor in $-\hat{K}$, then $H^i(L) = 0$ for $i \neq n$.

1. Introduction

1.1 Hypercomplex and hyperkähler manifolds

DEFINITION 1.1. Let M be a manifold, and let $I, J, K \in \text{End}(TM)$ be endomorphisms of the tangent bundle satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\text{Id}_{TM}.$$

The manifold (M, I, J, K) is called *hypercomplex* if the almost complex structures I, J, K are integrable. If, in addition, M is equipped with a Riemannian metric g which is Kähler with respect to I, J, K , the manifold (M, I, J, K, g) is called *hyperkähler*.

Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ on M :

$$\omega_I(\cdot, \cdot) := g(\cdot, I\cdot), \quad \omega_J(\cdot, \cdot) := g(\cdot, J\cdot), \quad \omega_K(\cdot, \cdot) := g(\cdot, K\cdot).$$

An elementary linear-algebraic calculation implies that the 2-form $\Omega := \omega_J + \sqrt{-1}\omega_K$ is of Hodge type $(2, 0)$ on (M, I) . This form is clearly closed and non-degenerate, hence it is a holomorphic symplectic form.

In algebraic geometry, the word ‘hyperkähler’ is essentially synonymous with ‘holomorphically symplectic’, due to the following theorem, which is implied by Yau’s solution of the Calabi conjecture.

THEOREM 1.2. *Let (M, I) be a compact, Kähler, holomorphically symplectic manifold. Then there exists a unique hyperkähler metric on (M, I) with the same Kähler class.*

Proof. For the proof, see [Yau78] and [Bes87]. □

REMARK 1.3. The hyperkähler metric is unique, but there could be several hyperkähler structures compatible with a given hyperkähler metric on (M, I) .

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DEFINITION 1.4. Let (M, I, J, K) be a compact, simply connected hyperkähler¹ manifold which satisfies $H^1(M) = 0$, $H^{2,0}(M, I) = \mathbb{C}$. Then M is called an *irreducible hyperkähler manifold*.

Remark 1.5. By Bogomolov’s decomposition theorem [Bog74], any compact holomorphic symplectic manifold admits a finite cover which is a product of several irreducible hyperkähler manifolds and a compact torus. This explains the term.

1.2 Vanishing theorems on hyperkähler manifolds

Using the argument which essentially belongs to the theory of hypercomplex manifolds, we are able to prove the following algebro-geometric statements.

THEOREM 1.6. Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, and L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$. Denote by

$$\overline{\mathcal{K}} \subset H^{1,1}(M, I) \cap H^2(M, \mathbb{R})$$

the closure of the dual Kähler cone of (M, I) (§ 5.2), and let $-\overline{\mathcal{K}}$ be the opposite cone. Then one of the following holds:

- (i) $c_1(L) \in \overline{\mathcal{K}}$; then $H^i(L) = 0$ for $i > \frac{1}{2} \dim_{\mathbb{C}} M$;
- (ii) $c_1(L) \in -\overline{\mathcal{K}}$; then $H^i(L) = 0$ for $i < \frac{1}{2} \dim_{\mathbb{C}} M$;
- (iii) $c_1(L)$ does not lie in $-\overline{\mathcal{K}} \cup \overline{\mathcal{K}}$; then $H^i(L) = 0$ for $i \neq \frac{1}{2} \dim_{\mathbb{C}} M$.

Proof. See Theorem 5.6. □

THEOREM 1.7. Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$, and B an arbitrary holomorphic vector bundle on (M, I) . Then there exists a sufficiently big number N_0 , such that for any integer $N > N_0$ one of the following holds:

- (i) $c_1(L) \in \overline{\mathcal{K}}$; then $H^i(L^N \otimes B) = 0$ for $i > \frac{1}{2} \dim_{\mathbb{C}} M$;
- (ii) $c_1(L) \in -\overline{\mathcal{K}}$; then $H^i(L^N \otimes B) = 0$ for $i < \frac{1}{2} \dim_{\mathbb{C}} M$;
- (iii) $c_1(L)$ does not lie in $-\overline{\mathcal{K}} \cup \overline{\mathcal{K}}$; then $H^i(L^N \otimes B) = 0$ for $i \neq \frac{1}{2} \dim_{\mathbb{C}} M$.

Proof. This is Theorem 5.8. □

The vanishing theorems have many interesting geometrical consequences. As an example, we give the following theorem (§ 6).

THEOREM 1.8. Let (M, I, J, K) be an irreducible hyperkähler manifold, and $X \subset (M, I)$ a subvariety of dimension $\dim_{\mathbb{C}} X > \frac{1}{2} \dim_{\mathbb{C}} M$. Assume that X is a complete intersection of ample divisors. Consider a holomorphic line bundle L on (M, I) with $c_1(L)$ numerically effective (that is, $c_1(L)$ lies in the closure of the Kähler cone of (M, I)) and $q(c_1(L), c_1(L)) = 0$, where q is the Bogomolov–Beauville–Fujiki bilinear form (Definition 4.4). Then the natural restriction map is surjective on holomorphic sections,

$$H^0(L^N) \rightarrow H^0(L^N|_X) \rightarrow 0,$$

for a sufficiently big power of L .

Proof. See Theorem 6.5. □

¹Or holomorphically symplectic Kähler; by Yau’s theorem, it is essentially the same notion.

2. Quaternionic Dolbeault complex

2.1 Weights of SU(2)-representations

It is well known that any finite-dimensional (continuous) irreducible representations of SU(2) over \mathbb{C} can be obtained as a symmetric power $S^i(V_1)$, where V_1 is a fundamental two-dimensional representation. We say that a representation W has weight i if it is isomorphic to $S^i(V_1)$. A representation is said to be *pure of weight i* if all its irreducible components have weight i . If all irreducible components of a representation W_1 have weight at most i , we say that W_1 is a *representation of weight at most i* . In a similar fashion one defines representations of weight at least i .

Remark 2.1. The Clebsch–Gordan formula (see [Hum72]) claims that the weight is *multiplicative*, in the following sense: if $i \leq j$, then

$$V_i \otimes V_j = \bigoplus_{k=0}^i V_{i+j-2k},$$

where $V_i = S^i(V_1)$ denotes the irreducible representation of weight i .

A subspace $W \subset W_1$ is *pure of weight i* if the SU(2)-representation $W' \subset W_1$ generated by W is pure of weight i .

2.2 Quaternionic Dolbeault complex: a definition

Let M be a hypercomplex (e.g. a hyperkähler) manifold, $\dim_{\mathbb{H}} M = n$. There is a natural multiplicative action of $SU(2) \subset \mathbb{H}^*$ on $\Lambda^*(M)$, associated with the hypercomplex structure.

The space $\Lambda^*(M)$ of differential forms is an infinite-dimensional representation of SU(2). However, the SU(2)-action is $C^\infty(M)$ -linear, because the group SU(2) acts on $\Lambda^*(M)$ fiberwise. Take a differential form $\gamma \in \Lambda^*(M)$ such that the restriction $\gamma|_x$ of γ to any fiber $\Lambda^*(M)|_x$ lies in an irreducible, weight k representation of SU(2) (this is possible, because the fiber $\Lambda^*(M)|_x$ is finite-dimensional). Then γ lies in a direct sum of weight k representations. If we pick γ in such a way that $\gamma|_x$ is a vector of highest (or lowest) weight with respect to some Cartan subalgebra in $\mathfrak{su}(2)$, then γ is also a vector of highest (or lowest) weight, and generates an irreducible weight k representation. It is easy to see that all irreducible subrepresentations of $\Lambda^*(M)$ are obtained this way. This gives the following claim.

