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ASYMPTOTOLOGY—A CAUTIONARY TALE

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Abstract

The art of asymptotology is a powerful tool in applied mathematics and theoretical physics, but can lead to erroneous conclusions if misapplied. A seemingly paradoxical case is presented in which a local analysis of an exactly solvable problem appears to find solutions to an eigenvalue problem over a continuous range of the eigenvalue, whereas the spectrum is known to be discrete. The resolution of the paradox involves the Stokes phenomenon. The example illustrates two of Kruskal's Principles of Asymptotology.

1. Introduction

In his 1963 pedagogical essay [7] Kruskal coined the term *asymptotology* to describe the "art of dealing with applied mathematical systems in limiting cases".

By "applied mathematical system" he means one that satisfies a *Principle of Classification (or Determinism)*. By this he means that the system must be completely specified mathematically so that a well-defined individual solution can be *determined* (or a family of solutions *classified*) for systematic asymptotic study.

He then formulates seven *Principles of Asymptotology*:

1. The Principle of Simplification;
2. The Principle of Recursion;
3. The Principle of Interpretation;
4. The Principle of Wild Behaviour;
5. The Principle of Annihilation;
6. The Principle of Maximal Balance;
7. The Principle of Mathematical Nonsense.

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In this paper we pose a simple mathematical problem—a second-order linear ordinary differential equation (ODE) eigenvalue problem whose spectrum is to be determined asymptotically in the limit $\epsilon \to 0$, where ϵ^2 is the coefficient of the second derivative term. This is used to illustrate some of the principles of asymptotology and to provide a cautionary example of some of the pitfalls.

Since 1963 a number of good texts have appeared on asymptotic methods in areas of applied mathematics. We shall use for reference the book of Bender and Orszag [1].

The genesis of the problem lies in an attempt to illustrate, in a brief one-page Comment [5], what we suggested was a flaw in a published [6] analysis of the spectrum of drift wave instabilities in a plasma with sheared drift velocity. The earlier authors [6] had concluded that the spectrum is continuous.

The physical problem is quite complicated and had not really been posed in such a way in [6] as to fully satisfy Kruskal's Principle of Classification. That is, it was arguably not yet an applied mathematical system and thus it was not perhaps surprising that the authors would come to grief in trying to apply asymptotic reasoning.

Our approach was to replace the original physical problem with a simpler, well-posed problem and to show that an analogous line of reasoning to that used in [6] would give erroneous results. It is true that our "counter example" is not very relevant to the original physical problem, and there is thus room for debate as to whether it catches its essence. However, we note that a much more physically motivated model equation has subsequently been analysed [3] with the same conclusion—there is no continuous spectrum in this problem.

2. The model problem

Consider the ODE defined on the real x-line

$$\epsilon^2 y''(x) + i\epsilon y'(x) + [\lambda - f(x)]\, y(x) = 0. \tag{1}$$

Specifically, for the function f we take

$$f(x) \equiv x^2/4 - ix/2. \tag{2}$$

Equation (1) is to be solved under the boundary conditions $y(x) \to 0$ as $x \to \pm\infty$ to determine the allowed (possibly complex) values of the eigenvalue λ.

The first order term can be removed by defining the new dependent variable $Y(x) \equiv \exp(ix/2\epsilon)y(x)$, leading to the standard form [1, (10.1.5)]

$$\epsilon^2 Y'' - Q(x)Y = 0, \tag{3}$$

where $Q(x|\lambda) \equiv (x - i)^2/4 - \lambda$.

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By transforming the independent variable using $u \equiv (x - i)/\epsilon^{1/2}$, defining $\psi(u) \equiv Y(x)$, and analytically continuing from the line $\operatorname{Im} u = -1/\epsilon^{1/2}$ to the real line, $\operatorname{Im} u = 0$, we find that (1) is nothing but a disguised version of the "quantum oscillator" equation

$$\psi''(u) + \left(E - u^2/4\right)\psi(u) = 0,$$

where $E \equiv \lambda/\epsilon$.

This equation can be solved [1, pp. 332–3, 573–4] in terms of the parabolic cylinder functions D_ν. When $E = n + 1/2$, where $n = 0, 1, \ldots$, there exists a solution satisfying the boundary conditions $\psi(u) \to 0$ as $u \to \pm\infty$:

$$\psi_n(u) = D_n(u) \equiv \operatorname{He}_n(u)\exp\left(-u^2/4\right),$$

He_n denoting a Hermite polynomial.

