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RESEARCH ARTICLE

Banach spaces for which the space of operators has $2^{\mathfrak{c}}$ closed ideals

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Abstract
We formulate general conditions which imply that $\mathcal{L}(X,Y)$, the space of operators from a Banach space X to a Banach space Y, has $2^{\mathfrak{c}}$ closed ideals, where $\mathfrak{c}$ is the cardinality of the continuum. These results are applied to classical sequence spaces and Tsirelson-type spaces. In particular, we prove that the cardinality of the set ofclosed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$ is exactly $2^{\mathfrak{c}}$ for all $1 < p < q < \infty$.

1. Introduction

Given Banach spaces X and Y, we call a subspace $\mathcal{J}$ of the space $\mathcal{L}(X,Y)$ of bounded operators an *ideal* if $ATB \in \mathcal{J}$ for all $A \in \mathcal{L}(Y)$, $T \in \mathcal{J}$ and $B \in \mathcal{L}(X)$. In the case that $X = Y$, this coincides with the standard algebraic definition of $\mathcal{J}$ being an ideal in the algebra of bounded operators $\mathcal{L}(X)$. In this paper we will only be considering closed ideals. For example, if X and Y are any Banach spaces, then the space of compact operators from X to Y and the space of strictly singular operators from X to Y are both closed ideals in $\mathcal{L}(X,Y)$. In the case of $X = Y = \ell_p$, the compact and strictly singular operators coincide and they are the only closed ideal in $\mathcal{L}(\ell_p)$ other than the trivial cases of $\{0\}$ and the entire space $\mathcal{L}(\ell_p)$. For $p \neq 2$, the situation for L_p is very different from that for ℓ_p. If X contains a complemented subspace Z such that Z is isomorphic to $Z \oplus Z$, then the closure of the set of operators in $\mathcal{L}(X)$ which factor through Z is a closed ideal, and moreover the map that associates this closed ideal with the isomorphism class of Z is injective. In the case $1 < p < \infty$ with $p \neq 2$, there are infinitely many (even uncountably many) distinct complemented subspaces of L_p which are isomorphic to their square [3], and thus there are infinitely many distinct closed ideals in $\mathcal{L}(L_p)$.

Obviously, constructing infinitely many closed ideals for $\mathcal{L}(\ell_p \oplus \ell_q)$ or $\mathcal{L}(\ell_p \oplus c_0)$ with $1 \leqslant p < q < \infty$ requires different techniques than just considering complemented subspaces, and it was a long outstanding question from Pietsch's book [21] whether these spaces have infinitely many distinct closed ideals. For the cases $1 \leqslant p < q < \infty$, the closures of the set of operators which factor through ℓ_p and the operators which factor through ℓ_q are distinct closed ideals (indeed, the only maximal ideals) in $\mathcal{L}(\ell_p \oplus \ell_q)$, and all other proper closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$ correspond to closed ideals in $\mathcal{L}(\ell_p, \ell_q)$. Progress on constructing new ideals in $\mathcal{L}(\ell_p, \ell_q)$ proceeded through building finitely many ideals at a time (see [23] and [25]) until it was shown using finite-dimensional versions of Rosenthal's $X_{p,w}$ spaces that there is chain of a continuum of distinct closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ for all $1 < p < q < \infty$ [27]. For

$1 < p < \infty$, $p \neq 2$, $\ell_p \oplus \ell_2$ is isomorphic to a complemented subspace of L_p, and thus there are at least a continuum of closed ideals in $\mathcal{L}\left(L_p\right)$. Other new constructions for building infinitely many closed ideals soon followed. Wallis observed [31] that the techniques of [27] extend to prove the existence of a chain of a continuum of closed ideals for $\mathcal{L}\left(\ell_p, c_0\right)$ in the range $1 < p < 2$, and for $\mathcal{L}\left(\ell_1, \ell_q\right)$ in the range $2 < q < \infty$. Then, using ordinal indices, Sirotkin and Wallis proved that there is an ω_1-chain of closed ideals in $\mathcal{L}\left(\ell_1, \ell_q\right)$ for $1 < q \leqslant \infty$ as well as in $\mathcal{L}\left(\ell_1, c_0\right)$ and in $\mathcal{L}\left(\ell_p, \ell_\infty\right)$ for $1 \leqslant p < \infty$ [28]. Using matrices with the restricted isometry property, both chains and anti-chains of a continuum of distinct closed ideals were constructed in $\mathcal{L}\left(\ell_p, c_0\right)$, $\mathcal{L}\left(\ell_p, \ell_\infty\right)$ and $\mathcal{L}\left(\ell_1, \ell_p\right)$ for all $1 < p < \infty$ [6].

Recently, using the infinite-dimensional $X_{p,w}$ spaces of Rosenthal and almost disjoint sequences of integers, Johnson and Schechtman proved that there are $2^{\mathfrak{c}}$ distinct closed ideals in $\mathcal{L}\left(L_p\right)$ for $1 < p < \infty$ with $p \neq 2$ [11]. In particular, the cardinality of the set of closed ideals in $\mathcal{L}\left(L_p\right)$ is exactly $2^{\mathfrak{c}}$.

The goal for this paper is to present a general method for proving when $\mathcal{L}(X, Y)$ contains $2^{\mathfrak{c}}$ distinct closed ideals for some Banach spaces X and Y with unconditional finite-dimensional decompositions. Given a collection of operators $(T_N)_{N \in [\mathbb{N}]^\omega}$ from X to Y indexed by the set of all infinite subsets of the natural numbers, we give sufficient conditions for there to exist an infinite subset L of $\mathbb{N}$ so that if $\mathcal{S} \subset [L]^\omega$ is a set of pairwise almost disjoint subsets of L, then for all $\mathcal{A}, \mathcal{B} \subset S$, if $M \in \mathcal{A} \setminus \mathcal{B}$, the operator T_M is not contained in the smallest closed ideal containing $\{T_N : N \in \mathcal{B}\}$. Thus, $\mathcal{L}(X, Y)$ contains $2^{\mathfrak{c}}$ closed ideals. We are able to apply this method to prove in particular that the cardinality of the set of closed ideals in $\mathcal{L}\left(\ell_p \oplus \ell_q\right)$ is exactly $2^{\mathfrak{c}}$ for all $1 < p < q < \infty$. It follows at once that $\mathcal{L}\left(L_p\right)$ contains exactly $2^{\mathfrak{c}}$ closed ideals for $1 < p \neq 2 < \infty$, and thus we have another proof of the aforementioned result of Johnson and Schechtman [11]. It is worth pointing out that they construct closed ideals using operators that are not even strictly singular (and on the other hand, their ideals do not contain projections onto non-Hilbertian subspaces). By contrast, our $2^{\mathfrak{c}}$ closed ideals are small in the sense that they consist of finitely strictly singular operators.

In [7] it was shown that there are $2^{\mathfrak{c}}$ distinct closed ideals in $\mathcal{L}\left(\ell_p, c_0\right)$, $\mathcal{L}\left(\ell_p, \ell_\infty\right)$ and $\mathcal{L}\left(\ell_1, \ell_p\right)$ for all $1 < p < \infty$. In this article, we will show that this result can also be obtained by our general construction.

Although our initial goals were to construct closed ideals between classical Banach spaces, the generality of our approach allows us to construct $2^{\mathfrak{c}}$ closed ideals in $\mathcal{L}(X, Y)$ when X and Y are exotic Banach spaces such as, for example, p-convexified Tsirelson spaces. In [2] it was shown that the projection operators in Tsirelson and Schreier spaces generate a continuum of distinct closed ideals. So again, an interesting distinction between these two methods is that the operators we use to generate ideals are finitely strictly singular, whereas the projections used in [2] are clearly not even strictly singular.

The paper is organised as follows. In the next section we give general conditions on Banach spaces X and Y that ensure that $\mathcal{L}(X, Y)$ contains $2^{\mathfrak{c}}$ closed ideals. We also prove two further results giving criteria that help with verifying those general conditions. Each one of these two results has applications that we present in the following two sections. In the final section we give further remarks and state some open problems.

2. General conditions for having $2^{\mathfrak{c}}$ closed ideals in $\mathcal{L}(\mathbf{X}, \mathbf{Y})$

Let X and Y be Banach spaces and let $\mathcal{T}$ be a subset of $\mathcal{L}(X, Y)$, the space of all bounded linear operators from X to Y. The *closed ideal generated by* $\mathcal{T}$ is the smallest closed ideal in $\mathcal{L}(X, Y)$ containing $\mathcal{T}$ and is denoted by $\mathcal{J}^{\mathcal{T}}(X, Y)$. That is, $\mathcal{J}^{\mathcal{T}}(X, Y)$ is the closure in $\mathcal{L}(X, Y)$ of the set

$$\left\{ \sum_{j=1}^{n} A_j T_j B_j : n \in \mathbb{N},\ \left(A_j\right)_{j=1}^{n} \subset \mathcal{L}(Y),\ \left(T_j\right)_{j=1}^{n} \subset \mathcal{T},\ \left(B_j\right)_{j=1}^{n} \subset \mathcal{L}(X) \right\}$$

consisting of finite sums of operators factoring through members of $\mathcal{T}$. When $\mathcal{T}$ consists of a single operator $T \in \mathcal{L}(X, Y)$, then we write $\mathcal{J}^{T}(X, Y)$ instead of $\mathcal{J}^{\{T\}}(X, Y)$.

In [6], for each $1 < p < \infty$, a collection $(T_N)_{N \subset \mathbb{N}} \subset \mathcal{L}\left(\ell_p, c_0\right)$ of operators was constructed such that $\mathcal{J}^{T_M}\left(\ell_p, c_0\right) \neq \mathcal{J}^{T_N}\left(\ell_p, c_0\right)$ whenever $M \triangle N$ is infinite. For a nonempty family $\mathcal{A}$ of subsets of $\mathbb{N}$, let $\mathcal{J}_{\mathcal{A}}$

be the closed ideal in $\mathcal{L}(\ell_p, c_0)$ generated by $\{T_N : N \in \mathcal{A}\}$. There are at most a continuum of closed ideals in $\mathcal{L}(\ell_p, c_0)$ that are generated by a single operator. However, it was observed in [7] that if $\mathcal{S}$ is an almost disjoint family of cardinality $\mathfrak{c}$ consisting of infinite subsets of $\mathbb{N}$, then $\{\mathcal{J}_{\mathcal{A}} : \mathcal{A} \subset \mathcal{S}, \mathcal{A} \neq \emptyset\}$ is a lattice of $2^{\mathfrak{c}}$ distinct closed ideals in $\mathcal{L}(\ell_p, c_0)$.

In this section, we will present a general condition which implies that $\mathcal{L}(X, Y)$ has $2^{\mathfrak{c}}$ closed ideals in the following framework, in which the example already given also sits.