CLAIM 2.2. *All irreducible components of $\Lambda^*(M)$ are finite-dimensional.*

Remark 2.3. From the above argument, we see that it makes sense to speak of the *weight* of $\Lambda^*(M)$ and its subrepresentations. Clearly, $\Lambda^1(M)$ has weight 1. From the Clebsch–Gordan formula (Remark 2.1), it follows that $\Lambda^i(M)$ is an SU(2)-representation of weight at most i . Using the Hodge *-isomorphism $\Lambda^i(M) \cong \Lambda^{4n-i}(M)$, we find that, for $i > 2n$, $\Lambda^i(M)$ is a representation of weight at most $2n - i$.

Let $V^i \subset \Lambda^i(M)$ be a maximal SU(2)-invariant subspace of weight less than i . The space V^i is well defined, because it is a sum of all irreducible representations $W \subset \Lambda^i(M)$ of weight less than i . Since the weight is multiplicative (Remark 2.1), $V^* = \bigoplus_i V^i$ is an ideal in $\Lambda^*(M)$. We also have $V^i = \Lambda^i(M)$ for $i > 2n$ (Remark 2.3).

It is easy to see that the de Rham differential d increases the weight by 1 at most. Therefore, $dV^i \subset V^{i+1}$, and $V^* \subset \Lambda^*(M)$ is a differential ideal in the de Rham DG-algebra $(\Lambda^*(M), d)$.

DEFINITION 2.4 [Ver02]. Denote by $(\Lambda_+^*(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$. It is called the *quaternionic Dolbeault algebra* of M , or the *quaternionic Dolbeault complex* (qD-algebra or qD-complex for short).

The space $\Lambda_+^i(M)$ can be identified with the maximal subspace of $\Lambda^i(M)$ of weight i , that is, a sum of all irreducible subrepresentations of weight i . This way, $\Lambda_+^i(M)$ can be considered as a subspace in $\Lambda^i(M)$; however, this subspace is not preserved by the multiplicative structure and the differential.

Remark 2.5. The complex $(\Lambda_+^*(M), d_+)$ was constructed much earlier by Salamon [Sal07], in a different (and much more general) situation, and has been much studied since then [CS88, Bas92, LY97].

2.3 The Hodge decomposition on the quaternionic Dolbeault complex

Let (M, I, J, K) be a hypercomplex manifold, and L a complex structure induced by the quaternionic action, say, I, J or K . Consider the $U(1)$ -action on $\Lambda^1(M)$ provided by $\varphi \xrightarrow{\rho_L} \cos \varphi \text{ Id} + \sin \varphi \cdot L$. We extend this action to a multiplicative action on $\Lambda^*(M)$. Clearly, for a (p, q) -form $\eta \in \Lambda^{p,q}(M, L)$, we have

$$\rho_L(\varphi)\eta = e^{\sqrt{-1}(p-q)\varphi}\eta. \tag{2.1}$$

LEMMA 2.6. *Let (M, I, J, K) be a hypercomplex manifold and*

$$\rho_I, \rho_J, \rho_K$$

the homomorphisms

$$U(1) \rightarrow \text{Aut}(\Lambda^*(M))$$

constructed above. Then ρ_I, ρ_J, ρ_K generate the Lie group action

$$\text{SU}(2) \subset \text{Aut}(\Lambda^*(M))$$

associated with the hypercomplex structure.

Proof. Lemma 2.6 is clear. Indeed, the action of $\text{SU}(2)$ and ρ_I, ρ_J, ρ_K are defined on $\Lambda^*(M)$ by multiplicativity, hence it suffices to check that ρ_I, ρ_J, ρ_K generate the standard action of $\text{SU}(2)$ on $\Lambda^1(M)$. On $\Lambda^1(M)$, ρ_I, ρ_J, ρ_K act as quaternions $\cos \varphi + \sin \varphi \cdot I, \cos \varphi + \sin \varphi \cdot J, \cos \varphi + \sin \varphi \cdot K$, and they generate the group of unitary quaternions. \square

From Lemma 2.6, it is clear that ρ_L preserves components of weight i . We obtain that V^* is preserved by ρ_L , hence ρ_L acts on $\Lambda_+^*(M)$. Then, (2.1) gives a Hodge decomposition on $\Lambda_+^*(M)$:

$$\Lambda_+^i(M) = \bigoplus_{p+q=i} \Lambda_{+,L}^{p,q}(M).$$

The following result is implied immediately by the standard calculations from the theory of $\text{SU}(2)$ -representations.

PROPOSITION 2.7. *Let (M, I, J, K) be a hypercomplex manifold and*

$$\Lambda_+^i(M) = \bigoplus_{p+q=i} \Lambda_{+,I}^{p,q}(M)$$

the Hodge decomposition of qD -complex defined above. Then there is a natural isomorphism

$$\Lambda_{+,I}^{p,q}(M) \cong \Lambda^{0,p+q}(M, I). \tag{2.2}$$

Proof. The following lemma is clear. \square

LEMMA 2.8. *Let (M, I, J, K) be a hypercomplex manifold, $\dim_{\mathbb{H}} M = n$, and p an integer, $0 \leq p \leq 2n$. Then $\Lambda^{0,p}(M, I) \subset \Lambda^p(M)$ is pure of weight p .*

Proof. Consider the operator $W_I : \Lambda^*(M) \rightarrow \Lambda^*(M)$ mapping a form $\eta \in \Lambda^{p,q}(M, I)$ to $\sqrt{-1}(p-q)\eta$. Clearly, W_I acts as a generator of $\mathfrak{u}(1)$, with $\mathfrak{u}(1)$ associated to $\rho_I : U(1) \rightarrow \text{End}(\Lambda^*(M))$. By Lemma 2.6, $W_I \in \mathfrak{su}(2)$, where the $\mathfrak{su}(2)$ -action on $\Lambda^*(M)$ is associated with the standard action of $SU(2)$. Writing $\mathfrak{su}(2)$ explicitly in terms of generators W_I, W_J and W_K , we find that W_I generates a Cartan subalgebra of $\mathfrak{su}(2)$ (indeed, the corresponding Lie group is a maximal compact torus of $SU(2)$). Since the Cartan algebra $\mathbb{C} \cdot W_I$ acts on $\Lambda^{p,0}(M, I)$ with weight p , the space $\Lambda^{p,0}(M, I)$ is of weight at least p . On the other hand, $\Lambda^p(M)$ is a representation of weight at most p (Remark 2.3). Therefore, $\Lambda^{p,0}(M, I)$ is pure of weight p . \square

Remark 2.9. This argument also implies that $\Lambda^{0,p}(M, I)$ coincides with $\Lambda_{+,I}^{0,p}(M) \subset \Lambda_+^p(M)$ (here we consider $\Lambda_+^p(M)$ as a maximal $SU(2)$ -invariant subspace of weight p in $\Lambda^p(M)$).

Now, Proposition 2.7 is implied by the general machinery of $SU(2)$ -representations. If R is a finite-dimensional $SU(2)$ -representation of weight at most p , the Cartan algebra action splits R onto weight components $R = \bigoplus R_i, i = -p, -p + 2, \dots, p - 2, p$ the weights of the root $\sqrt{-1}W_I$ acting on R_i as a multiplication by i . Moreover, if R is pure of weight p , then all spaces R_i are naturally isomorphic, with isomorphism provided by the $SU(2)$ -action.

In the case $R = \Lambda_+^p(M)$, the decomposition $R = \bigoplus R_i$ is precisely the Hodge decomposition, hence the spaces $\Lambda_{+,I}^{p,q}(M)$ are naturally isomorphic for all $p, q \geq 0$ satisfying $p + q = i$. We have proved Proposition 2.7.

2.4 The Hodge decomposition on qD-complex: an explicit construction

The isomorphism (2.2) can be made explicit, and also multiplicative, in the following way. Let \mathfrak{R} be an irreducible two-dimensional representation of $SU(2)$. Clearly, any irreducible $SU(2)$ -representation of weight p is isomorphic to $S^p\mathfrak{R}$ (the p th symmetric power of \mathfrak{R}). Consider the root $\sqrt{-1}W_I \in \mathfrak{su}(2)$, constructed in § 2.3. The corresponding $\mathfrak{sl}(2)$ -triple can be written as

$$f = W_J + \sqrt{-1}W_K, \quad g = W_J - \sqrt{-1}W_K, \quad h = \sqrt{-1}W_I.$$

Let x, y be a basis in \mathfrak{R} , such that $hx = x, hy = -y, gx = y, fy = x$.