The boundary conditions for y are also satisfied on the original contour, $\operatorname{Im} u = -1/\epsilon^{1/2}$, so the eigenvalue spectrum of (1) is discrete and given by $\lambda = \epsilon E_n \equiv \epsilon(n + 1/2)$.

In terms of the original variables, the general solution in the complex plane (arbitrary $\lambda, z \in \mathbb{C}$) is

$$y(z) = \exp\left(-\frac{iz}{2\epsilon}\right)\left[A D_\nu\left(\frac{z-i}{\epsilon^{1/2}}\right) + B D_{-\nu-1}\left(\frac{i(z-i)}{\epsilon^{1/2}}\right)\right],$$

where $\nu \equiv \lambda/\epsilon - 1/2$ and A and B are arbitrary constants. The dominant asymptotic behaviour as $x \to \pm\infty$ is as $z^{-\nu-1}\exp[(z^2/4 - iz)/\epsilon]$, with the subdominant solution going as $z^\nu \exp(-z^2/4\epsilon)$.

The transformation $Y(x) \equiv \exp(ix/2\epsilon)y(x)$ puts the problem in a more standard form, and in our case leads to an exact solution. However, it is a trick that may not be available in more complex physical problems. In plasma physics we often have to deal with higher order differential equations or even integro-differential equations [2]. Thus, in order to gain the maximum insight into the general problem, we return to the form given in (1).

3. Boundary layer ordering

In this section we suppose that the solution of (1) is localised in a narrow region (an "internal boundary layer" [1, p. 455]) about an arbitrary point x_0 on the real line. Supposing that the width of this region is $O(\epsilon^{1/2})$ we define a stretched variable ξ by setting

$$x = x_0 + \epsilon^{1/2}\xi, \quad y(x) = y_0(\xi) + \epsilon^{1/2} y_{1/2}(\xi) + \cdots.$$

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In the boundary layer, $\xi = O(1)$.

We also expand the eigenvalue in a power series in $\epsilon^{1/2}$, $\lambda = \lambda_0 + \epsilon^{1/2}\lambda_1 + \cdots$, where the λ_j, $j = 0, 1, 2, \ldots$, are $O(1)$. In this ordering the derivative terms are small, so at lowest order we find the equation

$$(\lambda_0 - f_0) y_0(\xi) = 0,$$

where $f_0 \equiv f(x_0)$. (In general, $f_0^{(l)}$ denotes the l'th derivative at x_0, $f^{(l)}(x_0)$.) This can be satisfied for arbitrary $y_0(\xi)$ by choosing

$$\lambda_0 = f_0. \tag{4}$$

Proceeding to next order, we find a first order ODE determining y_0

$$i\frac{dy_0}{d\xi} - (f_0'\xi - \lambda_1) y_0 = 0.$$

This is solved by

$$y_0 = \text{const} \exp[i(\lambda_1 \xi - f_0' \xi^2/2)]. \tag{5}$$

Since $\operatorname{Im} f_0' = -1/2$, this solution approaches zero as $\xi \to \pm\infty$ for *arbitrary* λ_1. As concluded in [6] there appears to be a continuum of allowed solutions!

This is clearly wrong, since we have given the exact point spectrum in Section 2, but the error is quite subtle. For instance, we do *not* find a contradiction by proceeding to higher order in the $\epsilon^{1/2}$ expansion—the solution found above is a perfectly respectable asymptotic solution and each order decays exponentially to zero as $\xi \to \pm\infty$.

However, the boundary layer does not really extend to infinity—once ξ becomes $O(\epsilon^{-1/2})$ the ordering breaks down. Thus we must match to "outer solutions" on either side of the layer [1, pp. 421–84] to construct a *global* solution to the problem. Even though the solution vanishes exponentially towards both sides of the layer, there is still the possibility that this trend may reverse in the outer region, leading to violation of the boundary conditions by the global solution at "true infinity".

4. WKB method

By arranging that the first derivative term dominates the second derivative, the boundary layer ordering approach gives only a first order ODE at leading nontrivial order, and thus there is only one solution. The original problem was a second order ODE, the general solution being a linear superposition of two independent functions, so we have clearly lost one solution using this method. To recover it we need to find an asymptotic expansion method that respects Kruskal's *Principle of Maximal Balance*—"no term should be neglected without a good reason" [7, p. 34].