We are given two Banach spaces X and Y which are assumed to have *unconditional finite-dimensional decompositions* (UFDDs) (E_n) and (F_n), respectively. By this we mean that E_n is a finite-dimensional subspace of X for each $n \in \mathbb{N}$, that each element of x can be written in a unique way as $x = \sum_{n\in\mathbb{N}} x_n$ with $x_n \in E_n$ for each $n \in \mathbb{N}$ and that $\sum_{n\in\mathbb{N}} x_n$ converges unconditionally. We can therefore think of the elements $x \in X$ as sequences (x_n) with $x_n \in E_n$, which we call the *n-component* of x, for each $n \in \mathbb{N}$.

As in the case of unconditional bases, this implies that for $N \subset \mathbb{N}$, the map

$$P_N^X : X \to X, \qquad (x_n)_{n\in\mathbb{N}} \mapsto (x_n)_{n\in N},$$

where $(x_n)_{n\in N}$ is identified with the element in X whose m-component vanishes for $m \in \mathbb{N} \setminus N$, is well defined and uniformly bounded. It follows that for some $C > 0$ we have $\left\| \sum_{n\in\mathbb{N}} \sigma_n x_n \right\| \leqslant C \left\| \sum_{n\in\mathbb{N}} x_n \right\|$ for all $(x_n) \in X$ and all $(\sigma_n) \in \{\pm 1\}^{\mathbb{N}}$. In this case we say that (E_n) is a *C-unconditional finite-dimensional decomposition* (or *C-unconditional FDD*) of X. After renorming X, we can (and will) assume that $\left\|P_N^X\right\| = 1$ for a nonempty $N \subset \mathbb{N}$ and that moreover

$$\left\| \sum_{n\in\mathbb{N}} x_n \right\| = \left\| \sum_{n\in\mathbb{N}} \sigma_n x_n \right\| \tag{1}$$

for all $(x_n) \in X$ and all $(\sigma_n) \in \{\pm 1\}^{\mathbb{N}}$. We denote for $N \subset \mathbb{N}$ the image of X under P_N^X by X_N. Thus $X_N = P_N^X(X) = \overline{\operatorname{span}} \bigcup_{n\in N} E_n$ is 1-complemented in X and $(E_n : n \in N)$ is a 1-unconditional FDD of X_N. Similarly, for the space Y with UFDD (F_n), we define P_N^Y and Y_N for every $N \subset \mathbb{N}$. We further assume that $\left\|P_N^Y\right\| = 1$ for every nonempty $N \subset \mathbb{N}$ and that (F_n) is a 1-unconditional FDD of Y.

For each $n \in \mathbb{N}$ we are given a linear operator $T_n : E_n \to F_n$ and we assume that the linear operator

$$T : \operatorname{span} \bigcup_{n\in\mathbb{N}} E_n \to \operatorname{span} \bigcup_{n\in\mathbb{N}} F_n, \quad (x_n) \mapsto (T_n(x_n)),$$

extends to a bounded operator $T : X \to Y$. We then define for $N \subset \mathbb{N}$ the *diagonal operator* $T_N : X_N \to Y_N$ by $T_N = T \circ P_N^X = P_N^Y \circ T$. Note that $\|T_N\| \leqslant \|T\|$.

Our goal is to formulate conditions which imply that the following holds for some $\Delta > 0$:

$$\forall M, N \in [\mathbb{N}]^\omega, \text{ if } M \setminus N \in [\mathbb{N}]^\omega, \text{ then } \operatorname{dist}\left(T_M, \mathcal{J}^{T_N}\right) \geqslant \Delta. \tag{2}$$

Using an observation in [11], we can conclude that $\mathcal{L}(X, Y)$ has $2^{\mathfrak{c}}$ closed ideals when formula (2) holds.

Proposition 1. *Let X, Y and (T_n) be as before, and assume that condition (2) holds for some $\Delta > 0$. Let $\mathcal{S} \subset [\mathbb{N}]^\omega$ be an almost disjoint family of cardinality $\mathfrak{c}$. For $\mathcal{A} \subset \mathcal{S}$, let $\mathcal{J}_{\mathcal{A}}$ be the closed ideal in $\mathcal{L}(X, Y)$ generated by $\{T_N : N \in \mathcal{A}\}$. Then if $\mathcal{A}, \mathcal{B} \subset \mathcal{S}$ with $\mathcal{A} \neq \mathcal{B}$, then $\mathcal{J}_{\mathcal{A}} \neq \mathcal{J}_{\mathcal{B}}$. In particular, the cardinality of the set of closed ideals in $\mathcal{L}(X, Y)$ is $2^{\mathfrak{c}}$.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two different subsets of $\mathcal{S}$. Without loss of generality, we assume that there is an $M \in \mathcal{A} \setminus \mathcal{B}$. We claim that $T_M \notin \mathcal{J}_{\mathcal{B}}$, and that actually $\operatorname{dist}(T_M, \mathcal{J}_{\mathcal{B}}) \geqslant \Delta$.

Indeed, set $n \in \mathbb{N}$, $(A_j)_{j=1}^n \subset \mathcal{L}(Y)$, $(B_j)_{j=1}^n \subset \mathcal{L}(X)$ and $(N_j)_{j=1}^n \subset \mathcal{B}$. Put $N = \bigcup_{j=1}^n N_j$. It follows that

$$\sum_{j=1}^n A_j \circ T_{N_j} \circ B_j = \sum_{j=1}^n A_j \circ P_{N_j}^Y \circ T_N \circ B_j \in \mathcal{J}^{T_N}.$$

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Since $M \setminus N$ is infinite, it follows from formula (2) that

$$\left\| \sum_{j=1}^{n} A_j \circ T_{N_j} \circ B_j - T_M \right\| \geqslant \Delta.$$

Since $\mathcal{J}_{\mathcal{B}}$ is the closure of the set of operators of the form $\sum_{j=1}^{n} A_j \circ T_{N_j} \circ B_j$ with $n \in \mathbb{N}$, $(A_j)_{j=1}^{n} \subset \mathcal{L}(Y)$, $(B_j)_{j=1}^{n} \subset \mathcal{L}(X)$ and $(N_j)_{j=1}^{n} \subset \mathcal{B}$, we deduce our claim. □

In order to separate T_M from $\mathcal{J}^{T_N}$ if $M \setminus N$ is infinite, the following condition is sufficient:

For each $n \in \mathbb{N}$ there exist $l_n \in \mathbb{N}$ and vectors $(x_{n,j})_{j=1}^{l_n} \subset S_{E_n}$, $\left(y_{n,j}^*\right)_{j=1}^{l_n} \subset S_{F_n^*}$ so that

$$y_{n,j}^* \left(T_n \left(x_{n,j}\right)\right) \geqslant 1 \quad \text{for } n \in \mathbb{N} \text{ and } j = 1, 2, \ldots, l_n, \tag{3a}$$

$$\lim_{\substack{m\to\infty \\ m\in M\setminus N}} \frac{1}{l_m} \sum_{i=1}^{l_m} \left\|T_N \circ B(x_{m,i})\right\| = 0 \tag{3b}$$

whenever $M, N \in [\mathbb{N}]^{\omega}$ satisfy $M \setminus N \in [\mathbb{N}]^{\omega}$, and $B \in \mathcal{L}(X)$.

Indeed, for $n \in \mathbb{N}$ we define the functional $\Psi_n \in \mathcal{L}(X, Y)^*$ by

$$\Psi_n(S) = \frac{1}{l_n} \sum_{j=1}^{l_n} y_{n,j}^* \left(S\left(x_{n,j}\right)\right), \quad \text{for } S \in \mathcal{L}(X, Y).$$

Given $M, N \in [\mathbb{N}]^{\omega}$ with $M \setminus N \in [\mathbb{N}]^{\omega}$, we let Ψ be a w^*-accumulation point of $(\Psi_m : m \in M \setminus N)$. It follows from formula (3a) that

$$\Psi(T_M) \geqslant \liminf_{m\in M\setminus N} \Psi_m(T_M) \geqslant 1,$$

and for any $A \in \mathcal{L}(Y)$ and $B \in \mathcal{L}(X)$ it follows from equation (3b) that

$$\begin{aligned}
\left|\Psi(AT_N B)\right| &\leqslant \limsup_{m\in M\setminus N} \left| \frac{1}{l_m} \sum_{i=1}^{l_m} y_{m,i}^* \left(AT_N B(x_{m,i})\right) \right| \\
&= \limsup_{m\in M\setminus N} \left| \frac{1}{l_m} \sum_{i=1}^{l_m} A^* y_{m,i}^* \left(T_N B(x_{m,i})\right) \right| \\
&\leqslant \|A\| \limsup_{m\in M\setminus N} \frac{1}{l_m} \sum_{i=1}^{l_m} \|T_N B(x_{m,i})\| = 0.
\end{aligned}$$

Since $\|\Psi_n\| \leqslant 1$ for all $n \in \mathbb{N}$, it follows that $\|\Psi\| \leqslant 1$, which in turn implies condition (2) with $\Delta = 1$.

Remark. Some extension of this result is possible. Assume for example that formula (3) holds and that U is an isomorphism of Y into another Banach space Z. Then $\mathcal{L}(X, Z)$ also has at least $2^{\mathfrak{c}}$ distinct closed ideals. Indeed, by Hahn–Banach, there are functionals $z_{n,j}^* \in Z^*$ such that $U^*\left(z_{n,j}^*\right) = y_{n,j}^*$ for all $n \in \mathbb{N}$ and $j = 1, 2, \ldots, l_n$, and moreover, $C = \sup_{n,j} \left\|z_{n,j}^*\right\| < \infty$. If we now define $\Psi_n \in \mathcal{L}(X, Z)^*$ as before but replace $y_{n,j}^*$ with $z_{n,j}^*$, then the previous argument will show that condition (2) holds with $\Delta = 1/C$ if we replace T_N with $U \circ T_N$ for every $N \subset \mathbb{N}$.

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We now want to formulate conditions on the spaces X and Y and the operators $T_n\colon E_n \longrightarrow F_n$, $n \in \mathbb{N}$, which imply that condition (3) is satisfied. From now on we assume that for each $n \in \mathbb{N}$, the spaces E_n and F_n have 1-unconditional, normalised bases $(e_{n,j})_{j=1}^{\dim(E_n)}$ and $(f_{n,j})_{j=1}^{\dim(F_n)}$ with coordinate functionals $\left(e_{n,j}^*\right)_{j=1}^{\dim(E_n)}$ and $\left(f_{n,j}^*\right)_{j=1}^{\dim(F_n)}$, respectively.