Consider a hypercomplex manifold (M, I, J, K) . The bundle

$$\mathfrak{S} := \bigoplus_p S^p\mathfrak{R} \otimes \Lambda^{0,p}(M, I) \tag{2.3}$$

is equipped with a natural multiplicative structure (we assume that the elements of $S^p\mathfrak{R}$ and $\Lambda^{0,q}(M, I)$ commute). We define the following $SU(2)$ -action on \mathfrak{S} : $SU(2)$ acts trivially on $\Lambda^{0,p}(M, I)$, and in a standard way on $S^p\mathfrak{R}$.

Consider an isomorphism $\mathfrak{R} \otimes \Lambda^{0,1}(M, I) \rightarrow \Lambda^1(M)$ mapping $x \otimes \eta$ to $J(\eta)$ and $y \otimes \eta$ to η . This map is clearly $SU(2)$ -invariant. Using the multiplicative structure on \mathfrak{S} , it can be extended to an $SU(2)$ -invariant algebra homomorphism

$$\bigoplus_p S^p\mathfrak{R} \otimes \Lambda^{0,p}(M, I) \rightarrow \Lambda_+^*(M). \tag{2.4}$$

PROPOSITION 2.10. *In these assumptions, (2.4) is an algebra isomorphism.*

Proof. Let $\mathfrak{S}^p \subset \mathfrak{S}$ denote the grading p component. Bijectivity of the map (2.4) is checked in the same way as one proves Proposition 2.7: the Hodge components of \mathfrak{S}^p are all isomorphic, because \mathfrak{S}^p is a pure representation of weight p , and the same is true for $\Lambda_+^p(M)$. Therefore, it suffices to prove that the restriction of (2.4) to one Hodge component, say, $y^p\Lambda^{0,p}(M, I)$, induces an isomorphism

$$y^p\Lambda^{0,p}(M, I) \rightarrow \Lambda_{+,I}^{0,p}(M).$$

This is implied by the equality $\Lambda^{0,p}(M, I) = \Lambda_{+,I}^{0,p}(M)$ (Remark 2.9). \square

2.5 The $\bar{\partial}_J$ -operator

Let (M, I, J, K) be a hypercomplex manifold. We extend

$$J : \Lambda^1(M) \rightarrow \Lambda^1(M)$$

to $\Lambda^*(M)$ by multiplicativity. Recall that

$$J(\Lambda^{p,q}(M, I)) = \Lambda^{q,p}(M, I),$$

because I and J anticommute on $\Lambda^1(M)$. Denote by

$$\bar{\partial}_J : \Lambda^{p,q}(M, I) \rightarrow \Lambda^{p,q+1}(M, I)$$

the operator $J \circ \partial \circ J$, where $\partial : \Lambda^{p,q}(M, I) \rightarrow \Lambda^{p+1,q}(M, I)$ is the standard Dolbeault operator on (M, I) , that is, the $(1, 0)$ part of the de Rham differential. Since $\partial^2 = 0$, we have $\bar{\partial}_J^2 = 0$. Since I, J, K are integrable, the operators $d, d_I := I \circ d \circ I, d_J := J \circ d \circ J, d_K := K \circ d \circ K$ pairwise anticommute. Therefore, $\bar{\partial} = \frac{1}{2}(d - \sqrt{-1}d_I)$ anticommutes with $\bar{\partial}_J = \frac{1}{2}(d_J - \sqrt{-1}d_K)$. Writing the supercommutator as $\{\cdot, \cdot\}$, we express this as

$$\{\bar{\partial}_J, \bar{\partial}_J\} = 0, \quad \{\bar{\partial}_J, \bar{\partial}\} = 0. \tag{2.5}$$

2.6 The $\bar{\partial}, \bar{\partial}_J$ -bicomplex

Consider the quaternionic Dolbeault complex $(\Lambda_+^*(M), d_+)$ constructed in §2.2. Using the Hodge decomposition, we can represent this complex as

$$\begin{array}{ccccc}
 & & \Lambda_{+,I}^0(M) & & \\
 & & \swarrow & \searrow & \\
 & & d_{+,I}^{1,0} & & d_{+,I}^{0,1} \\
 & & \swarrow & \searrow & \\
 \Lambda_{+,I}^{1,0}(M) & & & & \Lambda_{+,I}^{0,1}(M) & \tag{2.6} \\
 \swarrow & & \searrow & \swarrow & \searrow & \\
 d_{+,I}^{1,0} & & d_{+,I}^{0,1} & & d_{+,I}^{1,0} & & d_{+,I}^{0,1} \\
 \swarrow & & \searrow & \swarrow & \searrow & \\
 \Lambda_{+,I}^{2,0}(M) & & \Lambda_{+,I}^{1,1}(M) & & \Lambda_{+,I}^{0,2}(M)
 \end{array}$$

where $d_{+,I}^{1,0}$ and $d_{+,I}^{0,1}$ are the Hodge components of the quaternionic Dolbeault differential d_+ , taken with respect to I .

Consider a hypercomplex manifold (M, I, J, K) . Let

$$\bigoplus_p S^p \mathfrak{K} \otimes \Lambda^{0,p}(M, I) \rightarrow \Lambda_+^*(M) \tag{2.7}$$

be the isomorphism constructed in Proposition 2.10. Writing the basis x, y of \mathfrak{K} as in the proof of Proposition 2.10, we may write the Hodge decomposition of (2.7) as

$$x^p y^q \Lambda^{0,p+q}(M, I) \cong \Lambda_{+,I}^{p,q}(M).$$

THEOREM 2.11. Under this correspondence, $d_+^{0,1}$ corresponds to $\bar{\partial}$ and $d_+^{1,0}$ to $\bar{\partial}_J$. This way the bicomplex (2.6) becomes equivalent to the bicomplex $(\Lambda^{0,p}(M, I), \bar{\partial}, \bar{\partial}_J)$ as follows:

$$\begin{array}{ccc}
 \Lambda_{+,I}^0(M) & & \Lambda_I^{0,0}(M) \\
 \swarrow d_{+,I}^{1,0} \quad \searrow d_{+,I}^{0,1} & & \swarrow x\bar{\partial}_J \quad \searrow y\bar{\partial} \\
 \Lambda_{+,I}^{1,0}(M) \quad \Lambda_{+,I}^{0,1}(M) & \cong & x\Lambda_I^{0,1}(M) \quad y\Lambda_I^{0,1}(M) \\
 \swarrow d_{+,I}^{1,0} \quad \searrow d_{+,I}^{0,1} \quad \swarrow d_{+,I}^{1,0} \quad \searrow d_{+,I}^{0,1} & & \swarrow x\bar{\partial}_J \quad \searrow y\bar{\partial} \quad \swarrow x\bar{\partial}_J \quad \searrow y\bar{\partial} \\
 \Lambda_{+,I}^{0,2}(M) \quad \Lambda_{+,I}^{1,1}(M) \quad \Lambda_{+,I}^{0,2}(M) & & x^2\Lambda_I^{0,2}(M) \quad xy\Lambda_I^{0,2}(M) \quad y^2\Lambda_I^{0,2}(M)
 \end{array} \tag{2.8}$$

