

## Appendix B

### Center of mass (C-M) motion

The center-of-mass (C-M) motion can, in fact, be handled correctly in the usual non-relativistic many-body problem. We follow the approach of [Fo69]. Introduce the usual C-M and internal coordinates as indicated in Fig. B.1.

$$\begin{aligned} \mathbf{X} &\equiv \frac{1}{A} \sum_{i=1}^A \mathbf{x}_i \\ \mathbf{x}'_i &\equiv \mathbf{x}_i - \mathbf{X} \quad ; i = 1, 2, \dots, A \end{aligned} \quad (\text{B.1})$$

It follows that [Fo69]

$$\begin{aligned} \sum_{i=1}^A \mathbf{x}'_i &= 0 \\ d^3x_1 d^3x_2 \cdots d^3x_A &= d^3(\mathbf{X}) d^3x'_1 d^3x'_2 \cdots d^3x'_A \delta^{(3)}\left(\sum_{i=1}^A \mathbf{x}'_i\right) \end{aligned} \quad (\text{B.2})$$

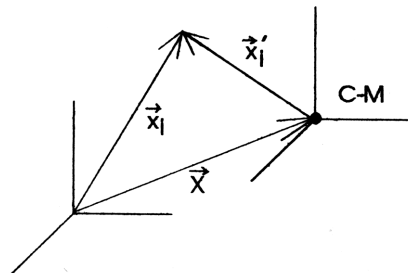


Fig. B.1. C-M and internal coordinates;  $i = 1, 2, \dots, A$  labels the particles.

The rewriting of the volume element in the second equation is particularly useful. The target wave function can be written quite generally as

$$\Psi_i(\mathbf{x}_1, \dots, \mathbf{x}_A) = \frac{1}{\sqrt{A^3 \Omega}} e^{i\mathbf{p} \cdot \mathbf{X}} \psi_i(\mathbf{x}'_1, \dots, \mathbf{x}'_A) \quad (\text{B.3})$$

Now the nuclear charge density operator, for example, is given as

$$\hat{\rho}(\mathbf{x}) = \sum_{i=1}^Z \delta^{(3)}(\mathbf{x} - \mathbf{x}_i) \quad (\text{B.4})$$

Its Fourier transform is written in terms of C-M and internal coordinates as

$$\int e^{-i\mathbf{q} \cdot \mathbf{x}} \hat{\rho}(\mathbf{x}) d^3x = e^{-i\mathbf{q} \cdot \mathbf{X}} \left( \sum_{i=1}^Z e^{-i\mathbf{q} \cdot \mathbf{x}'_i} \right) \quad (\text{B.5})$$

The integral over the C-M coordinate can be done in the big box of volume  $\Omega$  with p.b.c., and the result is

$$\langle f | \int e^{-i\mathbf{q} \cdot \mathbf{x}} \hat{\rho}(\mathbf{x}) d^3x | i \rangle = \delta_{\mathbf{p}, \mathbf{p}' + \mathbf{q}} \langle \psi_f | \int e^{-i\mathbf{q} \cdot \mathbf{x}} \hat{\rho}(\mathbf{x}) d^3x | \psi_i \rangle \quad (\text{B.6})$$

The remaining matrix element is now written in internal coordinates in the C-M system.

$$\begin{aligned} \langle \psi_f | \int e^{-i\mathbf{q} \cdot \mathbf{x}} \hat{\rho}(\mathbf{x}) d^3x | \psi_i \rangle &= \int d^3x'_1 \cdots d^3x'_A \delta^{(3)} \left( \sum_{i=1}^A \mathbf{x}'_i \right) \\ &\times \psi_f^*(\mathbf{x}'_1, \dots, \mathbf{x}'_A) \left( \sum_{i=1}^Z e^{-i\mathbf{q} \cdot \mathbf{x}'_i} \right) \psi_i(\mathbf{x}'_1, \dots, \mathbf{x}'_A) \end{aligned} \quad (\text{B.7})$$

Now use

$$\begin{aligned} \delta_{\mathbf{p}, \mathbf{p}' + \mathbf{q}}^2 &= \delta_{\mathbf{p}, \mathbf{p}' + \mathbf{q}} \\ \sum_f \delta_{\mathbf{p}, \mathbf{p}' + \mathbf{q}} &= \sum_f' \sum_{\mathbf{p}'} \delta_{\mathbf{p}, \mathbf{p}' + \mathbf{q}} = \sum_f' \end{aligned} \quad (\text{B.8})$$

Here  $\sum_f'$  goes over all internal quantum numbers. This allows one to write the sum over final states of the square of the matrix element as

$$\sum_f |\langle f | \int e^{-i\mathbf{q} \cdot \mathbf{x}} \hat{\rho}(\mathbf{x}) d^3x | i \rangle|^2 = \sum_f' |\langle \psi_f | \int e^{-i\mathbf{q} \cdot \mathbf{x}} \hat{\rho}(\mathbf{x}) d^3x | \psi_i \rangle|^2 \quad (\text{B.9})$$

Now the analysis proceeds as in the text.