We write for $n \in \mathbb{N}$ the operator $T_n\colon E_n \longrightarrow F_n$ as

$$T_n\colon E_n \longrightarrow F_n, \quad T_n(x) = \sum_{j=1}^{\dim(F_n)} x_{n,j}^*(x) f_{n,j},$$

where $x_{n,j}^* \in E_n^*$ for $n \in \mathbb{N}$ and $1 \leqslant j \leqslant \dim(F_n)$. In applications, we will define the T_n by choosing the $x_{n,j}^*$ so that

$$\text{the operator } T\colon X \longrightarrow Y,\ (x_n) \mapsto (T(x_n)),\ \text{is well defined and bounded.} \tag{4}$$

We secondly demand that $\dim(F_n) = l_n$ and that $y_{n,j}^* = f_{n,j}^*$ for $n \in \mathbb{N}$ and $j = 1, 2, \ldots, l_n$. Thus, in order to obtain formula (3a), we require

$$x_{n,j}^*\left(x_{n,j}\right) \geqslant 1 \quad \text{for all } n \in \mathbb{N} \text{ and } j = 1, 2, \ldots, l_n. \tag{5}$$

Finally, in order to satisfy equation (3b), we will ensure that for $m \in \mathbb{N}$ and any operator $B \in \mathcal{L}\left(E_m, X_{\mathbb{N}\setminus\{m\}}\right)$ with $\|B\| \leqslant 1$, it follows that

$$\frac{1}{l_m} \sum_{i=1}^{l_m} \left\| T_{N\setminus\{m\}} B(x_{m,i}) \right\| \leqslant \varepsilon_m, \tag{6}$$

where (ε_m) is a sequence in $(0,1)$ decreasing to 0 not depending on B. Now B can be written as the sum $B = B^{(1)} + B^{(2)}$, where $B^{(1)} \in \mathcal{L}(E_m, X_{\{1,2,\ldots,m-1\}})$ and $B^{(2)} \in \mathcal{L}(E_m, X_{\mathbb{N}\setminus\{1,2,\ldots,m\}})$.

It is not very hard to force formula (6) to hold for $B^{(1)}$ with $\varepsilon_m/2$ instead of ε_m: it will be enough to ensure that l_m is very large compared to $\dim\left(X_{\{1,2,\ldots,m-1\}}\right)$ and (see the proof of Proposition 2) that $\frac{1}{l_m} \sup_\pm \left\|\sum_{i=1}^{l_m} \pm x_{m,i}\right\|$ decreases to 0 for increasing m. To also ensure the necessary estimates for $B^{(1)}$, we will assume the following slightly stronger condition:

$$\lim_{m\to\infty} l_m = \infty \text{ and } \lim_{l\to\infty} \sup_{m\in\mathbb{N}, l_m \geqslant l} \frac{\varphi_m(l)}{l} = 0, \text{ where} \tag{7}$$

$$\varphi_m(l) = \sup\left\{ \left\| \sum_{i\in A} \sigma_i x_{m,i} \right\| : A \subset \{1, \ldots, \ell_m\},\ |A| \leqslant l,\ (\sigma_i)_{i\in A} \subset \{\pm 1\} \right\}.$$

Ensuring that formula (6) holds for $B^{(2)}$ is more complicated and will be done in two steps. The second of these two steps is more straighforward: it will be enough to assume that $T_{\mathbb{N}\setminus\{1,2,\ldots m\}}$ maps vectors with small coordinates into vectors with small norm (see Proposition 2(a) for the precise statement). The first step is then to assume (see Proposition 2(b)) that the set

$$\left\{(n,j) : n > m,\ 1 \leqslant j \leqslant l_n,\ \left| x_{n,j}^*\left(B^{(2)} x_{m,i}\right)\right| > \delta \text{ for some } 1 \leqslant i \leqslant l_m \right\}$$

has small cardinality compared to l_m. In many situations, guaranteeing that this set has small cardinality relative to l_m is the trickiest part, as $B^{(2)}$ is an arbitrary norm 1 operator. However, in Lemmas 3 and 4 we present conditions which imply this result and are stated in terms of only basic properties of the sequences $(x_{n,j})$ and $\left(x_{n,j}^*\right)$, as well as the Banach spaces X and Y.

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Of course, since for any $N \in [\mathbb{N}]^\omega$, X_N and Y_N are complemented subspaces of X and Y, respectively, we can pass to subsequences (E_{k_n}), (F_{k_n}) and (T_{k_n}) for which we are able to verify condition (2), in order to conclude that the lattice of closed ideals in $\mathcal{L}(X,Y)$ is of cardinality $2^{\mathfrak{c}}$. This follows from the following observation, whose verification is routine. Suppose that V and W are complemented subspaces of X and Y, respectively. For a closed ideal $\mathcal{J}$ in $\mathcal{L}(V,W)$, let $\widetilde{\mathcal{J}}$ be the closure in $\mathcal{L}(X,Y)$ of the set of operators of the form $\sum_{j=1}^n A_j S_j B_j$, where $n \in \mathbb{N}$, $(A_j)_{j=1}^n \subset \mathcal{L}(W,Y)$, $(S_j)_{j=1}^n \subset \mathcal{J}$ and $(B_j)_{j=1}^n \subset \mathcal{L}(X,V)$. Then $\widetilde{\mathcal{J}}$ is a closed ideal in $\mathcal{L}(X,Y)$ and the map $\mathcal{J} \mapsto \widetilde{\mathcal{J}}$ is injective.

Proposition 2. *Assume that the spaces X and Y, their 1-unconditional FDDs (E_n) and (F_n) and the operators $T_n\colon E_n \to F_n$, $n \in \mathbb{N}$, satisfy conditions (4), (5) and (7). Assume, moreover, that the following conditions hold:*

(a) *For all $\varepsilon > 0$ and all $M \in [\mathbb{N}]^\omega$, there are a $\delta > 0$ and $N \in [M]^\omega$ so that*

$$\forall x \in B_{X_N},\ \text{if} \sup_{n\in N, 1\leqslant j\leqslant l_n} \left|x^*_{n,j}(x)\right| \leqslant \delta, \text{ then } \|T_N(x)\| < \varepsilon.$$

(b) *For all $\delta, \varepsilon > 0$ and all $M \in [\mathbb{N}]^\omega$, there are $m \in M$ and $N \in [M]^\omega$ so that for every $B \in \mathcal{L}(E_m, X_N)$ with $\|B\| \leqslant 1$, we have*

$$\left|\left\{(n,j) : n \in N,\ 1 \leqslant j \leqslant l_n,\ \left|x^*_{n,j}(Bx_{m,i})\right| > \delta \text{ for some } 1 \leqslant i \leqslant l_m\right\}\right| < \varepsilon l_m.$$

Then there is a subsequence (k_n) of $\mathbb{N}$ so that for $\widetilde{E}_n = E_{k_n}$, $\widetilde{F}_n = F_{k_n}$, $\widetilde{T}_n = T_{k_n}$, $\tilde{l}_n = l_{k_n}$, $(\tilde{x}_{n,j})_{j=1}^{\tilde{l}_n} = (x_{k_n,j})_{j=1}^{l_{k_n}} \subset \widetilde{E}_n$ and $\left(\tilde{y}^*_{n,j}\right)_{j=1}^{\tilde{l}_n} = (f^*_{k_n,j})_{j=1}^{l_{k_n}} \subset \left(\widetilde{F}_n\right)^*$, condition (3) is satisfied. Hence, $\mathcal{L}(X,Y)$ contains $2^{\mathfrak{c}}$ closed ideals.

Proof. Let $(\varepsilon_r)_{r=1}^\infty \subset (0,1)$ be a sequence which decreases to 0. Put $k_0 = 0$ and $M_0 = \mathbb{N}$. We will inductively choose $k_r \in \mathbb{N}$ and $M_r \in [\mathbb{N}]^\omega$ so that for all $r \in \mathbb{N}$

$$\min(M_r) > k_r, \tag{8}$$

$$k_{r-1} < k_r,\ M_r \subset M_{r-1} \text{ and } k_r \in M_{r-1}, \tag{9}$$

$$\frac{1}{l_{k_r}} \sum_{i=1}^{l_{k_r}} \left\|B\left(x_{k_r,i}\right)\right\| \leqslant \varepsilon_r \text{ for all } B \in \mathcal{L}\left(E_{k_r}, X_{\{k_1,k_2,\ldots,k_{r-1}\}}\right),\ \|B\| \leqslant 1, \tag{10}$$

$$\frac{1}{l_{k_r}} \sum_{i=1}^{l_{k_r}} \left\|T_{M_r} B\left(x_{k_r,i}\right)\right\| \leqslant \varepsilon_r \text{ for all } B \in \mathcal{L}\left(E_{k_r}, X_{M_r}\right),\ \|B\| \leqslant 1. \tag{11}$$

Assume that for some $r \in \mathbb{N}$, we have already chosen suitable $k_1 < k_2 < \cdots < k_{r-1}$ and $\mathbb{N} = M_0 \supset M_1 \supset \cdots \supset M_{r-1}$. Put $C = \|T\|$. By using (a), we choose $\delta > 0$ and $M \in [M_{r-1}]^\omega$ so that

$$\|T_M(x)\| \leqslant \frac{\varepsilon_r}{2} \text{ for all } x \in B_{X_M}, \text{ with } \sup_{m\in M, 1\leqslant i\leqslant l_m} \left|x^*_{m,i}(x)\right| \leqslant \delta. \tag{12}$$

Note that condition (12) still holds if we replace M by any infinite subset of M.