Proof. Consider the action of $x\bar{\partial}_J + y\bar{\partial}$ on

$$\bigoplus_p S^p \mathfrak{K} \otimes \Lambda^{0,p}(M, I) \cong \Lambda_+^*(M)$$

defined as in (2.8). To prove Theorem 2.11, we need to show that

$$x\bar{\partial}_J + y\bar{\partial} = d_+ \tag{2.9}$$

Both of these operators satisfy the Leibniz rule, hence it suffices to check (2.9) on some set of multiplicative generators of $\Lambda_+^*(M)$. On $\Lambda_+^0(M)$, the equality (2.9) is clear from the definitions:

$$x\bar{\partial}_J + y\bar{\partial}|_{\Lambda_+^0(M)} = d_+|_{\Lambda_+^0(M)}. \tag{2.10}$$

It is easy to check that the space $\Lambda^0(M) \oplus d\Lambda^0(M)$ generates the algebra $\Lambda^*(M)$. Therefore, $\Lambda_+^0(M) \oplus d_+\Lambda_+^0(M)$ generates $\Lambda_+^*(M)$. To prove Theorem 2.11 it remains to show that

$$x\bar{\partial}_J + y\bar{\partial}|_{d_+\Lambda_+^0(M)} = d_+|_{d_+\Lambda_+^0(M)}. \tag{2.11}$$

Since $d_+^2 = 0$, we have $d_+|_{d_+\Lambda_+^0(M)} = 0$. By (2.5), we have

$$(x\bar{\partial}_J + y\bar{\partial})^2 = 0. \tag{2.12}$$

Using (2.10) and (2.12), we obtain

$$x\bar{\partial}_J + y\bar{\partial}|_{d_+\Lambda_+^0(M)} = x\bar{\partial}_J + y\bar{\partial}|_{x\bar{\partial}_J + y\bar{\partial}(\Lambda_+^0(M))} = 0.$$

Therefore, we have

$$x\bar{\partial}_J + y\bar{\partial}|_{d_+\Lambda_+^0(M)} = d_+|_{d_+\Lambda_+^0(M)} = 0.$$

This proves (2.11). Theorem 2.11 is proven. □

3. Kodaira identities for qD-complex

3.1 The Lefschetz-type $\mathfrak{sl}(2)$ -action on $\Lambda^{0,*}(M, I) \otimes \text{End}(B)$

Let (M, I, J, K) be a hyperkähler manifold, B a holomorphic Hermitian vector bundle on (M, I) , and $\Lambda^{0,*}(M, I) \otimes B$ the space of $(0, p)$ -forms with coefficients in B . Denote by $\bar{\Omega} \in \Lambda^{0,2}(M, I)$ the standard $(0, 2)$ -form $\omega_J + \sqrt{-1}\omega_K$ (see § 1.1).

Using a hyperkähler metric, we construct a natural Hermitian structure on $\Lambda^{0,*}(M, I) \otimes B$. Denote by $L_{\bar{\Omega}} : \Lambda^{q,p}(M, I) \rightarrow \Lambda^{q,p+2}(M, I)$ the operator of exterior multiplication by $\bar{\Omega}$, and by $\Lambda_{\bar{\Omega}} : \Lambda^{q,p}(M, I) \otimes B \rightarrow \Lambda^{q,p-2}(M, I) \otimes B$ its Hermitian adjoint. The same argument as proves the usual Lefschetz theorem about the $\mathfrak{sl}(2)$ -action (see [GH78]) can be used to prove the following linear-algebraic result, which is due to Fujiki [Fuj85].

PROPOSITION 3.1 [Fuj85]. *With the above assumptions, let*

$$H_{\overline{\Omega}} := [L_{\overline{\Omega}}, \Lambda_{\overline{\Omega}}]$$

be a commutator of $L_{\overline{\Omega}}, \Lambda_{\overline{\Omega}}$. Then $H_{\overline{\Omega}}$ is a scalar operator, multiplying a (q, p) -form by $n - p$, where $n = \frac{1}{2} \dim \mathbb{H}(M)$. Moreover, $L_{\overline{\Omega}}, \Lambda_{\overline{\Omega}}, H_{\overline{\Omega}}$ is an $\mathfrak{sl}(2)$ -triple.

Proof. For the proof, see [Ver96a, Theorem 4.2]. □

Let $\theta \in \Lambda^{0,1}(M, I) \otimes \text{End}(B)$ be a 1-form. Denote by

$$L_{\theta} : \Lambda^{q,p}(M, I) \otimes B \rightarrow \Lambda^{q,p+1}(M, I) \otimes B$$

the operator of multiplication by θ , and let

$$\Lambda_{\theta} : \Lambda^{q,p}(M, I) \rightarrow \Lambda^{q,p-1}(M, I)$$

be its Hermitian adjoint. Denote by θ_J the $(0, 1)$ -form $J(\overline{\theta})$.

CLAIM 3.2. *In the above assumptions, we have*

$$[L_{\overline{\Omega}}, \Lambda_{\theta}] = L_{\theta_J}. \tag{3.1}$$

Proof. This follows from a trivial computation. □

3.2 $\overline{\partial}, \overline{\partial}_J$ with coefficients in a bundle

Let (M, I, J, K) be a hyperkähler manifold, and B a holomorphic Hermitian vector bundle on (M, I) . Consider the standard (Chern) Hermitian connection ∇ on B , $\nabla = \nabla^{1,0} + \overline{\partial}$, where $\overline{\partial} : B \rightarrow B \otimes \Lambda^{0,1}(M, I)$ is the holomorphic structure operator. Denote by $\overline{\partial}_J : B \rightarrow B \otimes \Lambda^{0,1}(M, I)$ the composition of $\nabla^{1,0} : B \otimes \Lambda^{1,0}(M, I)$ and let

$$\text{Id}_B \otimes J : B \otimes \Lambda^{1,0}(M, I) \rightarrow B \otimes \Lambda^{0,1}(M, I)$$

be an endomorphism associated with $J \in \mathbb{H}$. We extend $\overline{\partial}, \overline{\partial}_J$ to operators

$$\overline{\partial}, \overline{\partial}_J : \Lambda^{0,p}(M, I) \otimes B \rightarrow \Lambda^{0,p+1}(M, I) \otimes B$$

using the Leibniz rule.

PROPOSITION 3.3. *With these assumptions, $\overline{\partial}^2 = \overline{\partial}_J^2 = 0$, and the anticommutator $\{\overline{\partial}, \overline{\partial}_J\}$ acts on $\Lambda^{0,*}(M, I)$ as a multiplication by an $\text{End}(B)$ -valued 2-form $\Theta_+ \in \Lambda^{0,2}(M, I) \otimes \text{End} B$. Moreover, under the identification*

$$\Lambda^{0,2}(M, I) \otimes \text{End} B \cong \Lambda_{+,J}^{1,1}(M) \otimes \text{End} B$$

(Proposition 2.7), Θ_+ corresponds to the $\Lambda_+^2(M)$ part of the curvature of B .

Proof. Let

$$\nabla_+ : \Lambda_+^p(M) \otimes B \rightarrow \Lambda_+^{p+1}(M) \otimes B$$

be the connection operator restricted to $\Lambda_+^*(M) \otimes B$, and $\nabla_+ = \nabla_+^{1,0} + \partial_+$ its Hodge decomposition. Clearly, ∇_+^2 is the $\Lambda_+^2(M)$ part of the curvature of B .

Now, Proposition 3.3 follows immediately from Theorem 2.11. Indeed, under the isomorphism (2.8), $x\overline{\partial}_J$ corresponds to $\nabla_+^{1,0}$; since the curvature of the Chern connection is of type $(1, 1)$, we have $(\nabla_+^{1,0})^2 = 0$, hence $\overline{\partial}_J^2 = 0$. Similarly, the operator $\{x\overline{\partial}_J, y\overline{\partial}\}$ under the isomorphism (2.8) corresponds to $\{\nabla_+^{1,0}, \partial_+\} = \nabla_+^2$. □

3.3 Kodaira relations for $\bar{\partial}, \bar{\partial}_J$

Let (M, I, J, K) be a hyperkähler manifold. Consider the bicomplex

$$(\Lambda^{0,*}(M, I), \bar{\partial}, \bar{\partial}_J),$$

constructed in § 2.6. Let

$$L_{\bar{\Omega}} : \Lambda^{0,p}(M, I) \rightarrow \Lambda^{0,p+2}(M, I)$$

be an operator of exterior multiplication by $\bar{\Omega}$ (§ 3.1), and

$$\bar{\partial}^*, \bar{\partial}_J^* : \Lambda^{0,p}(M, I) \rightarrow \Lambda^{0,p-1}(M, I)$$

the operators Hermitian adjoint to $\bar{\partial}, \bar{\partial}_J$.

