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ON A CHARACTERIZATION OF MAXIMAL IDEALS

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Let A be a commutative complex Banach algebra with identity e. Gleason [1] (cf. also Kahane and Żelazko [2]) has given the following characterization of maximal ideals in A.

THEOREM. *A subspace $X \subset A$ of codimension one is a maximal ideal in A if and only if it consists of non-invertible elements.*

The proofs given by Gleason and by Kahane and Żelazko are both based on the use of Hadamard's factorization theorem for entire functions. In this note we show that this can be avoided by using elementary properties of analytic functions.

Clearly any maximal ideal satisfies the above condition, so it suffices to show that if X is of codimension 1 and if X consists of non-invertible elements, then X is a maximal ideal in A. Since each element of A is of the form $x+\alpha e$ where $x \in X$ and α is scalar, we define a functional ϕ on A by putting $\phi(x+\alpha e)=\alpha$. Clearly ϕ is linear on A. Suppose that ϕ is not bounded. Then there exists a sequence $\{x_n+\alpha_n e\}$ such that

$$|\alpha_n| = |\phi(x_n+\alpha_n e)| > n\,\|x_n+\alpha_n e\|.$$

This implies that $\|x_n/\alpha_n+e\|<1/n$. Thus for large n, x_n/α_n is invertible which contradicts the hypothesis that X consists of non-invertible elements.

We now show that ϕ is multiplicative. Let $a \in A$ be such that $a\neq 0$. Put $\psi(\lambda) = \phi\,[\exp(\lambda a)]$ where λ is a complex variable. $\psi(\lambda)$ is an entire function since

$$\psi(\lambda) = \phi\,[\exp(\lambda a)] = \phi\left(\sum_0^\infty \frac{\lambda^n a^n}{n!}\right) = \sum_0^\infty \frac{\phi(a^n)}{n!}\lambda^n$$

which is dominated in absolute value by $\|\phi\| \exp\{|\lambda|\,\|a\|\}$. Since $\psi(\lambda)$ has no zeros it follows that $\log\psi(\lambda)$ is an entire function. We set

$$\log\psi(\lambda) = \sum_0^\infty c_n\lambda^n$$

so that

$$c_n = \frac{1}{2\pi i}\int_{|\lambda|=r} \frac{\log\psi(\lambda)}{\lambda^{n+1}}\,d\lambda \quad (n \geq 0).$$

Since $c_0=\log\psi(0)=\log\phi(e)=0$, for $n\geq 0$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\lambda|=r} \frac{\overline{\log\psi(\lambda)}}{\lambda^{n+1}}\,d\lambda &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_0^\infty \bar{c}_k r^k e^{-ik\theta}\right) r^{-n} e^{-in\theta}\,d\theta \\ &= \frac{1}{2\pi}\sum_0^\infty \bar{c}_k r^{k-n}\int_0^{2\pi} e^{-ik\theta-in\theta}\,d\theta \\ &= 0\end{aligned}$$

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so that

$$c_n = \frac{1}{\pi i}\int_{|\lambda|=r} \mathrm{Re}\,[\log \psi(\lambda)]\,\frac{d\lambda}{\lambda^{n+1}}$$

$$= \frac{1}{\pi i}\int_{|\lambda|=r} \log|\psi(\lambda)|\,\frac{d\lambda}{\lambda^{n+1}}.$$

Since $\log|\psi(\lambda)| \le \log\|\phi\| + |\lambda|\,\|a\|$, it follows that

$$|c_n| \le (\log\|\phi\| + \|a\|\,r)\frac{2}{r^n}$$

so that $c_n = 0$ for $n \ge 2$. Thus $\log \psi(\lambda) = c_1\lambda$. Hence

$$\psi(\lambda) = \sum_0^\infty \frac{c_1^n}{n!}\lambda^n.$$

But

$$\psi(\lambda) = \sum_0^\infty \frac{\phi(a^n)}{n!}\lambda^n$$

so that $\phi(a^n) = c_1^n = [\phi(a)]^n$ and, in particular, that $\phi(a^2) = [\phi(a)]^2$. Since

$$\phi[(a+b)^2] = [\phi(a+b)]^2 = [\phi(a)+\phi(b)]^2 = [\phi(a)]^2 + 2\phi(a)\phi(b) + [\phi(b)]^2$$

and

$$\phi((a+b)^2) = \phi(a^2+2ab+b^2) = \phi(a^2) + 2\phi(ab) + \phi(b^2),$$

it follows that $\phi(ab) = \phi(a)\,\phi(b)$ for all a, b in A. Since ϕ is a homomorphism with kernel X, it follows that X is a maximal ideal.

As shown by Zelazko [3], a modification of the reasoning in the last paragraph yields a proof of the above theorem when A is non-commutative.

References

1. A. M. Gleason, *A characterization of maximal ideals*, J. Analyse Math. **19** (1967), 171–172.

2. J.-P. Kahane and W. Żelazko, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math. **29** (1968), 339–343.

3. W. Żelazko, *A characterization of multiplicative linear functionals in complex Banach algebras*, Studia Math. **30** (1968), 83–85.

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