We now let $p \in \mathbb{N}$ be large enough so that there exists a sequence $\left(z^*_j\right)_{j=1}^p \subset S_{X_{\{k_1,k_2,\ldots,k_{r-1}\}}}$ which normalises the elements of $X_{\{k_1,k_2,\ldots,k_{r-1}\}}$ up to the factor 2 – that is,

$$\|x\| \leqslant \max_{1\leqslant j\leqslant p} 2\left|z^*_j(x)\right| \qquad \text{for all } x \in X_{\{k_1,k_2,\ldots,k_{r-1}\}}. \tag{13}$$

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We now apply equation (7) and choose $l \in \mathbb{N}$ and $m_1 > k_{r-1}$ large enough so that for all $m \geqslant m_1$ we have $l_m \geqslant l$, and if $A \subset \{1, 2, \ldots, l_m\}$ has $|A| \geqslant l$, then

$$\sup_{\pm} \left\| \sum_{i \in A} \pm x_{m,i} \right\| < \min\left(\frac{\delta}{C}, \frac{\varepsilon_r}{2p} \right) |A|. \tag{14}$$

For any $m \geqslant m_1$ and any $B \in \mathcal{L}\left(E_m, X_{\{k_1, k_2, \ldots, k_{r-1}\}}\right)$ with $\|B\| \leqslant 1$, it follows that

$$\begin{aligned}
\frac{1}{l_m} \sum_{i=1}^{l_m} \|B(x_{m,i})\| &\leqslant \frac{2}{l_m} \sum_{i=1}^{l_m} \sum_{j=1}^{p} \left| z_j^* B(x_{m,i}) \right| \\
&= \frac{2}{l_m} \sum_{j=1}^{p} z_j^* \circ B \left(\sum_{i=1}^{l_m} \sigma_{i,j} x_{m,i} \right) \\
&\left(\text{with } \sigma_{i,j} = \operatorname{sign}\left(z_j^* B(x_{m,i}) \right) \text{ for } 1 \leqslant i \leqslant l_m \text{ and } 1 \leqslant j \leqslant p \right) \\
&\leqslant \frac{2p}{l_m} \sup_{\pm} \left\| \sum_{i=1}^{l_m} \pm x_{m,i} \right\| \leqslant \varepsilon_r.
\end{aligned} \tag{15}$$

Thus formula (10) will hold for any $k_r \geqslant m_1$. We now apply assumption (b) and choose $k_r \in M$ and an infinite subset M_r of M with $m_1 \leqslant k_r < \min(M_r)$ so that for every $B \in \mathcal{L}\left(E_{k_r}, X_{M_r}\right)$ with $\|B\| \leqslant 1$ we have

$$\begin{aligned}
&|J(B)| < \frac{\varepsilon_r l_{k_r}}{2Cl}, \quad \text{where} \\
&J(B) = \left\{ (n, j) : n \in M_r,\ 1 \leqslant j \leqslant l_n,\ \left| x_{n,j}^* \left(B x_{k_r,i} \right) \right| > \delta \text{ for some } 1 \leqslant i \leqslant l_{k_r} \right\}.
\end{aligned} \tag{16}$$

We now verify condition (11) and complete the inductive construction. Set $B \in \mathcal{L}\left(E_{k_r}, X_{M_r}\right)$ with $\|B\| \leqslant 1$ and set $J = J(B)$. For each $(n, j) \in J$ we denote

$$I_{n,j} = \left\{ i \in \left\{1, 2, \ldots, l_{k_r}\right\} : \left| x_{n,j}^* \left(B x_{k_r,i} \right) \right| > \delta \right\}.$$

We now have for each $(n, j) \in J$ that

$$\begin{aligned}
C \sup_{\pm} \left\| \sum_{i \in I_{n,j}} \pm x_{k_r,i} \right\| &\geqslant \sup_{\pm} \left\| \sum_{i \in I_{n,j}} \pm T_{M_r} B x_{k_r,i} \right\| \\
&\geqslant \sup_{\pm} \sum_{i \in I_{n,j}} \pm f_{n,j}^* \left(T_{M_r} B x_{k_r,i} \right) \\
&\geqslant \left| I_{n,j} \right| \delta,
\end{aligned}$$

where we used the fact that $f_{n,j}^* \circ T_{M_r} = x_{n,j}^*$. On the other hand, we have by formula (14) that if $\left| I_{n,j} \right| \geqslant l$, then

$$\sup_{\pm} \left\| \sum_{i \in I_{n,j}} \pm x_{k_r,i} \right\| < \delta \left| I_{n,j} \right| / C.$$

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Thus, $|I_{n,j}| < l$ for all $(n, j) \in J$. We now set $I = \bigcup_{(n,j)\in J} I_{n,j}$ and calculate

$$\begin{aligned}
\sum_{i=1}^{l_{k_r}} \left\|T_{M_r} B\left(x_{k_r,i}\right)\right\| &\leqslant \sum_{i\in I} \left\|T_{M_r} B\left(x_{k_r,i}\right)\right\| + \sum_{i\notin I} \left\|T_{M_r} B\left(x_{k_r,i}\right)\right\| \\
&\leqslant \sum_{i\in I} \left\|T_{M_r} B\left(x_{k_r,i}\right)\right\| + \varepsilon_r l_{k_r}/2, && \text{by formula (12),} \\
&\leqslant \sum_{(n,j)\in J} \sum_{i\in I_{n,j}} \left\|T_{M_r} B\left(x_{k_r,i}\right)\right\| + \varepsilon_r l_{k_r}/2 \\
&\leqslant Cl|J| + \varepsilon_r l_{k_r}/2, && \text{as } |I_{n,j}| < l \text{ for all } (n,j) \in J, \\
&\leqslant \varepsilon_r l_{k_r}, && \text{by condition (16).}
\end{aligned}$$

Thus we have proven formula (11) and our induction is complete.

We now prove that condition (3) holds. Assumption (5) and the definition of T_n imply that formula (3a) holds with $y^*_{n,j} = f^*_{n,j}$. To verify equation (3b), we consider infinite subsets M and N of $\{k_r : r \in \mathbb{N}\}$ with $M \setminus N \in [\mathbb{N}]^\omega$. Set $B \in \mathcal{L}(X)$ and $m \in M \setminus N$. Define r by $m = k_r$. Let $N_{<m} = \{n \in N : n < m\}$ and $N_{>m} = \{n \in N : n > m\}$. We then have

$$\begin{aligned}
\frac{1}{l_m}\sum_{i=1}^{l_m} \left\|T_N B(x_{m,i})\right\| &\leqslant \frac{1}{l_m}\sum_{i=1}^{l_m} \left\|T_{N_{<m}} B(x_{m,i})\right\| + \frac{1}{l_m}\sum_{i=1}^{l_m} \left\|T_{N_{>m}} B(x_{m,i})\right\| \\
&\leqslant \frac{1}{l_{k_r}}\sum_{i=1}^{l_{k_r}} C\left\|P_{\{k_1,\ldots,k_{r-1}\}} B\left(x_{k_r,i}\right)\right\| + \frac{1}{l_{k_r}}\sum_{i=1}^{l_{k_r}} \left\|T_{M_r} B\left(x_{k_r,i}\right)\right\| \\
&\leqslant \varepsilon_r C\|B\| + \varepsilon_r\|B\|, && \text{by formulas (10) and (11).}
\end{aligned}$$

Hence we have

$$\lim_{m\to\infty} \frac{1}{l_m}\sum_{i=1}^{l_m} \left\|T_N B(x_{m,i})\right\| = 0$$

and equation (3b) is satisfied. □

As mentioned before, the key assumption in Proposition 2 is (b). We will now present conditions (Lemmas 3 and 4) which imply this assumption. We will later give applications in Sections 3 and 4.

For a Banach space Z with an unconditional basis (f_j), we define the *lower fundamental function* $\lambda_Z : \mathbb{N} \to \mathbb{R}$ *of* Z by

$$\lambda_Z(n) = \inf\left\{\left\|\sum_{j\in A} f_j\right\| : A \subset \mathbb{N},\ |A| \geqslant n\right\} \quad (n \in \mathbb{N}).$$

Lemma 3. *We are given* $\delta, \varepsilon \in (0, 1)$, $l \in \mathbb{N}$ *with* $\varepsilon l \geqslant 1$, *Banach spaces* G *and* Z *and a 1-unconditional basis* $(f_j)_{j=1}^\infty$ *for* Z *with biorthogonal functionals* $\left(f^*_j\right)_{j=1}^\infty$. *Assume that for some sequence* $(x_i)_{i=1}^l \subset S_G$ *we have*

$$\varphi(l)/\lambda_Z(\lfloor \varepsilon l \rfloor) < \delta, \tag{17}$$

where $\varphi(l) = \sup\{\|\sum_{i\in I} \sigma_i x_i\| : I \subset \{1, 2, \ldots, l\},\ (\sigma_i)_{i\in I} \subset \{\pm 1\}\}$. *Then for any* $B : G \to Z$ *with* $\|B\| \leqslant 1$, *we have*

$$\left|\left\{j \in \mathbb{N} : \left|f^*_j(Bx_i)\right| > \delta \text{ for some } 1 \leqslant i \leqslant l\right\}\right| \leqslant \varepsilon l.$$

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Proof. Fix an operator $B\colon G \to Z$ with $\|B\| \leqslant 1$ and set

$$I = \left\{i \in \{1, 2, \ldots, l\} : \left|f_j^*(Bx_i)\right| > \delta \text{ for some } j \in \mathbb{N}\right\},$$
$$J = \left\{j \in \mathbb{N} : \left|f_j^*(Bx_i)\right| > \delta \text{ for some } 1 \leqslant i \leqslant l\right\}.$$

We next fix independent Rademacher random variables $(r_i)_{i\in I}$ and establish the estimate

$$\mathbb{E}\left|\sum_{i\in I} r_i f_j^*\big(B(x_i)\big)\right| > \delta \quad \text{for all } j \in J. \tag{18}$$

To see this, fix $j \in J$ and set $y_i = f_j^*\big(B(x_i)\big)$ for $i \in I$. By the definition of J, there is an $i_0 \in I$ such that $|y_{i_0}| > \delta$. Thus, by Jensen's inequality we have

$$\mathbb{E}\left|\sum_{i\in I} r_i y_i\right| = \mathbb{E}\left|\sum_{i\in I} r_{i_0} r_i y_i\right| = \mathbb{E}\left|y_{i_0} + \sum_{i\in I, i\neq i_0} r_{i_0} r_i y_i\right|$$
$$\geqslant \left|y_{i_0} + \sum_{i\in I, i\neq i_0} \mathbb{E}(r_{i_0} r_i) y_i\right| = |y_{i_0}| > \delta.$$

We then calculate

$$\begin{aligned}
\varphi(l) &\geqslant \mathbb{E}\left\|\sum_{i\in I} r_i B(x_i)\right\|_Z, && \text{as } \|B\| \leqslant 1,\\
&= \mathbb{E}\left\|\sum_j \left|\sum_{i\in I} r_i f_j^*\big(B(x_i)\big)\right| f_j\right\|_Z, && \text{as } (f_j) \text{ is 1-unconditional},\\
&\geqslant \left\|\sum_j \mathbb{E}\left|\sum_{i\in I} r_i f_j^*\big(B(x_i)\big)\right| f_j\right\|_Z, && \text{by Jensen's inequality},\\
&\geqslant \delta\left\|\sum_{j\in J} f_j\right\|_Z, && \text{using formula (18) and the 1-unconditionality of } (f_j),\\
&\geqslant \delta\lambda_Z(|J|).
\end{aligned}$$

Since the lower fundamental function λ_Z is clearly increasing, it follows from assumption (17) that $|J| \leqslant \varepsilon l$. □

We now state and prove a very different condition that also implies Proposition 2(b). Here we use the notation and framework established on page 5.