The following proposition is well known.

PROPOSITION 3.4. *With these assumptions, the following commutator relations hold*

$$[L_{\bar{\Omega}}, \bar{\partial}^*] = \bar{\partial}_J, \quad [L_{\bar{\Omega}}, \bar{\partial}_J^*] = -\bar{\partial}. \tag{3.2}$$

Proof. The proof of (3.2) is essentially the same as the proof of the usual Kodaira relations; see e.g. [Ver96a, Proposition 4.2]. □

The same argument, applied locally to $\text{End}(B)$ -valued forms, gives the following theorem.

THEOREM 3.5. *Let (M, I, J, K) be a hyperkähler manifold, B a holomorphic Hermitian vector bundle on (M, I) ,*

$$\bar{\partial}, \bar{\partial}_J : \Lambda^{0,p}(M, I) \otimes B \rightarrow \Lambda^{0,p+1}(M, I) \otimes B$$

the operators constructed in § 3.2, and $\bar{\partial}^, \bar{\partial}_J^*$ the Hermitian adjoint operators. Then*

$$[L_{\bar{\Omega}}, \bar{\partial}^*] = \bar{\partial}_J, \quad [L_{\bar{\Omega}}, \bar{\partial}_J^*] = -\bar{\partial}. \tag{3.3}$$

Proof. For this, see [Ver96a]. □

3.4 Kodaira–Nakano identities

The following theorem is the qD-analogue of the usual Kodaira–Nakano identity (or, rather, the identity used in the proof of Kodaira–Nakano vanishing).

THEOREM 3.6. *Let (M, I, J, K) be a hyperkähler manifold, B a holomorphic Hermitian vector bundle on (M, I) ,*

$$\bar{\partial}, \bar{\partial}_J : \Lambda^{0,p}(M, I) \otimes B \rightarrow \Lambda^{0,p+1}(M, I) \otimes B$$

the operators constructed in § 3.2, and $\bar{\partial}^, \bar{\partial}_J^*$ the Hermitian adjoint operators. Consider the Laplacians*

$$\Delta_{\bar{\partial}} := \{\bar{\partial}, \bar{\partial}^*\}, \quad \Delta_{\bar{\partial}_J} := \{\bar{\partial}_J, \bar{\partial}_J^*\}$$

(here, as elsewhere, $\{\cdot, \cdot\}$ denotes the anticommutator). Then

$$\Delta_{\bar{\partial}} - \Delta_{\bar{\partial}_J} = [\Theta_+, \Lambda_{\bar{\Omega}}], \tag{3.4}$$

where

$$\Theta_+ : \Lambda^{0,p}(M, I) \otimes B \rightarrow \Lambda^{0,p+2}(M, I)$$

is an operator defined as

$$\Theta_+ := \{\bar{\partial}, \bar{\partial}_J\}$$

and identified with the $\Lambda_+^2(M) \otimes \text{End } B$ part of the curvature of B as in Proposition 3.3.

Proof. Using the graded Jacobi identity and Theorem 3.5, we obtain

$$[\Theta_+, \Lambda_{\bar{\Omega}}] = -[\Lambda_{\bar{\Omega}}, \{\bar{\partial}, \bar{\partial}_J\}] = \{\bar{\partial}, \bar{\partial}^*\} - \{\bar{\partial}_J, \bar{\partial}_J^*\} = \Delta_{\bar{\partial}} - \Delta_{\bar{\partial}_J}. \tag{3.4} \quad \square$$

4. Cohomology of hyperkähler manifolds

For the convenience of the reader, we recall here some well-known facts about the structure of $H^2(M)$ for M a compact, irreducible hyperkähler manifold; see [Bog78], [Bes87], [Bea83] and [Fuj85] for details.

4.1 $SU(2)$ -action on $H^2(M)$

Let (M, I, J, K, g) be a compact, irreducible hyperkähler manifold. Since g is Kähler with respect to (I, J, K) , we have

$$\nabla I = \nabla J = \nabla K = 0,$$

where ∇ denotes the Levi-Civita connection. Chern has shown that covariantly constant endomorphisms of $\Lambda^*(M)$ commute with the Laplacian (see [Bes87]). Then the $SU(2)$ -action generated by $I, J, K \in \mathbb{H}^*$ also commutes with the Laplacian. This gives an $SU(2)$ -action on the space of harmonic forms on M . Identifying the harmonic forms with cohomology, we obtain an $SU(2)$ -action on the cohomology as well.

Let $H^2(M) = H^2_+(M) \oplus H^2(M)_{SU(2)\text{-inv}}$ be a decomposition of $H^2(M)$ onto its weight 2 and weight 0 components. Using the weights of the Cartan algebra action as in the proof of Proposition 2.7, we find that

$$\dim H^{2,0}(M, I) = \dim H^{1,1}_+(M, I) = \dim H^{0,2}(M, I).$$

Since M is irreducible, $\dim H^{2,0}(M, I) = 1$ and the space $H^{1,1}_+(M, I)$ is one-dimensional. Let $H^2(M)_{SU(2)\text{-inv}}$ be the space of $SU(2)$ -invariant classes. It is easy to check that $SU(2)$ -invariant classes are all of type $(1, 1)$ (e.g. [Ver96a]).

Since $H^{1,1}_+(M, I)$ is one-dimensional and generated by the Kähler form ω_I , we have a decomposition

$$H^{1,1}(M, I) = \mathbb{C}\omega_I \oplus H^2(M)_{SU(2)\text{-inv}}. \tag{4.1}$$

Using the $\mathfrak{so}(1, 4)$ -action generated by the three Lefschetz $\mathfrak{sl}(2)$ -triples associated with the Kähler structures I, J, K as in [Ver90], we can easily show that an $SU(2)$ -invariant 2-form is primitive² (see e.g. [Ver96a]).

This gives the following well-known statement [Ver96a].

CLAIM 4.1. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold. Then the space $H^{1,1}_{\text{prim}}(M, I)$ of primitive classes in $H^{1,1}(M, I)$ coincides with the space $H^2(M)_{SU(2)\text{-inv}}$ of $SU(2)$ -invariant classes.*

Proof. Since all $SU(2)$ -invariant classes are primitive, $H^{1,1}_{\text{prim}}(M, I)$ contains $H^2(M)_{SU(2)\text{-inv}}$. Comparing the decomposition (4.1) with

$$H^{1,1}(M, I) = H^{1,1}_{\text{prim}}(M, I) \oplus \mathbb{C}\omega_I,$$

we find that $\dim H^{1,1}_{\text{prim}}(M, I) = \dim H^2(M)_{SU(2)\text{-inv}}$. □

4.2 The Bogomolov–Beauville–Fujiki form

Let (M, I, J, K) be a compact hyperkähler manifold, and $\Omega := \omega_J + \sqrt{-1}\omega_K$ the holomorphic symplectic form on (M, I) . Bogomolov [Bog78] defined the following bilinear symmetric 2-form

²Recall that the *primitive classes* [GH78] are cohomology classes which satisfy $\Lambda(\eta) = 0$, where $\Lambda : H^i(M) \rightarrow H^{i-2}(M)$ is the dual Lefschetz operator. A $(1, 1)$ -class is primitive if and only if it is orthogonal to the Kähler form with respect to the Riemann–Hodge pairing.

on $H^{1,1}(M, I)$:

$$\tilde{q}(\eta, \eta') := \int_M \eta \wedge \eta' \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1}, \tag{4.2}$$

where $n = \dim \mathbb{H}M$. Since $\Omega \wedge \overline{\Omega}$ is a positive $(2,2)$ -form, \tilde{q} is positive on the Kähler cone of (M, I) :

$$\forall \omega \in \mathcal{K}, \quad \tilde{q}(\omega, \omega) > 0. \tag{4.3}$$

An elementary linear-algebraic calculation similar to the proof of Riemann–Hodge bilinear relations implies that $\tilde{q}(\eta, \eta) < 0$ for η primitive. Therefore, \tilde{q} has signature $(+, -, -, -, \dots)$ on $H^{1,1}(M, I) \cap H^2(M, \mathbb{R})$.