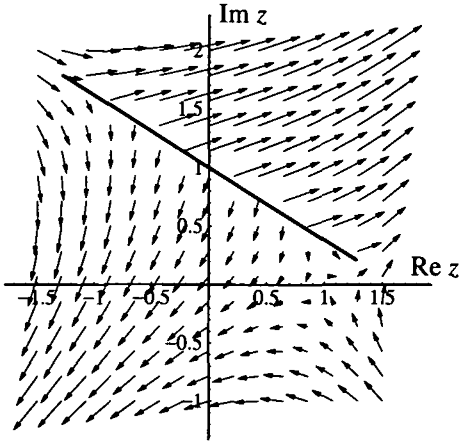

FIGURE 1. Polya plot [9] in the complex z-plane for the branch $q_-(z)$ in the case $\lambda = 1/4 - i/2$. The branch cut between the turning points at $i \pm 2\sqrt{\lambda}$ is also shown, as is the point $z = 1$ at which $q_-(z)$ vanishes.

Kruskal's *Principle of Wild Behaviour* [7, p. 30] tells us that solutions are lost by not allowing for sufficiently "wild behaviour" in the initial ansatz. In our case this wild behaviour is a rapid variation of the solution with respect to x—the Principle of Maximal Balance tells us to balance the first and second derivative terms in (1) and this is achieved by assuming variation on the ϵ scale rather than the $\epsilon^{1/2}$ scale assumed in the boundary layer ordering.

In discussing his Principle of Wild Behaviour, Kruskal suggests "to write the unknown as the exponential of a new unknown represented by a series, the dominant term of which must become infinite (in the limit as $\epsilon \to 0$)". This is the powerful Wentzel-Brillouin-Kramers (WKB) method, which includes boundary layer theory as a special case [1, p. 484].

Following Bender and Orszag [1, p. 485] we set

$$y = A \exp(S/\epsilon), \tag{6}$$

where A and S are slowly varying functions of z [that is, $A'/A = O(1)$, $S''/S' = O(1)$]. Defining $q(z) \equiv S'(z)$, at lowest order we find the "local dispersion relation"

$$q^2 + iq + \lambda - f = 0. \tag{7}$$

Solving this we find the *two* branches $q_\pm = -i/2 \pm [f(z) - \lambda - 1/4]^{1/2}$. With the choice of f in (2), we write these solutions in the form

$$q_\pm = -\frac{i}{2} \pm \frac{\sqrt{\lambda}}{2}\left(\frac{z-i}{\sqrt{\lambda}} - 2\right)^{1/2}\left(\frac{z-i}{\sqrt{\lambda}} + 2\right)^{1/2}. \tag{8}$$

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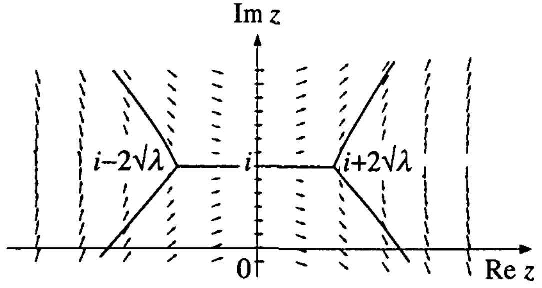

FIGURE 2. Stokes lines in the complex z-plane emanating from the turning points at $i \pm 2\sqrt{\lambda}$ for the differential equation in the transformed form, (3) in the case of real λ. Note that the real axis cuts two of these lines.

We take the branch cut for the square root $w^{1/2}$ of an arbitrary complex variable w to be along the negative real axis in the w-plane. With this convention, writing (8) in the form above defines the two branches on the z-plane cut by a straight line joining the branch points at $z = i \pm 2\sqrt{\lambda}$, as shown in Figure 1.

We see from (7) that q should have a zero at x_0 if $\lambda = f(x_0)$. This is the *same* condition, (4), as we found for the location of the boundary layer in Section 3. With the choice of f in (2), the branches $q_\pm(z)$ are found to have zeros at $z = i \mp (4\lambda - 1)^{1/2}$, respectively. Taking $\lambda = f(x_0)$, we find that it is the root $z = i + (4\lambda - 1)^{1/2}$ that gives the desired boundary layer point on the real line, $z = x_0$, the other root being complex. Thus the branch $q_-(z)$ should correspond to the boundary layer solution in the neighbourhood of x_0.

Expanding $q_\pm$ about x_0 and integrating to find the eikonal S, we find one branch, $S_-/\epsilon = i\lambda_1\xi - if_0'\xi^2/2$, that agrees with the exponent in the boundary layer solution, (5). However the other branch is given by $S_+/\epsilon = -i\xi/\epsilon^{1/2} - i\lambda_1\xi + if_0'\xi^2/2$, which violates the boundary layer ordering and was thus missed.