Lemma 4. *Let* $1 \leqslant s, t < \infty$ *and suppose the following hold:*

(a) *There is a constant* $c_1 > 0$ *so that* $\left(e_{m,i}^*\right)_{i=1}^{\dim(E_m)}$ *is* c_1*-dominated by the unit vector basis of* ℓ_s *for each* $m \in \mathbb{N}$*. That is,*

$$\left\|\sum_{i=1}^{\dim(E_m)} a_i e_{m,i}^*\right\| \leqslant c_1 \left(\sum_{i=1}^{\dim(E_m)} |a_i|^s\right)^{1/s} \quad \textit{for all scalars } (a_i)_{i=1}^{\dim(E_m)}.$$

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(b) *There is a constant $c_2 > 0$ so that for all $m, n \in \mathbb{N}$ with $m < n$ and all $A \subset \{1, 2, \ldots, l_n\}$ with $|A| \leqslant l_m$, the sequence $\left(x^*_{n,j}\right)_{j\in A}$ is c_2-weak ℓ_s. That is,*

$$\left(\sum_{j\in A}\left|x^*_{n,j}(x)\right|^s\right)^{1/s} \leqslant c_2\|x\| \quad \textit{for all } x \in E_n.$$

(c) *There is a constant $c_3 > 0$ so that if $z_n \in S_{E_n}$ for all $n \in \mathbb{N}$, then $(z_n)_{n=1}^\infty$ c_3-dominates the unit vector basis for ℓ_t. In other words,*

$$\left(\sum_{n\in\mathbb{N}}\left\|P^X_n x\right\|^t\right)^{1/t} \leqslant c_3\|x\| \quad \textit{for all } x \in X.$$

(d) $\lim_{m\to\infty}\left(\dim(E_m)\right)^{max(1,t/s)} l_m^{-1} = 0.$

Then for all $\delta, \varepsilon > 0$, there exists $m \in \mathbb{N}$ so that for all $N \in [\{n \in \mathbb{N} : n \geqslant m+1\}]^\omega$ and for all $B \in \mathcal{L}(E_m, X_N)$ with $\|B\| \leqslant 1$, the set

$$J = \left\{(n, j) : n \in N,\ 1 \leqslant j \leqslant l_n,\ \left|x^*_{n,j}(Bx_{m,i})\right| > \delta \textit{ for some } 1 \leqslant i \leqslant l_m\right\}$$

has $|J| \leqslant \varepsilon l_m$.

Proof. Set $0 < \delta, \varepsilon < 1$, $m \in \mathbb{N}$, $N \in [\{n \in \mathbb{N} : n \geqslant m+1\}]^\omega$ and $B \in \mathcal{L}(E_m, X_N)$ with $\|B\| \leqslant 1$. Let $H \subset J$ be such that $|H| \leqslant l_m$. Note that if we prove that $|H| < \varepsilon l_m$, then we have $|J| < \varepsilon l_m$. For each $n \in N$ denote $H_n = \left\{j \in \{1, 2, \ldots, l_n\} : (n, j) \in H\right\}$. We have

$$\begin{aligned}
\delta^s|H_n| &\leqslant \sum_{j\in H_n}\left\|B^* x^*_{n,j}\right\|^s \\
&= \sum_{j\in H_n}\left\|\sum_{i=1}^{\dim(E_m)}\left(B^* x^*_{n,j}(e_{m,i})\right) e^*_{m,i}\right\|^s \\
&\leqslant c_1^s \sum_{j\in H_n}\sum_{i=1}^{\dim(E_m)}\left|B^* x^*_{n,j}(e_{m,i})\right|^s && \text{by (a),} \\
&= c_1^s \sum_{i=1}^{\dim(E_m)}\sum_{j\in H_n}\left|x^*_{n,j}\left(P^X_n Be_{m,i}\right)\right|^s \\
&\leqslant c_1^s c_2^s \sum_{i=1}^{\dim(E_m)}\left\|P^X_n Be_{m,i}\right\|^s && \text{by (b).}
\end{aligned}$$

For the case where $t \leqslant s$, we may use the fact that $\left\|P^X_n Be_{m,i}\right\| \leqslant 1$ to obtain

$$\delta^s|H_n| \leqslant c_1^s c_2^s \sum_{i=1}^{\dim(E_m)}\left\|P^X_n Be_{m,i}\right\|^s \leqslant c_1^s c_2^s \sum_{i=1}^{\dim(E_m)}\left\|P^X_n Be_{m,i}\right\|^t. \tag{19}$$

For the case that $s < t$, Hölder's inequality gives

$$\delta^s|H_n| \leqslant c_1^s c_2^s \sum_{i=1}^{\dim(E_m)}\left\|P^X_n Be_{m,i}\right\|^s \leqslant c_1^s c_2^s\left(\dim(E_m)\right)^{\frac{t-s}{t}}\left(\sum_{i=1}^{\dim(E_m)}\left\|P_n Be_{m,i}\right\|^t\right)^{s/t}.$$

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By raising this inequality to the power t/s, we have, for $s < t$,

$$\delta^t |H_n| \leqslant \delta^t |H_n|^{t/s} \leqslant c_1^t c_2^t \big(\dim(E_m)\big)^{\frac{t-s}{s}} \sum_{i=1}^{\dim(E_m)} \left\|P_n^X B e_{m,i}\right\|^t . \tag{20}$$

We now finish the proof for the case where $t \leqslant s$; we will consider the remaining case later. Summing formula (19) over $n \in N$ gives

$$\begin{aligned} |H| &= \sum_{n\in N} |H_n| \\ &\leqslant \delta^{-s} c_1^s c_2^s \sum_{i=1}^{\dim(E_m)} \sum_{n\in N} \left\|P_n^X B e_{m,i}\right\|_{E_n}^t \\ &\leqslant \delta^{-s} c_1^s c_2^s \sum_{i=1}^{\dim(E_m)} c_3^t \left\|B e_{m,i}\right\|^t && \text{by (c),} \\ &\leqslant \delta^{-s} c_1^s c_2^s c_3^t \dim(E_m). \end{aligned}$$

As $t \leqslant s$, we have by (d) that $\lim_{m\to\infty} \dim(E_m) l_m^{-1} = 0$. Hence, if $m \in \mathbb{N}$ is large enough, then $|H| < \varepsilon l_m$, and thus $|J| < \varepsilon l_m$ as well.

We now consider the remaining case, where $s < t$. By formula (20) we have

$$\begin{aligned} |H| &= \sum_{n\in N} |H_n| \\ &\leqslant \delta^{-t} c_1^t c_2^t \big(\dim(E_m)\big)^{\frac{t-s}{s}} \sum_{i=1}^{\dim(E_m)} \sum_{n\in N} \left\|P_n^X B e_{m,i}\right\|_{E_n}^t \\ &= \delta^{-t} c_1^t c_2^t c_3^t \big(\dim(E_m)\big)^{\frac{t-s}{s}} \sum_{i=1}^{\dim(E_m)} \left\|B e_{m,i}\right\|^t && \text{by (c),} \\ &\leqslant \delta^{-t} c_1^t c_2^t c_3^t \big(\dim(E_m)\big)^{\frac{t-s}{s}} \dim(E_m) \\ &= \delta^{-t} c_1^t c_2^t c_3^t \big(\dim(E_m)\big)^{\frac{t}{s}}. \end{aligned}$$

As $s < t$, we have by (d) that $\lim_{m\to\infty} \big(\dim(E_m)\big)^{\frac{t}{s}} l_m^{-1} = 0$. Hence, if $m \in \mathbb{N}$ is large enough, then $|H| < \varepsilon l_m$, and thus $|J| < \varepsilon l_m$. □

3. Applications I

In this section we apply the general process developed in Section 2 together with Lemma 3 to establish a class of pairs (X, Y) of Banach spaces for which $\mathcal{L}(X, Y)$ contains $2^{\mathfrak{c}}$ distinct closed ideals. We will then give a list of examples including classical ℓ_p-spaces and p-convexified Tsirelson spaces.

Theorem 5. *Let $1 < p \leqslant r < 2$ and $1 < r < q < \infty$. Let X be an unconditional sum of a sequence (E_n) of finite-dimensional Banach spaces satisfying a lower ℓ_r-estimate, and assume that the E_n contain uniformly complemented, uniformly isomorphic copies of ℓ_p^m. Let Y be an unconditional sum of a sequence (F_n) of finite-dimensional Banach spaces satisfying an upper ℓ_q-estimate. Then $\mathcal{L}(X, Y)$ contains $2^{\mathfrak{c}}$ distinct closed ideals.*

Let us first recall some of the terminology used here. To say that X is an *unconditional sum of a sequence* (E_n) of finite-dimensional Banach spaces means that X consists of all sequences (x_n) with

$x_n \in E_n$ for all $n \in \mathbb{N}$, and there is an unconditional basis (u_n) of some Banach space such that the norm of an element (x_n) of X is given by

$$\|(x_n)\| = \left\|\sum_n \|x_n\| u_n\right\|.$$

If the (u_n) is a C-unconditional basis, then we say that X is a *C-unconditional sum of* (E_n). In this case (E_n) is a UFDD for X, but the converse is not true in general.

We say that X *satisfies a lower ℓ_r-estimate* if (u_n) dominates the unit vector basis of ℓ_r – that is, if for some $c > 0$ and for all $(x_n) \in X$, the estimate

$$\left\|\sum_n x_n\right\| \geqslant c\left(\sum_n \|x_n\|^r\right)^{1/r}$$

holds. In this case we say that X *satisfies a lower ℓ_r-estimate with constant c*. An upper ℓ_r-estimate is defined analogously in the obvious way. To say that the E_n contain uniformly complemented, uniformly isomorphic copies of ℓ_p^m means that for some $C > 0$ and for all $m \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and a projection $P_n \colon E_n \to E_n$ with $\|P_n\| \leqslant C$ whose image is C-isomorphic to ℓ_p^m.

The special case of $X = \ell_p$ and $Y = \ell_q$ was treated in [27], where the existence of a continuum of distinct closed ideals was established. Here we shall make use of finite-dimensional versions of Rosenthal's $X_{p,w}$ spaces, which were also the main ingredient in [27]. We begin by recalling the definition and relevant properties.

Given $2 < p < \infty$, $0 < w \leqslant 1$ and $n \in \mathbb{N}$, we denote by $E_{p,w}^{(n)}$ the Banach space $\left(\mathbb{R}^n, \|\cdot\|_{p,w}\right)$, where

$$\left\|(a_j)_{j=1}^n\right\|_{p,w} = \left(\sum_{j=1}^n |a_j|^p\right)^{\frac{1}{p}} \vee w\left(\sum_{j=1}^n |a_j|^2\right)^{\frac{1}{2}}.$$

We write $\left\{e_j^{(n)} : 1 \leqslant j \leqslant n\right\}$ for the unit vector basis of $E_{p,w}^{(n)}$, and we denote by $\left\{e_j^{(n)*} : 1 \leqslant j \leqslant n\right\}$ the unit vector basis of the dual space $\left(E_{p,w}^{(n)}\right)^*$, which is biorthogonal to the unit vector basis of $E_{p,w}^{(n)}$.