The form \tilde{q} is topological by its nature.

THEOREM 4.2 [Fuj85]. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold of real dimension $4n$. Then there exists a bilinear, symmetric non-degenerate 2-form $q : H^2(M, \mathbb{Q}) \otimes H^2(M, \mathbb{Q}) \rightarrow \mathbb{Q}$ such that*

$$\int_M \eta^{2n} = q(\eta, \eta)^n, \tag{4.4}$$

for all $\eta \in H^2(M)$. Moreover, q is proportional to the form (4.2) on $H^{1,1}(M)$, and has signature $(+, +, +, -, -, -, \dots)$.

Remark 4.3. If n is odd, (4.4) determines q uniquely, otherwise up to a sign. To choose a sign, we use (4.3).

DEFINITION 4.4. Let (M, I, J, K) be a compact, irreducible hyperkähler manifold. A *Bogomolov–Beauville–Fujiki form* on M is a form

$$q : H^2(M, \mathbb{Q}) \otimes H^2(M, \mathbb{Q}) \rightarrow \mathbb{Q}$$

which satisfies (4.4), and takes positive values on the Kähler cone of (M, I) . Such a form always exists and is unique, by Theorem 4.2.

Remark 4.5. The Bogomolov–Beauville–Fujiki form is integer, but not unimodular on $H^2(M, \mathbb{Z})$.

The Bogomolov–Beauville–Fujiki form can be expressed in terms of the $SU(2)$ -action on cohomology (§ 4.1) as follows.

CLAIM 4.6. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, and $(\cdot, \cdot)_{\mathcal{H}}$ the positive definite pairing on cohomology associated with the Euclidean metric on the space of harmonic forms induced by the Riemannian structure. Consider the form q' which is equal to $(\cdot, \cdot)_{\mathcal{H}}$ on the three-dimensional space generated by $\omega_I, \omega_J, \omega_K$, and to $-(\cdot, \cdot)_{\mathcal{H}}$ on its orthogonal complement. Then q' is proportional to the Bogomolov–Beauville–Fujiki form.*

Proof. For the proof, see e.g. [Ver96b, Theorem 2.1]. □

This immediately gives the following corollary.

COROLLARY 4.7. *Consider the natural $SU(2)$ -action on the cohomology of a hyperkähler manifold. Then the Bogomolov–Beauville–Fujiki form is $SU(2)$ -invariant.*

Using the Riemann–Hodge bilinear relations, we can express $(\cdot, \cdot)_{\mathcal{H}}$ in terms of the product structure on cohomology. Together with Claim 4.6, this gives

$$q'(\eta_1, \eta_2) = \int_X \omega_I^{2n-2} \wedge \eta_1 \wedge \eta_2 - \frac{2n-2}{(2n-1)^2} \cdot \frac{\int_X \omega_I^{2n-1} \eta_1 \cdot \int_X \omega_I^{2n-1} \eta_2}{\int_X \omega_I^n}, \tag{4.5}$$

for any $\eta_1, \eta_2 \in H^2(M)$ (see [Ver95, Claim 5.1]). This formula is due to Beauville.

The following claim follows directly from (4.5) and Claim 4.1.

CLAIM 4.8. *Let (M, I, J, K) be an irreducible, compact hyperkähler manifold, and $\eta \in H^{1,1}(M, I)$ a $(1, 1)$ -class. Then the following assertions are equivalent:*

- (i) $q(\eta, \omega_I) = 0$, where q is the Bogomolov–Beauville–Fujiki form, and ω_I the Kähler class of (M, I) ;
- (ii) η is primitive;
- (iii) η is $SU(2)$ -invariant.

Proof. The equivalence of assertions (ii) and (iii) is implied by Claim 4.1, and the equivalence of assertions (i) and (iii) by (4.5). □

5. The vanishing of cohomology

5.1 Cohomology vanishing for line bundles with $q(c_1(L), \omega) > 0$

The following result is implied immediately by the quaternionic Kodaira–Nakano identity (Theorem 3.6), in the same fashion as the usual Kodaira–Nakano vanishing follows from the usual Kodaira–Nakano identity.

PROPOSITION 5.1. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, $\dim_{\mathbb{R}} M = 4n$, and L a holomorphic line bundle on (M, I) , such that $q(c_1(L), \omega_I) > 0$, where ω_I is the Kähler class of (M, I) . Then the holomorphic cohomology $H^i(M, L)$ is zero for $i > n$.*

Proof. Let η be a harmonic form representing $c_1(L)$. We may choose the Hermitian structure on L in such a way that η is equal to the curvature of L (see [GH78]). Let ω denote the Kähler form of (M, I) . Abusing the notation, we denote the Kähler class of (M, I) by the same letter.

The cohomology class

$$\kappa := c_1(L) - \frac{q(c_1(L), \omega)}{q(\omega, \omega)}\omega$$

clearly satisfies $q(\kappa, \omega) = 0$. Therefore, κ is $SU(2)$ -invariant (Claim 4.8). Since η is harmonic, the harmonic form

$$\eta - \frac{q(c_1(L), \omega)}{q(\omega, \omega)}\omega \tag{5.1}$$

representing κ is also $SU(2)$ -invariant. Let $\tilde{\omega}$ be the form

$$\omega \in \Lambda_{+, I}^{1,1}(M)$$

considered as an element in $\Lambda^{0,2}(M, I)$ using the isomorphism constructed in Proposition 2.7. By Proposition 3.3,

$$\Theta_+ = \{\bar{\partial}, \bar{\partial}_J\} = \frac{q(c_1(L), \omega)}{q(\omega, \omega)}\tilde{\omega}. \tag{5.2}$$

Clearly, $\omega_I, \omega_J, \omega_K$ form a three-dimensional irreducible $SU(2)$ -invariant subspace of $\Lambda^2(M)$. A trivial calculation is used to show that $\tilde{\omega}$ is in fact equal to $\bar{\Omega}$. This gives

$$\Theta_+ = \{\bar{\partial}, \bar{\partial}_J\} = \lambda L_{\bar{\Omega}}, \tag{5.3}$$

where $\lambda = q(c_1(L), \omega)/q(\omega, \omega)$ is a positive constant. Comparing (5.3), Kodaira–Nakano identity (3.4) and the quaternionic Lefschetz theorem (Proposition 3.1), we obtain

$$\Delta_{\bar{\partial}} - \Delta_{\bar{\partial}_J} = [\Theta_+, \Lambda_{\bar{\Omega}}] = \lambda H_{\bar{\Omega}}. \tag{5.4}$$

On $(0, i)$ -forms this operator acts as $(i - n)\lambda$. Given a harmonic form

$$\nu \in \ker \Delta_{\bar{\partial}} \subset \Lambda^{0,i}(M, I) \otimes L,$$

we can obtain

$$0 = \Delta_{\bar{\partial}}(\nu) = \Delta_{\bar{\partial}_J}(\nu) + (i - n)\lambda\nu. \tag{5.5}$$

Since

$$(\Delta_{\bar{\partial}_J}(\nu), \nu) = (\bar{\partial}_J \eta, \bar{\partial}_J \eta) + (\bar{\partial}_J^* \eta, \bar{\partial}_J^* \eta) \geq 0, \tag{5.6}$$

(5.5) leads to $(\nu, \nu) = 0$ for $i > n$. The harmonic $(0, i)$ -forms are identified with the i th holomorphic cohomology of L as usual. We have proved Proposition 5.1. \square

Remark 5.2. Let W be a Hermitian vector space. A *positive operator* $A : W \rightarrow W$ is an operator which satisfies $(A(x), x) \geq 0$ for all $x \in W$. Operator A is *positive definite* if this inequality is strict for all non-zero x . From (5.6), we obtain that the Laplacians $\Delta_{\bar{\partial}_J}$ and $\Delta_{\bar{\partial}}$ are positive. If

$$\Delta_{\bar{\partial}} = \Delta_{\bar{\partial}_J} + A,$$

where A is positive definite, then $\ker \Delta_{\bar{\partial}} = 0$. This argument is used quite often in geometry and analysis.