In summary, assume the current density has the form

$$\hat{J}_v(\mathbf{x}) = \sum_{i=1}^A [j_v(\mathbf{x}_i)\delta^{(3)}(\mathbf{x} - \mathbf{x}_i)] \tag{B.10}$$

Its Fourier transform can then be written

$$\int e^{-i\mathbf{q}\cdot\mathbf{x}} \hat{J}_v(\mathbf{x}) d^3x = \sum_{i=1}^A [j_v(\mathbf{x}_i)e^{-i\mathbf{q}\cdot\mathbf{x}_i}] \tag{B.11}$$

Assume further that this expression can be written in terms of C-M and internal coordinates as

$$\int e^{-i\mathbf{q}\cdot\mathbf{x}} \hat{J}_v(\mathbf{x}) d^3x = e^{-i\mathbf{q}\cdot\mathbf{X}} \sum_{i=1}^A [j_v(\mathbf{x}'_i)e^{-i\mathbf{q}\cdot\mathbf{x}'_i}] \tag{B.12}$$

This holds true in the following cases:

- It is true for the charge density operator [see Eq. (B.5)];
- It is true for the spin current density [see Eq. (9.17)];
- It is true for the *transverse part* of the convection current density.

We give a proof of this third case. Consider the transverse part of the current defined by (here  $\lambda = \pm 1$ )

$$\hat{\mathbf{J}}(\mathbf{x}) \cdot \mathbf{e}_{\mathbf{q}\lambda}^\dagger = \sum_{i=1}^Z \left[ \frac{\mathbf{p}(i)}{m}, \delta^{(3)}(\mathbf{x} - \mathbf{x}_i) \right]_{\text{sym}} \cdot \mathbf{e}_{\mathbf{q}\lambda}^\dagger \tag{B.13}$$

Since  $m\dot{\mathbf{x}}_i = m\dot{\mathbf{X}} + m\dot{\mathbf{x}}'_i$ , it follows that

$$\mathbf{p}(i) = \frac{1}{A}\mathbf{p} + \mathbf{p}'(i) \tag{B.14}$$

Now note that the transverse part of the convection current from the C-M, when the target is initially at rest, satisfies

$$\frac{1}{2A}(\mathbf{p} + \mathbf{p}') \cdot \mathbf{e}_{\mathbf{q}\lambda}^\dagger = -\frac{1}{2A}\mathbf{q} \cdot \mathbf{e}_{\mathbf{q}\lambda}^\dagger = 0 \tag{B.15}$$

Hence the C-M momentum does not contribute, and one can rewrite Eq. (B.13) as

$$\hat{\mathbf{J}}(\mathbf{x}) \cdot \mathbf{e}_{\mathbf{q}\lambda}^\dagger = \sum_{i=1}^Z \left[ \frac{\mathbf{p}'(i)}{m}, \delta^{(3)}(\mathbf{x} - \mathbf{x}_i) \right]_{\text{sym}} \cdot \mathbf{e}_{\mathbf{q}\lambda}^\dagger \tag{B.16}$$

Thus the stated result is established.

In conclusion, it follows that

$$\begin{aligned} \overline{\sum}_i \sum_f |\langle f | \int e^{-i\mathbf{q}\cdot\mathbf{x}} \hat{\rho}(\mathbf{x}) d^3x | i \rangle|^2 &= \tag{B.17} \\ & \overline{\sum}_i \sum_f' |\langle \psi_f | \int e^{-i\mathbf{q}\cdot\mathbf{x}} \hat{\rho}(\mathbf{x}) d^3x | \psi_i \rangle|^2 \\ \overline{\sum}_i \sum_f \sum_{\lambda=\pm 1} |\langle f | \int e^{-i\mathbf{q}\cdot\mathbf{x}} \hat{\mathbf{J}}(\mathbf{x}) \cdot \mathbf{e}_{\mathbf{q}\lambda}^\dagger d^3x | i \rangle|^2 &= \\ & \overline{\sum}_i \sum_f' \sum_{\lambda=\pm 1} |\langle \psi_f | \int e^{-i\mathbf{q}\cdot\mathbf{x}} \hat{\mathbf{J}}(\mathbf{x}) \cdot \mathbf{e}_{\mathbf{q}\lambda}^\dagger d^3x | \psi_i \rangle|^2 \end{aligned}$$

All of the subsequent analysis proceeds exactly as in the text.

A few comments are relevant. These are *exact* relations within non-relativistic quantum mechanics. The matrix elements are computed in the internal space according to Eq. (B.7); however, there is an A-body constraint  $\delta^{(3)}(\sum_{i=1}^A \mathbf{x}_i)$  in them. One usually does not deal correctly with this A-body constraint in calculations involving one or more valence particles, but there are models, such as the harmonic oscillator model, where it is possible to do so.