Given $1 < p < 2$, $0 < w \leqslant 1$ and $n \in \mathbb{N}$, we fix once and for all a sequence $f_j^{(n)} = f_{p,w,j}^{(n)}$, $1 \leqslant j \leqslant n$, of independent symmetric, 3-valued random variables with $\left\|f_j^{(n)}\right\|_{L_p} = 1$ and $\left\|f_j^{(n)}\right\|_{L_2} = \frac{1}{w}$ for $1 \leqslant j \leqslant n$ (these two equalities determine the distribution of a 3-valued symmetric random variable). We then define $F_{p,w}^{(n)}$ to be the subspace span $\left\{f_j^{(n)} : 1 \leqslant j \leqslant n\right\}$ of L_p. It follows from the work of Rosenthal [22] that there exists a constant $K_p > 0$ dependent only on p so that for all scalars $(a_j)_{j=1}^n$ we have

$$\frac{1}{K_p}\left\|\sum_{j=1}^n a_j e_j^{(n)*}\right\| \leqslant \left\|\sum_{j=1}^n a_j f_j^{(n)}\right\|_{L_p} \leqslant \left\|\sum_{j=1}^n a_j e_j^{(n)*}\right\|, \tag{21}$$

where $\left\{e_j^{(n)*} : 1 \leqslant j \leqslant n\right\}$ is the unit vector basis of the dual space $\left(E_{p',w}^{(n)}\right)^*$ as already defined and p' is the conjugate index of p. Since the random variables $f_j^{(n)}$ are 3-valued, $F_{p,w}^{(n)}$ is a subspace of the span of indicator functions of 3^n pairwise disjoint sets. Thus, we can and will think of $F_{p,w}^{(n)}$ as a subspace of $\ell_p^{3^n}$. The following result follows directly from Rosenthal's work [22]:

Proposition 6 ([27, Proposition 1])**.** *Set $1 < p < 2$, $0 < w \leqslant 1$ and $n \in \mathbb{N}$. Then the following are true:*

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(i) $\left\{f_j^{(n)} : 1 \leqslant j \leqslant n\right\}$ *is a normalised,* 1*-unconditional basis of* $F_{p,w}^{(n)}$.

(ii) *There exists a projection* $P_{p,w}^{(n)} : \ell_p^{3^n} \to \ell_p^{3^n}$ *onto* $F_{p,w}^{(n)}$ *with* $\left\|P_{p,w}^{(n)}\right\| \leqslant K_p$.

(iii) *For each* $1 \leqslant k \leqslant n$ *and for every* $A \subset \{1, \dots, n\}$ *with* $|A| = k$*, we have*

$$\frac{1}{K_p} \cdot \left(k^{\frac{1}{p}} \wedge \tfrac{1}{w} k^{\frac{1}{2}}\right) \leqslant \left\|\sum_{j \in A} f_j^{(n)}\right\| \leqslant k^{\frac{1}{p}} \wedge \tfrac{1}{w} k^{\frac{1}{2}}.$$

The lower estimate of the lower fundamental function in the following lemma follows easily from [27, Lemma 3] and its proof:

Lemma 7. *Given an increasing sequence* (k_n) *in* $\mathbb{N}$ *and a decreasing sequence* (w_n) *in* $(0,1]$*, let* $1 < p \leqslant r < 2$ *and let* Z *be a* 1*-unconditional sum of* $\left(F_{p,w_n}^{(k_n)}\right)$ *satisfying a lower* ℓ_r*-estimate with constant* 1*. Then with respect to the unconditional basis* $\left(f_j^{(k_n)} : n \in \mathbb{N},\ 1 \leqslant j \leqslant k_n\right)$ *of* Z*, for all* $m \in \mathbb{N}$ *we have*

$$\lambda_Z(m) \geqslant \frac{1}{K_p}\left(\left(\frac{m}{2}\right)^{1/r} \wedge \left(\sum_{j=1}^{s-1} \frac{k_j}{w_j^2} + \frac{t}{w_s^2}\right)^{1/2}\right),$$

where $s = s(m) \in \mathbb{N}$ *is maximal so that* $\sum_{j=1}^{s-1} k_j \leqslant m/2$ *and* $t = m/2 - \sum_{j=1}^{s-1} k_j$*. In particular, if* $m \leqslant k_1$*, then*

$$\lambda_Z(m) \geqslant \frac{1}{2K_p}\left(m^{1/r} \wedge \frac{m^{1/2}}{w_1}\right).$$

Let us denote by $\left(e_{2,j}^{(n)}\right)_{j=1}^n$ the unit vector basis of ℓ_2^n. We will need the following lemma from [27]. Recall that p' is the conjugate index of p.

Lemma 8 ([27, Lemma 5]). *Given* $1 < p < 2$ *and* $p < q < \infty$*, set* $n \in \mathbb{N}$*,* $w \in (0,1]$ *and* $F = F_{p,w}^{(n)}$*. Set* $y = \sum_{j=1}^n y_j f_j^{(n)} \in F$ *with* $\|y\|_F \leqslant 1$ *and* $\tilde{y} = \sum_{j=1}^n y_j e_{2,j}^{(n)} \in \ell_2^n$*. If* $\|y\|_\infty = \max_j |y_j| \leqslant \sigma \leqslant 1$ *and* $w \leqslant \sigma^{\frac{1}{2} - \frac{1}{p'}} = \sigma^{\frac{1}{p} - \frac{1}{2}}$*, then*

$$\|\tilde{y}\|_{\ell_2^n}^q \leqslant D\sigma^s \cdot \|y\|_F^p,$$

where D *depends only on* p *and* q*, and* $s = \min\left\{\frac{q}{2} - \frac{p}{2}, \frac{q}{2} - \frac{q}{p'}\right\}$.

Proof of Theorem 5. Choose $\eta \in (0,1)$ so that $\eta < \frac{1}{r} - \frac{1}{2}$, and for each $n \in \mathbb{N}$, let $w_n = n^{-\eta}$. After passing to a complemented subspace of X using Proposition 6 and passing to an equivalent norm, we may assume that X is a 1-unconditional sum of (E_n) satisfying a lower ℓ_r-estimate with constant 1, where $E_n = F_{p,w_n}^{(n)}$ for all $n \in \mathbb{N}$. Also, using Dvoretzky's theorem, after passing to a subspace of Y with suitable renorming we may assume that Y is a 1-unconditional sum of (F_n) satisfying an upper ℓ_q-estimate with constant 1, where $F_n = \ell_2^n$ for all $n \in \mathbb{N}$ (compare the Remark following condition (3)).

We will now follow the scheme developed in Section 2. For each $m \in \mathbb{N}$ we let $l_m = m$, $x_{m,i} = f_i^{(m)} \in E_m$ and $f_{m,i} = e_{2,i}^{(m)} \in F_m$ for $1 \leqslant i \leqslant m$, and define $T_m \colon E_m \to F_m$ by $T_m(x) = \sum_{i=1}^m x_{m,i}^*(x) f_{m,i}$, where $x_{m,i}^*$ are the biorthogonal functionals to the 1-unconditional basis $(x_{m,i})_{i=1}^m$ of E_m. We are thus in the situation described in Proposition 2. It remains to verify assumptions (a) and (b) of the proposition as well as the general assumptions (4), (5) and (7).

Assumption (5) is clear. Next, it follows from Proposition 6(i) that $\sup_n \|T_n\|$ is bounded by the cotype-2 constant of L_p. Since $r < q$, it follows from the upper ℓ_q-estimate on Y and the lower ℓ_r-estimates of X that condition (4) holds.

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Using Proposition 6 again, we note that

$$\varphi_m(l) \leqslant l^{\frac{1}{p}} \wedge \frac{1}{w_m} l^{\frac{1}{2}}$$

for all $l \leqslant m$ in $\mathbb{N}$, and condition (7) follows.

We next turn to Proposition 2(a). Fix $\varepsilon > 0$ and $M \in [\mathbb{N}]^{\omega}$. Choose $\delta \in (0,1)$ so that $(D\delta^s)^{\frac{r}{p}} < \varepsilon^q$, where D and s are given by Lemma 8 with q replaced by $\frac{pq}{r}$. Then choose $N \in [M]^{\omega}$ so that $w_n \leqslant \delta^{\frac{1}{p}-\frac{1}{2}}$ for all $n \in N$. Now fix $x \in B_{X_N}$ with $\sup_{n\in N, 1\leqslant j\leqslant n} \left|x^*_{n,j}(x)\right| \leqslant \delta$. Writing $x = \sum_{n\in N}\sum_{j=1}^{n} a_{n,j}x_{n,j}$, we have $\left|a_{n,j}\right| \leqslant \delta$ for all $n \in N$ and $1 \leqslant j \leqslant n$. It follows from Lemma 8 that

$$\left(\sum_{j=1}^{n}\left|a_{n,j}\right|^2\right)^{\frac{pq}{2r}} \leqslant D\delta^s \left\|\sum_{j=1}^{n} a_{n,j}x_{x,j}\right\|_{E_n}^{p},$$

and hence

$$\left(\sum_{j=1}^{n}|a_{n,j}|^2\right)^{\frac{q}{2}} \leqslant (D\delta^s)^{\frac{r}{p}} \left\|\sum_{j=1}^{n} a_{n,j}x_{x,j}\right\|_{E_n}^{r}$$

for every $n \in N$. Summing over $n \in N$ and using the lower ℓ_r-estimate of X and the upper ℓ_q-estimate of Y, we obtain

$$\|T_N(x)\|_Y^q \leqslant (D\delta^s)^{\frac{r}{p}} \|x\|_{X_N}^r < \varepsilon^q,$$

which completes the proof of Proposition 2(a).

To verify Proposition 2(b), we fix $\delta, \varepsilon \in (0,1)$ and $M \in [\mathbb{N}]^{\omega}$. We first choose $m \in M$ so that $m\varepsilon \geqslant 1$ and

$$\frac{2K_p m^{\eta+\frac{1}{2}}}{\widetilde{m}^{\frac{1}{r}}} < \delta \quad \text{where } \widetilde{m} = \lfloor \varepsilon m \rfloor.$$

We then choose $N \in [M]^{\omega}$ so that $n = \min(N)$ satisfies $\widetilde{m}^{\frac{1}{r}} \leqslant \widetilde{m}^{\frac{1}{2}}/w_n$. We now apply Lemma 3 with $G = E_m$, $l = m$ and $Z = X_N$. First note that by Proposition 6(iii), we have

$$\varphi_m(m) \leqslant m^{\frac{1}{p}} \wedge \frac{m^{\frac{1}{2}}}{w_m} = m^{\eta+\frac{1}{2}}.$$

On the other hand, it follows from Lemma 7 that

$$\lambda_{X_N}(\widetilde{m}) \geqslant \frac{1}{2K_p}\left(\widetilde{m}^{1/r} \wedge \frac{\widetilde{m}^{1/2}}{w_n}\right) = \frac{\widetilde{m}^{1/r}}{2K_p}$$

by the choice of N. Hence, $\varphi_m(m)/\lambda_{X_N}(\lfloor \varepsilon m \rfloor) \leqslant \frac{2K_p m^{\eta+\frac{1}{2}}}{\widetilde{m}^{\frac{1}{r}}} < \delta$ by the choice of m. An application of Lemma 3 shows that for any $B \in \mathcal{L}(E_m, X_N)$ with $\|B\| \leqslant 1$, we have

$$\left|\left\{(n,j) : n \in N,\ 1 \leqslant j \leqslant n,\ \left|x^*_{n,j}(Bx_{m,i})\right| > \delta \text{ for some } 1 \leqslant i \leqslant m\right\}\right| < \varepsilon m.$$

This shows that Proposition 2(b) holds, and the proof of the theorem is thus complete. □

Remark. It is not difficult to prove (compare [27, Proposition 8]) that the $2^{\mathfrak{c}}$ closed ideals constructed in the proof of Theorem 5 are all contained in the ideal in the space of finitely strictly singular operators.