We have now found the missing solution, but this does not in itself resolve the paradox: does not the S_- branch by itself provide a solution that satisfies the global boundary conditions? If this branch provided a global asymptotic solution, then this would be true. However, it has been known since the work of Stokes in the last century that the coefficients of the two asymptotic representations may change as we cross a *Stokes line* in the complex plane [1, pp. 112–117], where the roles of the dominant and subdominant solutions interchange. (Sometimes this is called an anti-Stokes line if a factor i is included in the exponent in (6) [8].) In our problem, if the real axis crosses one or more Stokes lines, then the $S_\pm$ solutions will couple and the S_- branch alone will not represent a global solution.

The Stokes phenomenon has been much discussed in texts such as [1] and we do not intend to review this here. However, we remark that White's [8] general

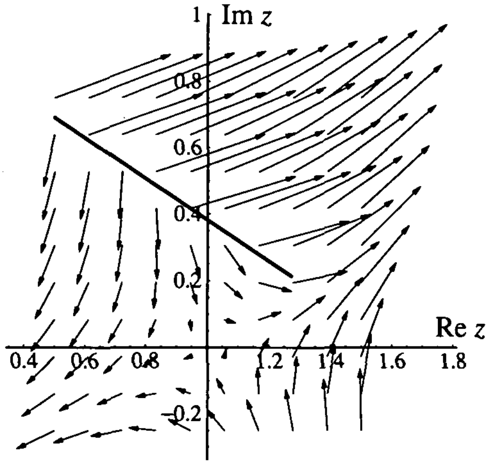

FIGURE 3. Detail of Figure 1 in the neighbourhood of the "internal boundary layer" at $z = 1$.

graphical computer method for determining Stokes lines in second order ODEs in the standard form of (3) has been found very useful in practical problems, see for example [4]. White's approach is based on plotting vectors representing the direction of $i\,Q^{-1/2}$ (or $i\,\bar{Q}^{1/2}$) on a grid in the complex plane. Along curves everywhere tangent to these vectors, the real part of S does not change, so neither solution is dominant or subdominant. These vectors thus represent the directions of local Stokes lines. Fortran and C versions of White's program exist, but in this paper we have used the `PlotPolyaField` function in *Mathematica* [9] to implement the strategy.

Figure 2 shows such a plot and also sketches in the Stokes curves emanating from the turning points at $z = i \pm 2\lambda^{1/2}$ (that is, the points where $Q(z) = 0$). It is seen that the real line [the domain of the ODE, (1)] crosses two of these lines, and thus the localised S_+ asymptotic solution is not a global solution. That is, both $S_\pm$ solutions need to be taken into account in determining the eigenvalue on the original domain.

Figures 1 and 3 show "Polya plots", made using `PlotPolyaField`, of $iq_-(z)$, with q_- as defined in (8) for the case $\lambda = 1/4 - i/2$. Using $\lambda = f(x_0)$ we see that this choice corresponds to putting the boundary layer at $x_0 = 1$. The vectors are in the local direction of $i\bar{q}_-(z)$ (where the bar denotes complex conjugation) with the lengths proportional to $1 + |q_-(z)|$.

As discussed in [2], the general condition for a turning point at z_0 is not $q(z_0) = 0$, but the requirement that that there be a branch point at z_0 (that is, a point where two branches of the local dispersion relation merge). These two conditions are the same in the standard form, (3), but not in general.

For instance, the boundary layer point x_0 defined in (4) is not a branch point, although $q_-(x_0)$ vanishes there, and it is therefore *not* a turning point. This is not

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surprising, since it is an artifact of the transformation $Y(x) \equiv \exp(ix/2\epsilon)y(x)$. On the other hand, the branch points at $i \pm 2\lambda^{1/2}$ remain invariant under this transformation, as does the "global dispersion relation" [2, (18)]

$$\oint_C iq_-(z, \lambda)\, dz = (2n + 1)\pi,$$

where the contour C encloses the branch cut shown in Figure 1.

5. Conclusion

By setting up a well defined "applied mathematical system" in accordance with Kruskal's zeroth law of asymptotology, the Principle of Classification, we have been able to shed light on a controversy in the plasma physics literature. We have found that at least two of Kruskal's Principles of Asymptotology, the Principle of Wild Behaviour and the Principle of Maximal Balance, come into play in this problem.

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