Within the framework of many particles in a harmonic oscillator potential, the center-of-mass motion can be taken into account by writing [E155, Ta58]

$$f(\kappa) = f_{\text{CM}}(\kappa) F_{\text{SM}}(\kappa) \tag{B.18}$$

Here  $F_{\text{SM}}(\kappa)$  is calculated with  $A$  independent nucleons in a harmonic oscillator shell-model potential, and  $f(\kappa)$  is the transition form factor calculated with an intrinsic wave function with coordinates measured with respect to the center-of-mass; this is clearly what one is after. The C-M correction factor is

$$\begin{aligned} f_{\text{CM}}(\kappa) &= \exp\left(\frac{y}{A}\right) \\ y &\equiv \left(\frac{\kappa b_{\text{osc}}}{2}\right)^2 \\ \hbar\omega_{\text{osc}} &= \frac{\hbar^2}{mb_{\text{osc}}^2} \end{aligned} \tag{B.19}$$

Note that the correction factor goes as  $1/A$  where  $A$  is the number of nucleons. In calculations, this additional factor can always be conveniently lumped, together with the single-nucleon form factor of chapter 19, into an effective Mott cross section

$$\bar{\sigma}_{\text{M}} \equiv f_{\text{SN}}^2(\kappa) f_{\text{CM}}^2(\kappa) \sigma_{\text{Mott}} \tag{B.20}$$

Unless stated otherwise, this expression is used in all the traditional nuclear physics calculations carried out in this text.

We proceed to demonstrate the result in Eq. (B.19) in the case of four nucleons in the 1s state of the three-dimensional simple harmonic oscillator.<sup>1</sup> The independent-particle wave function in this case is

$$\Psi_{\text{SM}} \sim \exp\left(-\frac{1}{2b_{\text{osc}}^2} \sum_{i=1}^A \mathbf{r}_i^2\right) \quad (\text{B.21})$$

The norm is discussed below. Introduce C-M and relative coordinates according to

$$\begin{aligned} \mathbf{R} &\equiv \frac{1}{A} \sum_{i=1}^A \mathbf{r}_i \\ \mathbf{r}'_i &\equiv \mathbf{r}_i - \mathbf{R} \end{aligned} \quad (\text{B.22})$$

Use the simple, crucial identity

$$\sum_{i=1}^A \mathbf{r}_i^2 = \sum_{i=1}^A \mathbf{r}'_i{}^2 + A\mathbf{R}^2 \quad (\text{B.23})$$

Hence

$$\Psi_{\text{SM}} \sim \exp\left(-\frac{A\mathbf{R}^2}{2b_{\text{osc}}^2}\right) \psi_{\text{int}}(\mathbf{r}'_i) \quad (\text{B.24})$$

Now compute the charge form factor

$$\begin{aligned} F_{\text{SM}}(\kappa) &\equiv \frac{1}{Z} \langle \Psi_{\text{SM}} | \sum_{i=1}^Z e^{-i\mathbf{q}\cdot\mathbf{r}_i} | \Psi_{\text{SM}} \rangle \\ &\sim \frac{1}{Z} \int d^3R e^{-i\mathbf{q}\cdot\mathbf{R}} \exp\left(-\frac{A\mathbf{R}^2}{b_{\text{osc}}^2}\right) \langle \psi_{\text{int}} | \sum_{i=1}^Z e^{-i\mathbf{q}\cdot\mathbf{r}'_i} | \psi_{\text{int}} \rangle \end{aligned} \quad (\text{B.25})$$

The Fourier transform of the gaussian is immediately performed to give

$$F_{\text{SM}}(\kappa) = \exp\left(-\frac{b_{\text{osc}}^2 \mathbf{q}^2}{4A}\right) \frac{1}{Z} \langle \psi_{\text{int}} | \sum_{i=1}^Z e^{-i\mathbf{q}\cdot\mathbf{r}'_i} | \psi_{\text{int}} \rangle \quad (\text{B.26})$$

One can now check the normalization. Set  $\kappa \equiv |\mathbf{q}| = 0$ , and since both the shell-model and internal wave function are normalized, the overall factor is correct. This result is now solved for the true internal form factor

$$\begin{aligned} F_{\text{SM}}(\kappa) &= \exp\left(-\frac{y}{A}\right) f_{\text{int}}(\kappa) \\ f_{\text{int}}(\kappa) &= f_{\text{CM}}(\kappa) F_{\text{SM}}(\kappa) \end{aligned} \quad (\text{B.27})$$

<sup>1</sup> For the extension, see [de66].

This is the stated result. Note that  $f_{\text{int}}(\kappa)$  is now calculated with true internal wave functions, the true internal volume element (see above discussion), and with the constraint  $\sum_{i=1}^A \mathbf{r}'_i = 0$  thereby incorporated. The physics of this result is the following. The independent-particle model includes motion of the center-of-mass. This smears out the charge (probability) density. The true internal density is more compact, and hence its form factor falls off more slowly with  $\kappa$ . With many nucleons, the C-M motion does not smear out the density as much. Of course, this discussion is still all within the framework of the harmonic oscillator shell model. The extension to other forms of the potential, and especially to the fully relativistic case, is still an open problem.