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Corollary 9. *Let $1 < p < q < \infty$ and let p' and q' denote the conjugate indices of p and q, respectively. Let X be one of the spaces ℓ_p, T_p or $T^*_{p'}$. Let Y be one of the spaces ℓ_q, T_q or $T^*_{q'}$. Then $\mathcal{L}(X,Y)$ has exactly $2^{\mathfrak{c}}$ closed ideals. It follows that $\mathcal{L}(X \oplus Y)$ also has exactly $2^{\mathfrak{c}}$ closed ideals.*

Proof. We recall the following properties of the p-convexified Tsirelson space T_p, which can be found in [4]. Its unit vector basis (t_n) is normalised, 1-unconditional and dominated by the unit vector basis of ℓ_p, and dominates the unit vector basis of ℓ_r whenever $p < r < \infty$. Moreover, given a sequence (I_n) of consecutive intervals of positive integers with $1 \in I_1$, if we let $E_n = \operatorname{span}\{t_i : i \in I_n\}$ and pick any $k_n \in I_n$ for every $n \in \mathbb{N}$, then T_p is isomorphic to the unconditional sum of (E_n) with respect to the unconditional basis $\left(t_{k_n}\right)$. It follows from Theorem 5 that $\mathcal{L}(X,Y)$ has exactly $2^{\mathfrak{c}}$ closed ideals when $1 < p < 2$, and the same then holds by duality when $2 \leqslant p < \infty$.

It follows by standard basis techniques that every operator from Y to X is compact. Hence nontrivial closed ideals in $\mathcal{L}(X,Y)$ correspond to nontrivial closed ideals in $\mathcal{L}(X \oplus Y)$ as follows. We think of operators on $X \oplus Y$ as 2×2 matrices in the obvious way. Given a nontrivial closed ideal $\mathcal{J}$ in $\mathcal{L}(X,Y)$, it is easy to see that

$$\widetilde{\mathcal{J}} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in \mathcal{K}(X),\ B \in \mathcal{L}(Y,X),\ C \in \mathcal{J},\ D \in \mathcal{K}(Y) \right\}$$

is a closed ideal in $\mathcal{L}(X \oplus Y)$, and moreover, the map $\mathcal{J} \mapsto \widetilde{\mathcal{J}}$ is injective. It follows that $\mathcal{L}(X \oplus Y)$ also has $2^{\mathfrak{c}}$ closed ideals, and this completes the proof of the theorem. □

As mentioned in the introduction, this result implies the recent result of Johnson and Schechtman [11] that $\mathcal{L}\left(L_p\right)$ contains $2^{\mathfrak{c}}$ closed ideals for $1 < p \neq 2 < \infty$.

Corollary 10. *Let $1 < p \neq 2 < \infty$. The algebra $\mathcal{L}\left(L_p\right)$ of operators on L_p contains exactly $2^{\mathfrak{c}}$ closed ideals.*

4. Applications II

As in the previous section, we will apply the general process developed in Section 2 to establish a class of pairs (X,Y) of Banach spaces for which $\mathcal{L}(X,Y)$ contains $2^{\mathfrak{c}}$ distinct closed ideals. However, we will be using Lemma 4 in this section as opposed to Lemma 3.

Let $1 \leqslant p < q \leqslant \infty$. Suppose that $\left(\ell_2^n\right)_{n=1}^{\infty}$ is a UFDD for a Banach space X with a lower ℓ_p-estimate and that $\left(\ell_\infty^n\right)_{n=1}^{\infty}$ is a UFDD for a Banach space Y with an upper ℓ_q-estimate. We will prove that $\mathcal{L}(X,Y)$ contains $2^{\mathfrak{c}}$ distinct closed ideals. As $\left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_p}$ is complemented in ℓ_p for all $1 < p < \infty$, we obtain that $\mathcal{L}\left(\ell_p, c_0\right)$ contains $2^{\mathfrak{c}}$ distinct closed ideals for all $1 < p < \infty$, which proves that our general setup incorporates the results presented in [7]. By duality, we obtain that $\mathcal{L}\left(\ell_1, \ell_p\right)$ and $\mathcal{L}\left(\ell_p, \ell_\infty\right)$ each contain $2^{\mathfrak{c}}$ distinct closed ideals. Hence, the cardinality of the set of closed ideals is exactly $2^{\mathfrak{c}}$ for each of $\mathcal{L}\left(\ell_p \oplus c_0\right)$, $\mathcal{L}\left(\ell_p \oplus \ell_\infty\right)$ and $\mathcal{L}\left(\ell_1 \oplus \ell_p\right)$ for all $1 < p < \infty$. Note that we also obtain that the cardinality of the set of closed ideals in $\mathcal{L}\left(\left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_1} \oplus c_0\right)$ is $2^{\mathfrak{c}}$; however, we are not able to conclude anything about $\mathcal{L}(\ell_1 \oplus c_0)$, as the finite-dimensional spaces ℓ_2^n are not uniformly complemented in ℓ_1.

In the previous section, for each $n \in \mathbb{N}$, the operator $T_n : E_n \to \ell_2^n$ was the formal identity between two n-dimensional Banach spaces. Now we will choose sequences $k_1 < l_1 < k_2 < l_2 < \cdots$ and operators $T_n : \ell_2^{k_n} \to \ell_\infty^{l_n}$. When considered as a matrix, each T_n will be much taller than it is wide.

Set $1 \leqslant p < \infty$. The probabilistic proofs for the existence of restricted isometry property matrices from compressed sensing [5] show that there exist sequences $k_1 < l_1 < k_2 < l_2 < \cdots$ with $\lim_{n\to\infty} k_n^{\max(1,p/2)} l_n^{-1} = 0$ such that if unit vectors $\left(x_{n,j}\right)_{j=1}^{l_n}$ are randomly chosen with uniform

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distribution in $\ell_2^{k_n}$, then with high probability we have, for all $J \subset \{1, 2, \ldots, l_n\}$ with $|J| \leqslant l_{n-1}$,

$$\frac{1}{2}\sum_{j\in J}|a_j|^2 \leqslant \Big\|\sum_{j\in J} a_j x_{n,j}\Big\|^2 \leqslant 2\sum_{j\in J}|a_j|^2 \text{ for all } (a_j)_{j\in J} \subset \mathbb{R}, \tag{22}$$

$$\sum_{j\in J}\left|\langle x, x_{n,j}\rangle\right|^2 \leqslant 2\|x\|^2 \text{ for all } x \in \ell_2^{k_n}. \tag{23}$$

We now show how this construction satisfies the conditions of Proposition 2 and Lemma 4.

Theorem 11. *Set $1 \leqslant p < q \leqslant \infty$. Suppose that $(\ell_2^n)_{n=1}^\infty$ is a UFDD for X with a lower ℓ_p-estimate and that $(\ell_\infty^n)_{n=1}^\infty$ is a UFDD for Y with an upper ℓ_q-estimate. Then $\mathcal{L}(X, Y)$ contains $2^{\mathfrak{c}}$ distinct closed ideals.*

Proof. Choose $k_1 < l_1 < k_2 < l_2 < \cdots$ in $\mathbb{N}$ with $\lim_{n\to\infty} k_n^{\max(1,p/2)} l_n^{-1} = 0$ and unit vectors $(x_{n,j})_{j=1}^{l_n} \subset \ell_2^{k_n}$ for all $n \in \mathbb{N}$ to satisfy formulas (22) and (23). Let $E_n = \ell_2^{k_n}$ and $F_n = \ell_\infty^{l_n}$ for all $n \in \mathbb{N}$. As E_n is a Hilbert space, we may take $\left(x_{n,j}^*\right)_{j=1}^{l_n} = (x_{n,j})_{j=1}^{l_n} \subset S_{E_n^*}$. Suppose that $C_1, C_2 > 0$ are such that if $(x_n)_{n=1}^\infty \in X$, then $(\sum\|x_n\|^p)^{1/p} \leqslant C_1\|(x_n)\|_X$, and if $(y_n)_{n=1}^\infty \in Y$, then $\|(y_n)\|_Y \leqslant C_2\,(\sum\|y_n\|^q)^{1/q}$.

For each $n \in \mathbb{N}$, we define the operator $T_n : \ell_2^{k_n} \to \ell_\infty^{l_n}$ by $x \mapsto \left(\langle x, x_{n,j}\rangle\right)_{j=1}^{l_n}$. We now show that the conditions of Proposition 2 are satisfied.

We have that condition (4) is satisfied, as if $(x_n) \in X$, then

$$\begin{aligned}\left\|T\left((x_n)\right)\right\|_Y &\leqslant C_2\left(\sum\|T_n x_n\|_\infty^q\right)^{1/q}\\ &= C_2\left(\sum \sup_{1\leqslant j\leqslant l_n}\left|\langle x_n, x_{n,j}\rangle\right|^q\right)^{1/q}\\ &\leqslant C_2\left(\sum\|x_n\|^q\right)^{1/q}\\ &\leqslant C_2\left(\sum\|x_n\|^p\right)^{1/p} \leqslant C_2C_1\|(x_n)\|_X.\end{aligned}$$

Thus, the map $(x_n) \mapsto T\left((x_n)\right)$ is well defined and bounded. Condition (5) is trivially satisfied, as $\left(x_{m,i}^*\right)_{i=1}^{l_m} = (x_{m,i})_{i=1}^{l_m}$ for all $m \in \mathbb{N}$.