Remark 5.3. Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, $\dim_{\mathbb{H}} M = n$, and L a holomorphic line bundle on (M, I) . Then Serre’s duality gives $H^i(M, L)^* \cong H^{n-i}(M, L^*)$, because the canonical class of M is trivial. Therefore, Proposition 5.1 implies that $H^i(M, L)$ vanishes for all $i < n$ if L is a holomorphic line bundle on (M, I) with $q(c_1(L), \omega) < 0$.

5.2 The dual Kähler cone and vanishing

Let (M, I) be a Kähler manifold.

DEFINITION 5.4. The *Kähler cone* $\mathcal{K} \subset H^{1,1}(M, I)$ for (M, I) is the set of all Kähler classes $\omega \in H_I^{1,1}(M, \mathbb{R})$, where $H_I^{1,1}(M, \mathbb{R})$ denotes the intersection $H^{1,1}(M, I) \cap H^2(M, \mathbb{R})$. Clearly, \mathcal{K} is a convex cone in $H_I^{1,1}(M, \mathbb{R})$.

Now, let (M, I, J, K) be a hyperkähler manifold, $\mathcal{K} \subset H_I^{1,1}(M, \mathbb{R})$ the Kähler cone of (M, I) , and $q : H_I^{1,1}(M, \mathbb{R}) \times H_I^{1,1}(M, \mathbb{R}) \rightarrow \mathbb{R}$ the Bogomolov–Beauville–Fujiki form. We define the *dual Kähler cone*

$$\mathcal{K}^\sim \subset H_I^{1,1}(M, \mathbb{R})$$

as

$$\mathcal{K}^\sim := \{x \in H_I^{1,1}(M, \mathbb{R}) \mid \forall y \in \mathcal{K}, \quad q(x, y) > 0\}.$$

It is an open, convex cone. Since a product of two Kähler forms is positive, we have $\mathcal{K}^\sim \supset \mathcal{K}$.

Denote by $\bar{\mathcal{K}}^\sim$ the closure of \mathcal{K}^\sim in $H_I^{1,1}(M, \mathbb{R})$, and by $-\bar{\mathcal{K}}^\sim$ the opposite cone. Clearly,

$$\bar{\mathcal{K}}^\sim := \{x \in H_I^{1,1}(M, \mathbb{R}) \mid \forall y \in \mathcal{K}, \quad q(x, y) \geq 0\}$$

and

$$-\bar{\mathcal{K}}^\sim := \{x \in H_I^{1,1}(M, \mathbb{R}) \mid \forall y \in \mathcal{K}, \quad q(x, y) \leq 0\}.$$

Proposition 5.1 immediately leads to the following corollary.

COROLLARY 5.5. *Let (M, I) be a compact, irreducible, holomorphically symplectic Kähler manifold, $\dim_{\mathbb{C}} M = 2n$, and L a holomorphic line bundle on (M, I) with $c_1(L) \notin \bar{\mathcal{K}}^\sim$. Then the holomorphic cohomology $H^i(M, L)$ is zero for all $i < n$.*

Now we can prove the main result of this paper.

THEOREM 5.6. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, and L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$. Then one of the following holds:*

- (i) $c_1(L) \in \overline{\mathcal{K}}^\sim$; then $H^i(L) = 0$ for $i > \frac{1}{2} \dim_{\mathbb{C}} M$;
- (ii) $c_1(L) \in -\overline{\mathcal{K}}^\sim$; then $H^i(L) = 0$ for $i < \frac{1}{2} \dim_{\mathbb{C}} M$;
- (iii) $c_1(L)$ does not lie in $-\overline{\mathcal{K}}^\sim \cup \overline{\mathcal{K}}^\sim$; then $H^i(L) = 0$ for $i \neq \frac{1}{2} \dim_{\mathbb{C}} M$.

Proof. Denote $\frac{1}{2} \dim_{\mathbb{C}} M$ by n . Theorem 5.6(iii) is a direct consequence of Corollary 5.5. Indeed, in this case

$$H^i(L) = 0 \quad \text{for } i < n$$

and

$$H^i(L^*) = 0 \quad \text{for } i < n, \tag{5.7}$$

because the Chern classes of both L and L^* do not lie in $\overline{\mathcal{K}}^\sim$. However, by Serre’s duality, (5.7) is equivalent to

$$H^i(L) = 0 \quad \text{for } i > n.$$

Let us prove Theorem 5.6(i). Since $c_1(L) \in \overline{\mathcal{K}}^\sim$, we may assume that $q(c_1(L), \omega) \geq 0$ for all $\omega \in \mathcal{K}$. Unless $q(c_1(L), \omega) = 0$ for all Kähler classes ω , the assertion of Theorem 5.6(i) is obtained from Proposition 5.1. However, if $q(c_1(L), \omega) = 0$ for all Kähler classes, $c_1(L) = 0$, because q is non-degenerate and the Kähler classes generate $H_I^{1,1}(M, \mathbb{R})$. Theorem 5.6(i) is obtained from (ii) by Serre’s duality. \square

The classes $\eta \notin -\overline{\mathcal{K}}^\sim \cup \overline{\mathcal{K}}^\sim$ can also be characterized as follows.

CLAIM 5.7. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, and $\eta \in H_I^{1,1}(M, \mathbb{R})$ a non-zero cohomology class. Then the following conditions are clearly equivalent:*

- (i) $\eta \notin -\overline{\mathcal{K}}^\sim \cup \overline{\mathcal{K}}^\sim$;
- (ii) $q(\eta, \omega_1) > 0$ and $q(\eta, \omega_2) < 0$ for some Kähler forms ω_1, ω_2 on (M, I) ;
- (iii) η is primitive with respect to some Kähler form on (M, I) ; or, equivalently, $q(\eta, \omega) = 0$ (see Claim 4.8);
- (iv) the class η is $SU(2)$ -invariant with respect to some hyperkähler structure (I, J', K') on M .

Proof. The equivalence of conditions (i) and (ii) is clear. The equivalence of conditions (iii) and (iv) is implied by Claim 4.8. The implication (ii) \Rightarrow (iii) is clear, because the Kähler cone is connected, hence from $q(\eta, \omega_1) > 0$ and $q(\eta, \omega_2) < 0$ it follows that $q(\eta, \omega_3) = 0$ for some Kähler form. Finally, (iii) \Rightarrow (ii) is obtained as follows: Given a Kähler class ω , with $q(\eta, \omega) = 0$, take a neighbourhood U of ω in the Kähler cone. The function $U \xrightarrow{v} \mathbb{R}, v(\omega') = q(\eta, \omega')$, is non-zero and linear, hence it takes positive and negative values in any open neighbourhood of ω . \square

5.3 Cohomology vanishing for vector bundles of arbitrary rank

A version of Theorem 5.6 can be stated for holomorphic bundles of arbitrary rank, as follows.