To prove condition (7), fix $n \in \mathbb{N}$ and let $l \in \mathbb{N}$ be such that $l \geqslant l_n > k_n$. Given $m \in \mathbb{N}$ with $l_m \geqslant l$ and $A \subset \{1, 2, \ldots, l_m\}$ with $|A| = l$, set $t_n = \lceil l/k_n \rceil$. Partition A into $(A_j)_{j=1}^{t_n}$ such that $|A_j| \leqslant k_n$ for all $1 \leqslant j \leqslant t_n$. By formula (22), we have, for all $1 \leqslant j \leqslant t_n$,

$$\Big\|\sum_{i\in A_j}\sigma_i x_{m,i}\Big\|^2 \leqslant 2\left|A_j\right| \quad \text{for all } (\sigma_i)_{i\in A_j} \subset \{\pm 1\}.$$

Thus, for all $(\sigma_i)_{i=1}^l \subset \{\pm 1\}$, we have

$$\begin{aligned}\Big\|\sum_{i\in A}\sigma_i x_{m,i}\Big\| &\leqslant \sum_{j=1}^{t_n}\Big\|\sum_{i\in A_j}\sigma_i x_{m,i}\Big\|\\ &\leqslant \sum_{j=1}^{t_n} 2^{1/2}\left|A_j\right|^{1/2}\\ &\leqslant t_n 2^{1/2} k_n^{1/2}\\ &< (2l/k_n)2^{1/2}k_n^{1/2} < 4lk_n^{-1/2}.\end{aligned}$$

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Thus, for

$$\varphi_m(l) = \sup\left\{\left\|\sum_{i\in A}\sigma_i x_{m,i}\right\| : A \subset \{1,2,\ldots,l_m\},\ |A| \leqslant l,\ (\sigma_i)_{i\in A} \subset \{\pm 1\}\right\},$$

we have $\frac{\varphi_m(l)}{l} < 4k_n^{-1/2}$. Hence, $\lim_{l\to\infty} \sup_{m\in\mathbb{N},\ l_m\geqslant l} \dfrac{\varphi_m(l)}{l} = 0$, and we have condition (7).

We next verify Proposition 2(a). Fix $\varepsilon > 0$. There exists $\delta > 0$ such that if $(a_j) \in \ell_p$ with $\|(a_j)\|_{\ell_p} \leqslant C_1$ and $|a_j| \leqslant \delta$ for all $j \in \mathbb{N}$, then $\|(a_j)\|_{\ell_q} < C_2^{-1}\varepsilon$. Let $x = (x_n) \in S_X$ such that $\sup_{1\leqslant j\leqslant l_n} |\langle x_n, x_{n,j}\rangle| \leqslant \delta$ for all $n \in \mathbb{N}$. Thus, we have

$$\left(\sum_{n=1}^{\infty} \sup_{1\leqslant j\leqslant l_n} |\langle x_n, x_{n,j}\rangle|^p\right)^{1/p} \leqslant \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p} \leqslant C_1 \tag{24}$$

and

$$\begin{aligned}\|T((x_n))\|_Y &\leqslant C_2\left(\sum \|T_n x_n\|_\infty^q\right)^{1/q}\\ &= C_2\left(\sum \sup_{1\leqslant j\leqslant l_n} |\langle x_n, x_{n,j}\rangle|^q\right)^{1/q}\\ &< \varepsilon, \qquad \text{by formula (24) and our assumption on } \delta.\end{aligned}$$

Finally, it follows from formulas (22) and (23) that the conditions of Lemma 4 are satisfied for $s = 2$ and $t = p$. This in turn implies Proposition 2(b), and thus the proof is complete. □

Remark. The remark following the proof of Theorem 5 applies here, too. The closed ideals constructed are all contained in the ideal in the space of finitely strictly singular operators.

Theorem 11 gives the following immediate corollary:

Corollary 12. *Set* $1 < p < \infty$. *Then* $\mathcal{L}(\ell_p, c_0)$, $\mathcal{L}(\ell_1, \ell_p)$ *and* $\mathcal{L}(\ell_p, \ell_\infty)$ *each contain* $2^{\mathfrak{c}}$ *distinct closed ideals.*

Proof. We have by Theorem 11 that $\mathcal{L}\left(\left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_p}, c_0\right)$ contains $2^{\mathfrak{c}}$ distinct closed ideals, and $\left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_p}$ is isomorphic to ℓ_p for $1 < p < \infty$. By duality, we have that $\mathcal{L}(\ell_1, \ell_p)$ and $\mathcal{L}(\ell_p, \ell_\infty)$ each contain $2^{\mathfrak{c}}$ distinct closed ideals. □

In the previous section we deduced from our results that the cardinality of the lattice of closed ideals in $\mathcal{L}(L_p)$, $1 < p \neq 2 < \infty$, is $2^{\mathfrak{c}}$. Note that the Hardy space H_1 and its predual VMO can be seen as the 'well-behaved' limit cases of the L_p-spaces. For example, ℓ_2 is complemented in both spaces, and H_1 contains a complemented copy of ℓ_1 and VMO a complemented copy of c_0 (compare [19] and [20, page 125]). Thus, we deduce the following corollary:

Corollary 13. *The cardinality of the lattice of closed ideals in* $\mathcal{L}(VMO)$ *and* $\mathcal{L}(H_1)$ *is* $2^{\mathfrak{c}}$.

5. Final remarks and open problems

If one considers only Banach spaces X with a countable unconditional basis or, more generally, with a UFDD, then the cardinalities κ for which we know examples of Banach spaces X with a UFDD for which the number of nontrivial proper closed ideals in $\mathcal{L}(X)$ is κ are only the following three:

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$\kappa = 1$: For $X = \ell_p$, $1 \leqslant p < \infty$, or $X = c_0$, the closed ideal in the space of compact operators is the only nontrivial proper closed ideal [8].

$\kappa = 2$: For the spaces $X = \left(\bigoplus \ell_2^n\right)_{c_0}$ and its dual $X^* = \left(\bigoplus \ell_2^n\right)_{\ell_1}$, there are exactly two nontrivial proper closed ideals – the compacts and the closure of operators which factor through c_0 or ℓ_1, respectively [13, 14].

$\kappa = 2^{\mathfrak{c}}$: $\mathcal{L}(X)$ has $2^{\mathfrak{c}}$ closed ideals for the spaces listed in the previous two sections. In addition to these spaces, it was recently observed by Johnson [11] that $\mathcal{L}(T)$ and $\mathcal{L}\left(T^{(p)}\right)$, where T is a Tsirelson space and $T^{(p)}$ its p-convexification for $1 < p < \infty$, also have $2^{\mathfrak{c}}$ closed ideals.[1]

This raises the following questions:

Problem 14. *Are there Banach spaces X with a countable unconditional basis or unconditional UFDD for which the cardinality of the nontrivial proper closed ideals in $\mathcal{L}(X)$ is strictly between 2 and $2^{\mathfrak{c}}$? Can this cardinality be any natural number, countably infinite or $\mathfrak{c}$?*

An interesting space in the context of this question is $c_0 \oplus \ell_1$. According to [28], $\mathcal{L}(c_0 \oplus \ell_1)$ contains an ω_1-chain of closed ideals.

Problem 15. *What is the cardinality of the lattice of the closed ideals in the space of the operators on $c_0 \oplus \ell_1$?*

Another space of interest for Problem 14 is the Schreier space. In [2] it was shown that the space of operators on this space has continuum many maximal ideals.

Problem 16. *What is the cardinality of the lattice of closed ideals in the space of operators on Schreier spaces?*

Among the class of general separable Banach spaces, there are more examples for which the lattice of closed ideals in their space of operators, or at least its cardinality, is determined. Such a list can be found in [12]. For example, Mankiewicz [15] constructed a separable, superreflexive space X, with an FDD, for which there are $2^{\mathfrak{c}}$ multiplicative functional on $\mathcal{L}(X)$. Then, of course, the nullspaces of these functionals are closed ideals in $\mathcal{L}(X)$

Based on the construction by Argyros and Haydon [1] of a space on which all operators are compact perturbations of multiples of the identity, Tarbard [29] constructed for each $n \in \mathbb{N}$ a space X_n for which $\mathcal{L}(X_n)$ has exactly n nontrivial proper closed ideals. There are also Banach spaces X for which the cardinality of the lattice of closed ideals in $\mathcal{L}(X)$ is exactly $\mathfrak{c}$. Indeed, suppose that A is a separable Banach algebra isomorphic to the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ for a Banach space X which has the approximation property (to ensure that $\mathcal{K}(X)$ is the smallest nontrivial closed ideal). Then, as observed in [12], the closed ideals in $\mathcal{L}(X)$ arise from preimages of closed ideals in A. Examples of separable Banach spaces X for which $\mathcal{L}(X)/\mathcal{K}(X)$ has exactly continuum many closed ideals were constructed, for instance, in [30], [18] and [17], as explained in [9]. An example of a space X for which the number of closed ideals is infinite but countable seems to be missing.

Problem 17. *Are there separable Banach spaces X for which $\mathcal{L}(X)$ has countably infinitely many closed ideals?*

A candidate for such a space is $C[0,\alpha]$, where α is a large enough countable ordinal. But already for $\alpha = \omega^\omega$ (the first ordinal α for which $C[0,\alpha]$ is not isomorphic to c_0), the answer to the following question is not known:

Problem 18. *For a countable ordinal α, what are the closed ideals in $\mathcal{L}\big(C[0,\alpha]\big)$? What is the cardinality of the lattice of these ideals?*

[1]Moreover, during Kevin Beanland's talk at the Banach Space Webinar on 3 April 2020, Johnson noted that the case $p = 2$ was already covered using methods developed in [11].

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Another space of interest is $L_1[0,1]$. It was shown by Johnson, Pisier and Schechtman [10] that $\mathcal{L}(L_1[0,1])$ has at least $\mathfrak{c}$ closed ideals.

Problem 19. *How many closed ideals does* $\mathcal{L}(L_1[0,1])$ *have?*

As alluded to by our terminology, the closed ideals in $\mathcal{L}(X,Y)$ for Banach spaces X and Y form a lattice with respect to inclusion and with lattice operations given by $\mathcal{I} \wedge \mathcal{J} = \mathcal{I} \cap \mathcal{J}$ and $\mathcal{I} \vee \mathcal{J} = \overline{\mathcal{I} + \mathcal{J}}$ for closed ideals $\mathcal{I}$ and $\mathcal{J}$. In the foregoing problems, we have only asked about the cardinality of this lattice. As a future ambitious target, one could study the lattice structure.

After the submission of this paper, additional examples of spaces X were discovered for which $\mathcal{L}(X)$ has $2^{\mathfrak{c}}$ closed ideals:

- Manoussakis and Pelczar-Barwacz [16] showed that if X is a Schreier space of finite order or X is the arbitrarily distortable space constructed by the second author [24], then $\mathcal{L}(X)$ has $2^{\mathfrak{c}}$ closed ideals. Thus Problem 16 is only open for Schreier spaces of infinite order.
- The second author recently proved [26] that modified Schreier families of any order coincide with the usual Schreier families. It follows from this that the arguments of [11] extend to all higher-order Tsirelson spaces and their convexifications.

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