THEOREM 5.8. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$, and B an arbitrary holomorphic vector bundle on (M, I) . Then there exists a sufficiently big number N_0 , such that for any integer $N > N_0$ one of the following holds:*

- (i) $c_1(L) \in \overline{\mathcal{K}}^\sim$; then $H^i(L^N \otimes B) = 0$ for $i > \frac{1}{2} \dim_{\mathbb{C}} M$;
- (ii) $c_1(L) \in -\overline{\mathcal{K}}^\sim$; then $H^i(L^N \otimes B) = 0$ for $i < \frac{1}{2} \dim_{\mathbb{C}} M$;
- (iii) $c_1(L)$ does not lie in $-\overline{\mathcal{K}}^\sim \cup \overline{\mathcal{K}}^\sim$; then $H^i(L^N \otimes B) = 0$ for $i \neq \frac{1}{2} \dim_{\mathbb{C}} M$.

Proof. The proof of Theorem 5.8 is similar to the Kodaira–Nakano vanishing for vector bundles of arbitrary rank. The same argument as used in the proof of Theorem 5.6 can be employed to deduce Theorem 5.8 from the following statement. \square

PROPOSITION 5.9. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$, and B an arbitrary holomorphic vector bundle on (M, I) . Assume that $q(c_1(L), \omega) > 0$, where ω is the Kähler form of (M, I) . Then there exists a sufficiently big number N_0 , such that, for any integer $N > N_0$, $H^i(L^N \otimes B) = 0$ for all $i > \frac{1}{2} \dim_{\mathbb{C}} M$.*

Proof. To prove Proposition 5.9, we use the formula (3.4) again:

$$\Delta_{\bar{\partial}} - \Delta_{\bar{\partial}_J} = -[\Theta_+, \Lambda_{\bar{\Omega}}], \tag{5.8}$$

where $\Delta_{\bar{\partial}}, \Delta_{\bar{\partial}_J}$ are the Laplacians on $L^N \otimes B$, and Θ_+ is the $\Lambda_{+,I}^{1,1}(M) \otimes \text{End}(L^N \otimes B)$ part of the curvature of $L^N \otimes B$, considered as an operator on $\Lambda^{(0,*)}(M) \otimes (L^N \otimes B)$ as in the proof of the quaternionic Dolbeault Kodaira–Nakano identity (Theorem 3.6). Since the curvature is additive on tensor product, we have

$$\Theta_+ = \Theta_B + N\Theta_L,$$

where Θ_B, Θ_L are $\Lambda_+^2(M)$ parts of the curvatures of B and L . The same argument as used in the proof of (5.2) implies that $\Theta_L = \lambda\bar{\Omega}$, where $\lambda = q(c_1(L), \omega)/q(\omega, \omega)$. Then, as the Lefschetz formula (Proposition 3.1) implies,

$$-[\Theta_+, \Lambda_{\bar{\Omega}}] = -[\Theta_B, \Lambda_{\bar{\Omega}}] + V,$$

where V is a scalar operator acting on $(0, i)$ -forms as $\lambda(i-n)N$, $n = \frac{1}{2} \dim_{\mathbb{C}} M$. Clearly, $-[\Theta_B, \Lambda_{\bar{\Omega}}] + V$ is positive definite for N sufficiently big and $i > n$. From Remark 5.2 we obtain immediately that $\ker \Delta_{\bar{\partial}} = 0$ whenever $-[\Theta_B, \Lambda_{\bar{\Omega}}] + V$ is positive definite. This proves Proposition 5.9. \square

6. Vanishing of cohomology and nef classes with $q(\eta, \eta) = 0$

6.1 Nef classes on hyperkähler manifolds

DEFINITION 6.1. Let M be a Kähler manifold, and $\eta \in H^{1,1}(M)$ a real $(1, 1)$ -class. Then η is called *nef* (numerically effective) if η belongs to a closure $\bar{\mathcal{K}}$ of the Kähler cone \mathcal{K} of M . The closure $\bar{\mathcal{K}}$ is called the *nef cone*. A nef line bundle on M is a line bundle with $c_1(L)$ nef; a nef divisor D is one with nef cohomology class.

Consider a compact, irreducible hyperkähler manifold (M, I, J, K) . Let L be a holomorphic line bundle on (M, I) which is nef and satisfies

$$q(c_1(L), c_1(L)) = 0.$$

It was conjectured [GHJ02, Saw03] that L is base point free, that is, defines a holomorphic map

$$(M, I) \rightarrow \mathbb{P}H^0(L^N) \tag{6.1}$$

for N sufficiently big. If this is true, then (6.1) is a Lagrangian fibration onto its image [Mat99]. A special case of this conjecture was recently proven by Matsushita [Mat06]. This motivates our interest in the geometry of nef classes satisfying $q(\eta, \eta) = 0$.

PROPOSITION 6.2. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, $\eta \in H_I^{1,1}(M, \mathbb{R})$ a non-zero nef class on (M, I) , satisfying $q(\eta, \eta) = 0$, and ω a Kähler class on (M, I) . Then we have the following:*

- (i) $q(\omega, \eta) > 0$;
- (ii) choose a positive real number $\varepsilon < q(\eta, \omega)/q(\omega, \omega)$ then $\eta - \varepsilon\omega$ lies outside of $\bar{\mathcal{K}} \cup -\bar{\mathcal{K}}$.

Proof. The proof is clear. □

6.2 A vanishing theorem and its applications

From Proposition 6.2, the following theorem is apparent.

THEOREM 6.3. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, $\dim_{\mathbb{H}} M = n$, and L a non-trivial holomorphic bundle on (M, I) which is nef and satisfies $q(c_1(L), c_1(L)) = 0$. Consider an ample line bundle H on (M, I) . Then there exists N_0 such that, for all integers $N > N_0$,*

$$H^i(L^N \otimes H^*) = 0 \quad \text{for } i \neq n. \tag{6.2}$$

Proof. Let $N_0 = 1/\varepsilon$, where

$$\varepsilon = \frac{q(c_1(L), c_1(H))}{q(c_1(H), c_1(H))}.$$

Then $Nc_1(L) - c_1(H) \notin \overline{\mathcal{K}} \cup -\overline{\mathcal{K}}$ as follows from Proposition 6.2. The vanishing of (6.2) then follows from Theorem 5.6. □

Theorem 6.3 has an immediate corollary.

COROLLARY 6.4. *Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, $\dim_{\mathbb{H}}(M) > 1$, L a non-trivial holomorphic bundle on (M, I) which is nef and satisfies $q(c_1(L), c_1(L)) = 0$, and D an ample divisor on (M, I) . Then, for sufficiently big $N > N_0$, the natural restriction map*

$$H^0(L^N) \rightarrow H^0(L^N|_D)$$

is surjective.

Proof. The following exact sequence is well known:

$$0 \rightarrow L^N(-D) \rightarrow L^N \rightarrow L^N|_D \rightarrow 0.$$

By Theorem 5.6, $H^1(L^N(-D)) = 0$. Then the long exact sequence of cohomology gives

$$0 \rightarrow H^0(L^N(-D)) \rightarrow H^0(L^N) \rightarrow H^0(L^N|_D) \rightarrow 0.$$

This proves Corollary 6.4. □

Corollary 6.4 can be generalized as follows.

THEOREM 6.5. *Let (M, I, J, K) be an irreducible hyperkähler manifold, and $X \subset (M, I)$ a subvariety of dimension $\dim_{\mathbb{C}} X > \frac{1}{2} \dim_{\mathbb{C}} M$. Assume that X is a complete intersection of ample divisors. Consider a holomorphic line bundle L on (M, I) with $c_1(L)$ nef and $q(c_1(L), c_1(L)) = 0$. Then the natural restriction map is surjective on holomorphic sections:*

$$H^0(L^N) \rightarrow H^0(L^N|_X) \rightarrow 0,$$

for a sufficiently big power of L .

Proof. The proof follows the standard pattern of how to generalize an assertion for hypersurfaces to one for complete intersections. □

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